Automated Exchange Economies*

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Abstract:
The canonical mechanism for financial asset exchange is the limit-order book. In decentralized blockchain ledgers (DeFi), costs and delays in appending new blocks to the ledger render a limit-order book impractical. Instead, a "pricing curve" is specified (e.g., the "constant product pricing function") and implemented using smart contracts deployed to the ledger. We develop a framework to study the equilibrium properties of such markets. Our framework provides new insights into how informational frictions distort liquidity provision in DeFi markets.

1 Introduction

Decentralized finance, or DeFi, summarizes classical financial intermediation activities that take place on blockchains and in other decentralized marketplaces. DeFi represents a growing and sizable portion of trade in blockchain-based assets. One of the core activities of DeFi is automated market-making. Automated markets are smart-contract-based exchanges that facilitate trading or swaps of tokens that are native to a decentralized distributed ledger. Due to the costly nature of blockchain-based communication, running a limit order book on a blockchain is prohibitively expensive. As a consequence, we have observed a rapid pace of innovation and development in efforts to provide intermediation services, known as automated market makers (AMMs), on blockchains.

While AMMs economize on transaction costs, by design, they are passive and do not react to the information available in the market. As a result, they are susceptible to losses; what is sometimes referred to as impermanent losses or losses due to adverse selection; see Glosten and Milgrom (1985). In this paper, we undertake a systematic analysis of the design of AMMs.

One might expect these automated markets to resemble a central limit order book that runs as code—smart contracts—that happens to be executed on the blockchain by the decentralized network of ledger validators or "miners." A challenge to establishing a central limit order book is that this method of intermediation requires an incredibly high volume of messages to be recorded. Each bid–even each update to a bid such as a modified price or a cancellation—is a message that changes the state of the blockchain ledger. Since each change of the blockchain state requires a (non-negligible) transaction cost, replicating a central limit order book on a blockchain is prohibitively expensive.

To facilitate trading on blockchains then, AMMs, such as Uniswap or Curve, have instead deployed ad-hoc pricing functions to the ledger that define terms of trade between liquidity providers (depositors) and liquidity takers. Specifically, suppose liquidity providers have contributed quantities Q_A and Q_B of tokens A and B and a liquidity taker would like to swap q_A units of token A for token B. Then the AMM pricing function G

dictates that the liquidity taker may withdraw $q_B = G(q_A; Q_A, Q_B)$ units of token B (for some arbitrarily specified function G.[1] A portion of the contributed tokens q_A augment the quantity of tokens in the pool Q_A and a portion is paid out to liquidity providers as fees. A commonly used functional form is the geometric mean function wherein the geometric mean of the pre-trade positions of the AMM in the two tokens equals to that of the post-trade positions—save for the fees charged by the AMM.

While we observe a great deal of trial and error of different pricing functions, fee structures, and code development for AMMs, there is little systematic analysis of the underlying market micro-structure of AMMs. What are the gains to trade between liquidity providers and liquidity takers? What are the potential losses from the inflexibility of the functional form in the face of informed traders? How does the design of the AMM pricing function impact AMM volume and the division of surplus between liquidity providers and liquidity takers? And finally, what is the optimal design of AMM pricing, which is robust to a variety of beliefs about the potential returns to holding tokens? There is a nascent literature studying the outcomes of AMMs and, specifically, the opportunity cost of providing liquidity—sometimes referred to as impermanent loss. See Milionis et al. (2022) for a recent example. In this paper, we develop the first comprehensive framework to examine these questions.

Our proposed framework begins by specifying potential gains to trade between liquidity providers and liquidity takers—something essentially absent from the emerging literature on AMMs. These gains arise in our model due to heterogeneous beliefs in the (relative) value of a pair of tokens as in Harrison and Kreps (1978). Implicitly, we think of liquidity providers as "slow" traders who are less able to obtain transaction priority on the blockchain and, therefore, less able to take advantage of high-frequency arbitrage opportunities.[2] Such agents are the natural liquidity providers in our environment.

Liquidity takers, on the other hand, we model as "fast" and able to attain priority for

[1] In theory, one could represent a central limit order book in this fashion where the function G depends on the entire set of messages—bids and asks—relayed to the exchange. However, here, consistent with what we observe at AMMs, we focus on G that depend solely on quantities

[2] Existing work on impermanent loss implicitly assumes the opportunity cost of liquidity provision is the ability to profit from such high-frequency opportunities Milionis et al. (2022).

blockchain execution. In our framework, liquidity takers may be "informed" or "uninformed,", giving rise to a classic form of adverse selection in asset markets—see Glosten and Milgrom (1985). While we use the language of informed and uninformed trading, our preferred interpretation is rather that uninformed traders trade for reasons that are orthogonal to liquidity providers' beliefs about the value of the tokens. Instead, informed traders trade for reasons that are correlated with (changes) in liquidity providers' beliefs about the value of the tokens.

To the extent that AMM pricing cannot flexibly react to the news or information available in the market, and to the extent that informed traders are able to trade at the AMM *before* the liquidity providers may withdraw their deposits—again, liquidity providers are slow traders relative to liquidity takers—informed traders create losses for liquidity providers.

We explore how this form of adverse selection distorts the amount of liquidity contributed by providers who must balance profits they earn from uninformed liquidity takers (noise traders) with the losses that arise from trading with informed liquidity takers. Our results provide modern analogs to those in Glosten and Milgrom (1985) in a smart contract setting and offer a new interpretation of impermanent loss—committing to trade with informed liquidity takers at "stale" prices—stemming from a traditional notion of adverse selection. While in Glosten and Milgrom (1985) liquidity providers distort *prices* to protect themselves from informed trading losses, such distortions may only manifest in the quantities of deposits liquidity providers post in the AMM.

The conventional wisdom shared as guidance on major AMMs is that liquidity providers should deposit liquidity in equal US dollar value amounts. For example, Uniswap explains that liquidity providers "are incentivized to deposit an equal value of both tokens into the pool. To see why, consider the case where the first liquidity provider deposits tokens at a ratio different from the current market rate. This immediately creates a profitable arbitrage opportunity, which is likely to be taken by an external party." [3] This simple logic, while correct, ignores the concept that some fast traders may trade for reasons or-

[3] See Uniswap-V2 (2023) https://docs.uniswap.org/contracts/v2/concepts/core-concepts/pools

thogonal to current market prices should they require liquidity. Effectively, conventional wisdom assumes there is a single agreed-upon "market price".

Instead, we argue that (fast) liquidity takers may have heterogeneous beliefs or heterogeneous reasons for trade and therefore, depositing at equal *value* may not be optimal. Indeed, we show that in any equilibrium where liquidity providers in sum earn strictly positive profits, they prefer to distort their deposit ratio away from equal value. Such changes allow them to earn higher profits per trade should the first trader be uninformed. We find that in if the fraction of informed traders is such that the profits from uninformed traders exactly balance the losses from informed traders, then it is optimal for liquidity providers to deposit tokens in equal value (according to the liquidity providers' expected valuation of the tokens).

We go on to explore how the shape of the pricing function G impacts gains to trade and liquidity provider's profits. Analogous to results in Milionis, Moallemi and Roughgarden (2023), we find that in the presence of only uninformed traders, convex prices impede ex post trading volumes and reduce ex-ante profits of liquidity providers. Hence, in such a case, linear pricing is optimal. However, the presence of informed traders complicates this analysis because convex prices also limit the losses liquidity providers realize from informed trades. Nonetheless, we show that reducing the (local) convexity of the pricing function improves liquidity provider's profits as long as liquidity provision is profitable. Specifically, we construct a perturbation of the pricing function that decreases its convexity around the liquidity provider's deposit point and scales the gains from uninformed trades at the same rate as losses from adverse selection. If the original CPMM function induces positive ex-ante gains for the liquidity traders, then less locally convex prices increase ex-ante gains for both liquidity providers and liquidity traders, thus improving efficiency.

Recently, a few papers have analyzed the theoretical properties of constant function market makers. Starting from Angeris and Chitra (2020), they show how this class of mechanisms can reflect "true" prices. Further, Angeris et al. (2021) presents a more specific analysis of Uniswap, in which they show that the exchange rate in such a market matches the exogenous prices up to the interval of fee level. Similar to our paper, Aoyagi (2022) considers the effect of information asymmetry on these types of markets and shows that the equilibrium liquidity supply size is stable. However, they work in a competitive liquidity provision environment in which one token in the pool is stable. Naturally, the expected return of the liquidity providers will be zero due to perfect competition. Our framework works on two risky assets, and the rate of returns is robust to competition. On the side of liquidity traders, Capponi and Jia (2021) focuses on the competition among arbitrageurs. This competition allows them to consider the joint determination of gas fees and pool size. More recently, Lehar and Parlour (2023) characterize the endogenous liquidity supply for each set of markets and solve for the optimal venue choice of a liquidity trader. For the returns of the liquidity providers, Milionis et al. (2022) conducts a continuous-time Black-Scholes analysis on the stablecoin pool and decomposes the return of LP into risky and predictable components.

In terms of the design and efficiency of the price function, Park (2023) points out that the CPMM violates some desirable properties and causes economically meaningless and costly trading, such as front running. Bergault et al. (2023) shows that the return of LP is always smaller than holding by duality theorem and a constant product formula with a proportional fee is not efficient from the mean-variance perspective. Goyal et al. (2023) focus on the design of convex pricing functions that maximize the fraction of trades that with only uninformed trades. Milionis, Moallemi and Roughgarden (2023) uses the optimal auction framework to show that a linear price curve maximizes the expected return of the liquidity provider. We achieve the same results and provide additional conditions under our framework.

2 Model

In this section, we introduce a model of blockchain-based automated exchange. The primary role of the automated exchange market is to promote efficient portfolio allocations between heterogeneous investors. We introduce frictions to the model to capture key frictions present in existing automated, blockchain-based exchanges.

2.1 Setup

Consider a static economy populated by two risk-neutral agents: a liquidity provider (LP) and a liquidity taker (LT). While we model these as individual agents, we think of the LP and LT as representative of a pool of potential liquidity providers and takers, respectively. The representative liquidity provider is initially endowed with a portfolio of tokens (E_A, E_B). We assume the LT has deep pockets and cares only about her net trading profits. These tokens represent digital units of account in a blockchain-based ledger such as the Ethereum blockchain, such as units of the Ethereum cryptocurrency and units of the stablecoin USDC on the Ethereum blockchain.

At the end of the period, each token i yields a stochastic value v_i to its owner. In our static economy, we interpret the value v_i as either the future "price" of token i or possibly the service flow attainable by holding 1 unit of token i. For example, 1 unit of the Ethereum cryptocurrency may be "spent" on the execution of smart contracts on the Ethereum blockchain or 1 unit of the stablecoin USDC may be redeemed for 1 US dollar by trading with the company Circle who issues USDC. Cicle (2023)

The LP initially believes v_i is distributed according to the distribution F_i with expected value $\mu_i = \mathbb{E}[v_i]$.

Blockchain-based automated markets (AMMs) rely on smart contracts—digital software deployed to a blockchain ledger and executed by a network of validators or miners. Each automated market's software embeds a pricing function that takes as inputs the amount of liquidity deposited by liquidity providers and the amount of a given token a liquidity taker wishes to deposit (or withdraw) and yields as an output the amount of the other token the liquidity taker may withdraw (or must deposit).

Suppose the LP has deposited a portfolio (e_A, e_B) with the smart contract of the AMM. We let $G(\cdot)$ the embedded pricing function. That is, if the LT wishes to deposit (withdraw)

q_A units of token A then the function specifies an amount q_B units of token B that the LT may withdraw (deposit) where

$$q_B = G(q_A | e_A, e_B).$$

We use the convention that if $q_A > 0$—the LT deposits token A—then $q_B < 0$—the LT may withdraw token B—and vice versa. Most AMM price functions also have the property that q_B is decreasing in q_A so that the LT must pay more of token A per unit of token B she wishes to withdraw. The price function G also requires $q_i \leqslant e_i$, at least implicitly, as capacity constraints.

The most common implementation of automated markets imposes the constant product market maker (CPMM):

$$(e_A + q_A)(e_B - q_B) = e_A e_B. \tag{1}$$

It is originally proposed by Bergault et al. (2023) and then is adopted in Uniswap-V2 (2023).

To motivate gains to trade in the automated market, we assume that after the LP deposits liquidity with the AMM, the LT realizes a shock to her beliefs about the distribution of values each token delivers. In particular, the LT believes the stochastic value of each token ν_i is distributed according to the distribution H_i, possibly different from F_i. Let $\hat{\mu}_i$ denote the mean value of ν_i under the distribution H_i.

Following Glosten and Milgrom (1985), we assume that these information events may be one of two types. The first type of information event—analogous to uninformed trading in Glosten and Milgrom (1985)—represents a case where the LT's new beliefs are uncorrelated with the LP's beliefs. That is, the LP believes the value of each token i will remain distributed according to F_i while the LT believes the value of each token i is distributed according to H_i. When $\mu_i \neq \hat{\mu}_i$ under such an event, there are gains to trade between the LP and the LT. Following the literature, we interpret such an event as a "noise" trade where trade occurs for reasons orthogonal to the LP's beliefs about the potential returns to her tokens. We let $\pi \in [0, 1]$ denote the probability of this first type of information event which we describe as a *trade for tastes* or *uninformed trade*.

Instead, the second type of information event—analogous to informed trading in Glosten and Milgrom (1985)—represents a case where the LT's new beliefs are correlated with the LP's beliefs. That is, in such a case we assume that both the LP and the LT now believe the value of each token is distributed according to H_i. We impose the timing restriction that despite the fact that there are no gains to trade in this case between the LP and the LT, the LT is a "fast trader" and knows how to prioritize their transaction on the blockchain to trade at the AMM before the LP may modify their liquidity position. We interpret this type of information event as an aggregate information event in the sense that the information that arrives may be public and impacts both the LP and the LT's beliefs. For simplicity, we assume that in such an event, the LP's beliefs are identical to the LT's beliefs ex-post. Given our timing restriction, however, the LP views this outcome as a form of adverse selection. This second type of information event occurs with probability $1 - \pi$.

We now summarize the timing of the environment. At the beginning of the economy, the LP deposits a portfolio (e_A, e_B) with the AMM given a pricing function $G(\cdot)$. Next, the type of information event is realized according to π and the LT realizes a shock to her beliefs specified by H_i. The LT then chooses an amount to trade with the AMM. Finally, values and payoffs are realized according to the terminal portfolios of the LP and LT.

Next, we define the problem of the liquidity taker and the liquidity provider working backwards from the LT's problem. We maintain the Constant Product Market Making rule specified in Equation (1) through Section 2.2, 2.3, 3 below.

2.2 The Liquidity Taker's Problem

The LT—whether in an uninformed or informed trading event—observes liquidity on deposit at the AMM as well as her realization of $\hat{\mu}_i$. From her own perspective, the LT perceives an arbitrage opportunity as prices at the AMM do not reflect her realized beliefs of the token values.

The LT maximizes the expected value of her tokens:

$$\begin{aligned} &\max_{q_A, q_B} -\hat{\mu}_A q_A + \hat{\mu}_B q_B \\ &\text{s.t. } (e_A + q_A)(e_B - q_B) = e_A e_B \end{aligned} \tag{2}$$

When $q_A > 0$, the LT's problem given in (2) represents a case where the LT "buys" token B from the AMM by depositing token A. She may wish to set $q_A < 0$ in which case she buys token A from the exchange by depositing some amount of token B. The constraint represents the effective price she faces on any trade. Under the Constant Product rule, the LT would have to deposit infinitely much of one token to withdraw all of the other (i.e. setting $q_B = e_B$, requires $q_A \to -\infty$) and hence the implicit capacity constraints are slack under such a rule.

The solution to the LT's problem is straightforward and satisfies

$$e_A + q_A = \sqrt{\frac{\hat{\mu}_B}{\hat{\mu}_A} e_A e_B}, \quad e_B - q_B = \sqrt{\frac{\hat{\mu}_A}{\hat{\mu}_B} e_A e_B}. \tag{3}$$

More importantly, for any beliefs $\hat{\mu}_i$, she will trade up until the relative price at the AMM equals her relative valuation of the tokens or

$$\frac{\hat{\mu}_B}{\hat{\mu}_A} = \frac{e_A + q_A}{e_B - q_B}. \tag{4}$$

If we let $x_A = e_A + q_A$ and $x_B = e_B - q_B$ denote the LP's post-trade portfolio, then (1) and (4) imply that LP's post-trade portfolio satisfies

$$x_A x_B = e_A e_B \tag{5}$$

$$\hat{\mu}_A x_A = \hat{\mu}_B x_B. \tag{6}$$

The liquidity provider internalizes that for any realization of beliefs of the LT, $\hat{\mu}_i$, her

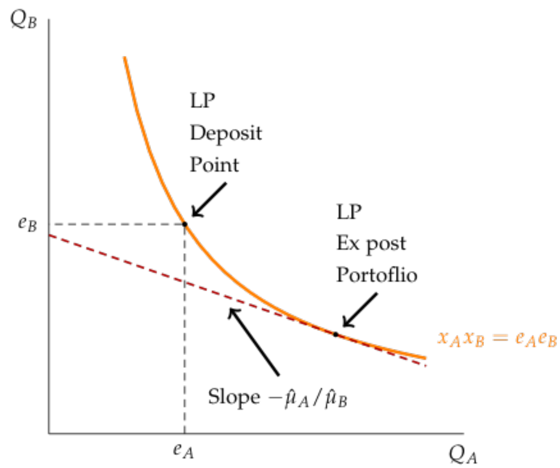

Figure 1: Liquidity Taker's Optimal Trade

ex-post portfolio will satisfy (5)–(6). We may represent this behavior graphically as in Figure 1.

The convex curve represents the constant product market-making rule, and the point (e_A, e_B) represents the liquidity deposited by the LP. Any trade by the LT will move the LP's ex-post portfolio along the convex curve. Once the LT realizes her beliefs $\hat{\mu}_i$, she will trade up until the relative price at the AMM equals her relative valuation of the tokens (represented by the dashed line with slope $-\hat{\mu}_A/\hat{\mu}_B$.

2.3 The Liquidity Provider's Problem

Anticipating the behavior of the liquidity taker, the LP chooses her liquidity deposit to solve the following program.

$$\begin{aligned} \max_{e_A, e_B} \; & \pi(\mu_A \mathbb{E}[x_A - e_A] + \mu_B \mathbb{E}[x_B - e_B]) + \\ & (1-\pi)(\mathbb{E}\hat{\mu}_A[x_A - e_A] + \mu_B \mathbb{E}\hat{\mu}_B[x_B - e_B]) \\ \text{s.t. } & (5)\text{–}(6), \\ & 0 \leqslant e_i \leqslant E_i, \quad \forall i \end{aligned} \tag{7}$$

where π is the probability of an uninformed trading event. Notice, regardless of whether the LP experiences an uninformed or informed trading event, the beliefs of the liquidity taker will result in an ex-post portfolio of the LP according to (5)–(6). These events differ, however, in how the LP perceives the value of these ex-post portfolios. When the LT represents an uninformed trade, the LP continues to value her ex-post portfolio according to her prior beliefs, μ_i. Instead, when the LT represents an informed trade, the LP values her ex-post portfolio according to the realized beliefs of the LT, $\hat{\mu}_i$. As we show below, the LP will trade off profits she earns on uninformed trades with losses on informed trades. Unlike in standard models of exchange subject to adverse selection where market makers post prices that reflect the extent of adverse selection, blockchain market makers must distort their quantity choices for liquidity provision to protect themselves from possible adverse selection.

3 Equilibrium AMM Liquidity Provision

In this section, we examine equilibrium liquidity provision by liquidity providers in our model. Our notion of equilibrium is standard subgame perfect equilibrium. We examine the usefulness of the conventional wisdom from existing automated marketplaces—that liquidity providers *should* deposit liquidity in equal (dollar) values—and find that such behavior is optimal for the representative liquidity provider only under special circumstances. We demonstrate how adverse selection distorts the quantities of liquidity deposited by providers on automated exchanges. To ease the analysis, we first consider two special cases of our model—when all trade is uninformed and when all trade is informed—before turning to the general case.

3.1 Liquidity Provision with Uninformed Trade Only

Suppose first that $\pi = 1$ so that there are only uninformed trades. Using straightforward algebra, the LP's problem (7) simplifies to

$$\max_{e_A, e_B} \mu_A \left(\mathbb{E}\sqrt{\frac{\hat{\mu}_B}{\hat{\mu}_A} e_A e_B} - e_A \right) + \mu_B \left(\mathbb{E}\sqrt{\frac{\hat{\mu}_A}{\hat{\mu}_B} e_A e_B} - e_B \right)$$
$$\text{s.t. } 0 \leqslant e_i \leqslant E_i, \quad \forall i.$$

Since the LP's deposit quantities, e_i, are not random, her objective may be written as

$$\left(\mathbb{E}\omega + \mathbb{E}\frac{1}{\omega} - 2 \right) \sqrt{\mu_A e_A}\sqrt{\mu_B e_B} - \left(\sqrt{\mu_A e_A} - \sqrt{\mu_B e_B}\right)^2 \tag{8}$$

where $\omega = \sqrt{\frac{\hat{\mu}_A/\mu_A}{\hat{\mu}_B/\mu_B}}$. Equation (8) shows how an LP facing only uninformed trade chooses the optimal liquidity to provide. By changing the quantities of tokens A and B she deposits, she adjusts the position of the pricing curve the LT will face ex-post.

To better understand (8), consider one possible (suboptimal) deposit choice for the LP: an equal value deposit, or e_A and e_B that satisfy $\mu_A e_A = \mu_B e_B$. Notice that all possible ex-post portfolios for the LP lie on the constant product price function that runs through the point (e_A, e_B). Moreover, at (e_A, e_B), the constant product price function has slope $-\mu_A/\mu_B$. Since the constant product price function is convex, any trade by the LT will appear to happen at favorable prices from the perspective of the LP—that is, terms of trade are better than $-\mu_A/\mu_B$ for the LP regardless of whether the LT is buying token A or token B. As a result, for such a deposit choice, the LP only stands to gain and suffers no losses.

Figure 2 illustrates this result graphically. Given the LP's beliefs are fixed, facing only uninformed trades, the straight (blue) line with slope $-\mu_A/\mu_B$ reflects the LP's indifference curve. Since all terminal portfolios lie on the constant product price function, and this function lies above the LP's preferences, such a deposit choice by the LP ensures the LP only stands to gain from trade.

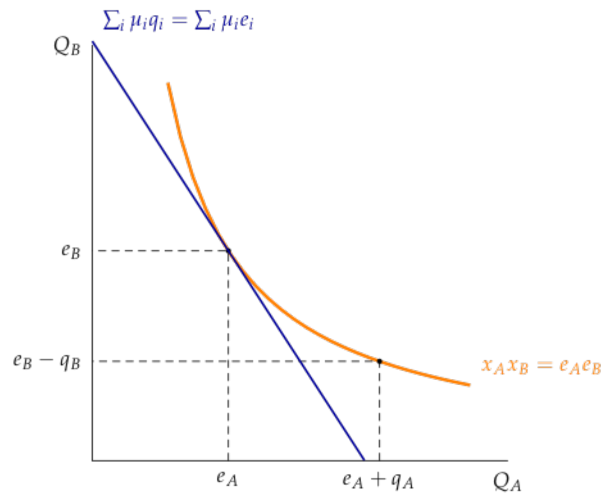

Figure 2: Liquidity Provider's "No-Loss" Deposit Choice

Should the LP provide liquidity different from an equal value deposit, then for small differences in beliefs from her own, the constant produce price function will provide prices that appear unfavorable from the perspective of the LP and yield second-order losses. For this reason, the LP faces a loss function—the second term in (8)—that depends on how her portfolio differs from an equal value ($\mu_A e_A = \mu_B e_B$) portfolio.

To the extent $\hat{\mu}_i$ differs from μ_i, there are gains to trade. The value of these gains depend on the term $\mathbb{E}\omega + \mathbb{E}\frac{1}{\omega} - 2 \geqslant 0$. (The inequality follows directly from Jensen's inequality.) As a result, from any equal value deposit, a small perturbation that raises e_A or e_B on the margin will induce second-order losses but incur first-order gains. As a result, equal-value deposits are generically not optimal for the LP. In general, the LP desires to provide as much liquidity as possible to facilitate gains to trade, and thus, her budget constraint must bind (either $e_A = E_A$ or $e_B = E_B$). We then have the following proposition.

Proposition 1: Optimal Liquidity with only Uninformed Trade. With only uninformed trade, the optimal liquidity deposit satisfies:

$$\begin{cases} e_A^* = E_A,\ e_B^* = \min\left\{ \left(\frac{\mathbb{E}\omega + \mathbb{E}\frac{1}{\omega}}{2}\right)^2 \frac{\mu_A}{\mu_B} E_A, E_B \right\}, & \text{if } \mu_A E_A \leqslant \mu_B E_B \\ e_A^* = \min\left\{ \left(\frac{\mathbb{E}\omega + \mathbb{E}\frac{1}{\omega}}{2}\right)^2 \frac{\mu_B}{\mu_A} E_B, E_A \right\},\ e_B^* = E_B, & \text{if } \mu_A E_A > \mu_B E_B. \end{cases}$$

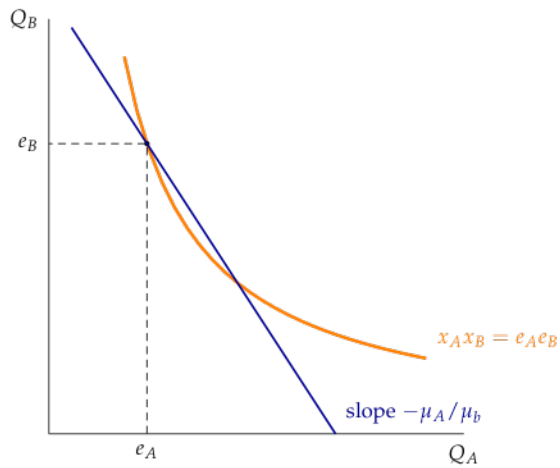

Figure 3: Liquidity Provider's Optimal Deposit Choice

Generically, then, the LP will prefer a deposit choice different from the equal value portfolio to maximize intermediation profits with uninformed traders. Such a choice is illustrated in Figure 3 where, according to Proposition 1 typically, we expect either $e_A = E_A$ or $e_B = E_B$.

3.2 Liquidity Provision with Informed Trade Only

Suppose next that $\pi = 0$ so that there are only informed trades. The LP's problem (7) simplifies to

$$\max_{e_A, e_B} \mathbb{E}\hat{\mu}_A \left(\sqrt{\frac{\hat{\mu}_B}{\hat{\mu}_A} e_A e_B} - e_A \right) + \mathbb{E}\hat{\mu}_B \left(\sqrt{\frac{\hat{\mu}_A}{\hat{\mu}_B} e_A e_B} - e_B \right) \tag{9}$$

$$\text{s.t. } 0 \leqslant e_i \leqslant E_i, \quad \forall i \tag{10}$$

If we impose a mild assumption that $\hat{\mu}_i$ is a mean preserving spread of μ_i, i.e. $\mathbb{E}\frac{\hat{\mu}_i}{\mu_i} = 1$, the LP's objective in this case may be written as

$$(2\mathbb{E}\psi - 2)\sqrt{\mu_A e_A}\sqrt{\mu_B e_B} - \left(\sqrt{\mu_A e_A} - \sqrt{\mu_B e_B}\right)^2 \tag{11}$$

where $\psi = \sqrt{\frac{\hat{\mu}_A}{\mu_A}\frac{\hat{\mu}_B}{\mu_B}}$. Equation (11) shows how an LP facing only informed trade chooses the optimal liquidity to provide.

Since the LP and the LT hold the same ex-post belief, any gains of the LT must reflect losses borne by the LP. Moreover, since the LT only trades when it is beneficial for herself, all trades hurt the LP. As a result, the case of only informed trading reflects a case of pure adverse selection and induced losses for the LP relative to what the value of her wealth would have been had she simply held her portfolio rather than providing liquidity.[4]

Mathematically, the Cauchy-Schwarz inequality implies $\mathbb{E}\psi \leqslant \sqrt{\mathbb{E}\frac{\hat{\mu}_A}{\mu_A}\mathbb{E}\frac{\hat{\mu}_B}{\mu_B}}$ and holds with equality only when $\hat{\mu}_A$ and $\hat{\mu}_B$ are perfectly correlated. Since we impose $\mathbb{E}\hat{\mu}_i/\mu_i = 1$, the above inequality implies $\mathbb{E}\psi \leqslant 1$. Therefore, the LP's objective function is necessarily non-positive for any deposit amount, yielding our next proposition.

Proposition 2: No Liquidity Provision with Only Informed Trade. The optimal liquidity deposit satisfies:

$$e_A^* = e_B^* = 0.$$

3.3 Liquidity Provision with Uninformed and Informed Trading

We now use these results to understand better the general problem (7) with arbitrary π. We once again simplify the LP's objective function as

$$\left[\pi\left(\mathbb{E}\omega + \mathbb{E}\frac{1}{\omega}\right) + (1-\pi)2\mathbb{E}\psi - 2\right]\sqrt{\mu_A e_A}\sqrt{\mu_B e_B} - \left(\sqrt{\mu_A e_A} - \sqrt{\mu_B e_B}\right)^2. \quad (12)$$

As before, we may write the LP's objective as the sum of a revenue function less losses that depend on how the LP's deposit portfolio differs from an equal value portfolio. The revenue function now reflects the probability of realizing an informed versus an uninformed trade. Similar to the previous cases, when uninformed trades occur the LP realizes profits and when informed trades occur, the LP realizes losses. If the gains from uninformed trades are larger than the loss from informed trades, i.e. $\pi\left(\mathbb{E}\omega + \mathbb{E}\frac{1}{\omega}\right) + (1-\pi)2\mathbb{E}\psi \geqslant 2$, then the LP will be willing to provide as much liquidity as possible—up to their ex-ante

[4]Since we implicitly assume LPs are "slow" traders, we do not consider the opportunity cost of trading at an AMM herself. See Milionis et al. (2022) for such an analysis.

resource constraint. Otherwise, the LP will optimally choose to provide no liquidity. We summarize this result in the next proposition.

Proposition 3: Optimal Liquidity. The optimal liquidity deposit with π proportion of uninformed trade and $1-\pi$ proportion of informed trade satisfies

$$\begin{cases} e_A^* = E_A,\ e_B^* = \min\left\{\left(\pi\left(\frac{\mathbb{E}_U\omega+\mathbb{E}_U\frac{1}{\omega}}{2}\right)+(1-\pi)\mathbb{E}_I\psi\right)^2\frac{\mu_A}{\mu_B}E_A, E_B\right\}, & \text{if } \mu_A E_A \leqslant \mu_B E_B \\ e_A^* = \min\left\{\left(\pi\left(\frac{\mathbb{E}_U\omega+\mathbb{E}_U\frac{1}{\omega}}{2}\right)+(1-\pi)\mathbb{E}_I\psi\right)^2\frac{\mu_B}{\mu_A}E_B, E_A\right\},\ e_B^* = E_B, & \text{if } \mu_A E_A > \mu_B E_B \end{cases}$$

if $\pi\left(\mathbb{E}\omega+\mathbb{E}\frac{1}{\omega}\right)+(1-\pi)2\mathbb{E}\psi \geqslant 2$ and

$$e_A^* = e_B^* = 0$$

otherwise.

Given optimal liquidity provision, we next explore the optimality of the conventional wisdom that liquidity providers should deposit portfolios with equal values.

We write $\Pi = \pi\left(\frac{\mathbb{E}_U\omega+\mathbb{E}_U\frac{1}{\omega}}{2}\right)+(1-\pi)\mathbb{E}_I\psi$ to represent the LP's expected profit margin from liquidity provision. According to Proposition 3, if $\Pi > 1$, then the optimal value ratio $\mu_A e_A^*/\mu_B e_B^*$ satisfies

$$\frac{\mu_A e_A^*}{\mu_B e_B^*} = \begin{cases} \frac{1}{\Pi^2} & \text{if } E_A \leqslant \frac{1}{\Pi^2}\frac{\mu_B}{\mu_A}E_B \\ \frac{\mu_A E_A}{\mu_B E_B} & \text{if } \frac{1}{\Pi^2}\frac{\mu_B}{\mu_A}E_B < E_A < \Pi^2\frac{\mu_B}{\mu_A}E_B \\ \Pi^2 & \text{if } \Pi^2\frac{\mu_B}{\mu_A}E_B \leqslant E_A \end{cases}. \tag{13}$$

For $\Pi > 1$, unless $\mu_A E_A = \mu_B E_B$ then the optimal deposit ratio is always different from 1. However, Proposition 3 also reveals that as $\Pi \to 1$ then $\mu_A e_A^* \to \mu_B e_B^*$ for all values of E_A, E_B. In other words, only when the gains from uninformed trades exactly offset the losses from informed trades, then it is optimal for the LP to deposit a portfolio with equal values.

We note that the LP's expected profit margin Π is increasing in the probability that trades are uninformed, π. Hence, there is a minimal value π such that $\Pi = 1$. We then have the following Corollary.

Corollary 1: Optimal Value Share. Let $\underline{\pi}$ be such that $\Pi = 1$ and assume $\mu_A E_A \neq \mu_B E_B$.[5] The equal value deposit $\mu_A e_A = \mu_B e_B$ is optimal only when $\pi = \underline{\pi}$.

3.4 Break Even Proportion of Uninformed Trading

The threshold $\underline{\pi}$ also sheds light on the extent to which liquidity provision is profitable. The value of π such that $\Pi = 1$ depends critically on the distribution of the LT's beliefs specified by H_i. Since the term $\omega + \frac{1}{\omega}$ is not globally convex in $\hat{\mu}_i$, a mean preserving spread of the LT's beliefs $\hat{\mu}_i$ could increase or decrease the threshold $\underline{\pi}$. We instead explore how the profitability of liquidity provision varies with the distribution of the LT's beliefs via a numerical example.

To simplify the numerical analysis, consider a special case where one token is a stablecoin whose value (purportedly) does not fluctuate over time such as USDC or Tether.[6] We let token B represent the stable coin and set $\hat{\mu}_B = \mu_B = 1$ and $h_B(\hat{\mu}_B) = 1$ if $\hat{\mu}_B = 1$. Then we have $\omega = \psi = \sqrt{\frac{\hat{\mu}_A}{\mu_A}}$. We assume $\frac{\hat{\mu}_A}{\mu_A}$ is a log-normally distributed random variable with $\mathbb{E}[\hat{\mu}_A/\mu_A] = 1$ and $Var[\hat{\mu}_A/\mu_A] = \sigma_A^2$. As a benchmark, we impose $\sigma_A^2 = 0.8$ consistent with variation in the daily price of ETH–the native cryptocurrency of the Ethereum blockchain–over the past five years.[7] Around this benchmark, we explore how changes in the variance of beliefs about ETH prices change the threshold probability for liquidity provision to be profitable, $\underline{\pi}$. We plot how this threshold varies with the variance of the LT's beliefs in Figure 4, which shows that increases in variance typically decrease this

[5] If the LP happens to be endowed with an equal value portfolio and profits from liquidity provision are increasing, then she may deposit in equal value simply because she is constrained. We rule out this uninteresting case with this assumption.

[6] In practice, the value of stablecoins do fluctuate at specific points in time, such as when USDC depegged for a short window in April 2023. For our example, we assume liquidity providers and takers believe the stablecoin peg will hold with certainty.

[7] Based on the Coinbase ETH index price obtained from `fred.stlouis.org`.

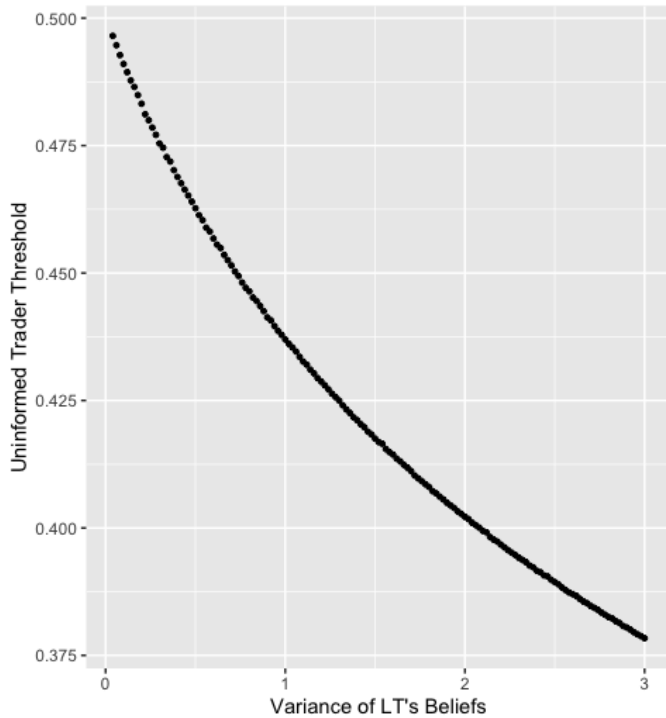

Figure 4: $\underline{\pi}$ against variance of $\hat{\mu}_A$

threshold. [8] In other words, liquidity provision becomes more profitable (LPs can tolerate more informed trading) as ETH price risk increases.

4 Efficiency Losses from Constant Product Market Making

In this section we examine how the shape of the AMM pricing function impacts gains to trade realized by liquidity providers. We focus on the (local) convexity of the CPMM price function and leave a full mechanism design perspective for future work (see Milionis, Moallemi and Roughgarden (2023) for such an approach applied in an environment with only one risky token and limit pocket for the traders.) Specifically, we consider perturbing the CPMM price formula and study a class of pricing functions given by

$$(e_A + (1-\tau)q_A)\,(e_B - (1-\tau)q_B) = e_A e_B \tag{14}$$

[8] We experimented with several other distributional assumptions for $\frac{\hat{\mu}_A}{\mu_A}$ and found similar results. Details are available upon request.

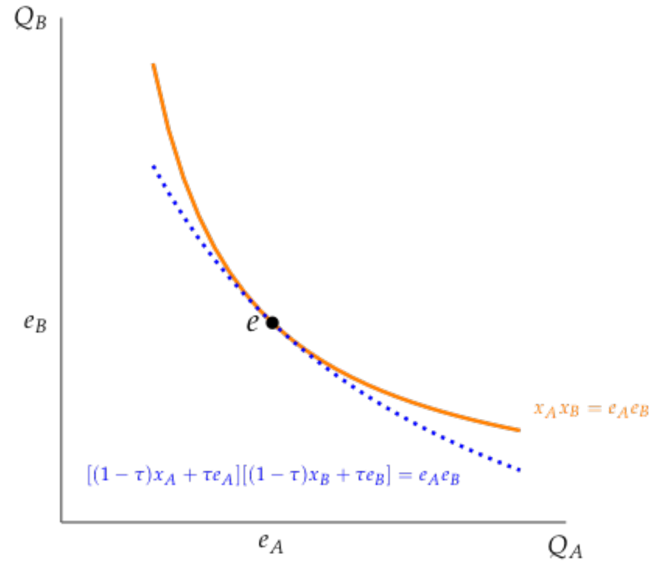

Figure 5: CPMM prices with $\tau = 0$ (the solid, orange curve) and with $\tau > 0$ (the dashed, blue curve).

where $\tau \in [0, 1)$. Notice that this class of price functions admits the CPMM function when $\tau = 0$. For values of q_i close to zero, an increase in τ reduces the convexity of the price function. For larger values of q_i, it is possible the price function becomes more convex. Moreover, for any $\tau > 0$, there exist values of q_i such that the implied ex-post portfolio of the LP would have a negative amount of token A or B so we must impose the boundary conditions, $e_A \geqslant q_A$ and $e_B \geqslant q_B$. Such boundary conditions also tend to increase the global convexity of the price function.

We illustrate how an increase in τ impacts the price function locally in Figure 5 below. The solid curve represents the standard CPMM with $\tau = 0$. Around a given deposit point, (e_A, e_B), the dashed curve represents how the CPMM function changes when τ increases.

If we impose the LP's ex-post token holdings ($x_A = e_A + q_A$ and $x_B = e_B - q_B$) then we may re-write (14) as

$$((1-\tau)x_A + \tau e_A)\,((1-\tau)x_B + \tau e_B) = e_A e_B. \tag{15}$$

The price function (14) is convex and smoothly decreasing when $x > 0$. The convexity of the function is decreasing in τ. The boundary conditions on q_i simply imply $x_i \geqslant 0$.

For a given the realization of the LT's beliefs, $(\hat{\mu}_A, \hat{\mu}_B)$, the LP's net proceeds from trade satisfy

$$x - e_A = \frac{1}{1-\tau}\left[\sqrt{\frac{\hat{\mu}_B}{\hat{\mu}_A} e_A e_B} - e_A\right], \quad y - e_B = \frac{1}{1-\tau}\left[\sqrt{\frac{\hat{\mu}_A}{\hat{\mu}_B} e_A e_B} - e_B\right]. \tag{16}$$

Since net proceeds for both tokens scale by the same factor $1/(1-\tau)$, the LP's expected returns also scale by $\frac{1}{1-\tau}$. Moreover, gains from uninformed trading and losses from informed trading scale by the same ratio so that the break-even proportion $\underline{\pi}$ does not change with τ. As a result, increased (local) convexity of the CPMM hinders trading volume and reduces gains to trade for both the LP and the LT.

However, eliminating (global) convexity of the CPMM is not costless. When $\tau > 0$, equation 15 has finite positive intercepts: $(0, \frac{1+\tau}{\tau} e_B)$ and $(\frac{1+\tau}{\tau} e_A, 0)$. For such values of τ, trading volume cannot increase beyond the two intercepts, even for more extreme beliefs of the LT. Holding the LP's choice of liquidity fixed, we argue that relaxing the local convexity of the pricing function may be detrimental to the LP's ex-ante profits.

To illustrate this, it is simplest to consider a piece-wise linear approximation to the convex pricing function that runs through the LP's (fixed) choice of liquidity deposit. With piece-wise linear prices, liquidity takers either do not trade or trade up to one of the intercept points. For example, suppose p_h represents the (minus the) slope of the price function for values of x_A between 0 and e_A the amount of token A deposited by the LP. If the beliefs of the LT are more optimistic than p_h (so if $\hat{\mu}_A/\mu_A > p_h$), then the LT will trade up to the intercept where $x_A = 0$—the LT will buy all of token A in the pool at the prevailing price, p_h. Otherwise, for $p_h > \hat{\mu}_A/\mu_A > 1$, the LT will not trade.

Consider a marginal increase in p_h (in absolute value). Such a change increases the region of no trade by the LT and thus reduces trading volume on the extensive margin. Recall that the LP only loses expected value from informed trades (and earns exactly zero losses on the marginal informed LT who is just indifferent between trading at p_h and not trading). Therefore, decreasing the volume of trade reduces the LP's expected losses from informed trading. Among uninformed trades, reducing volume is costly on the extensive margin, but raising the intercept implies the LP realizes increased gains to trade for all beliefs where the LT continues to trade. An analogous argument occurs if beliefs of the LT are sufficiently low so that the LT trades to the point where $x_B = 0$. Consequently, it is possible that the gains from increasing the global convexity of a piece-wise linear price function outweigh the costs, implying some degree of convexity is desirable. We show this result both for piece-wise linear prices as well as for the continuously differentiable price function in (14) in Appendix B.

If the distribution of the LT's beliefs has bounded support, then the potential losses from reduced (global) convexity for extremal beliefs may be limited with an appropriate choice of τ. In other words, when the LT's beliefs have bounded support, then there exists $\tau > 0$ that increases the LP's expected returns. In fact, we generalize these results beyond the CPMM formula in the next Proposition (proved in Appendix A).

Proposition 4: Pareto Improvement. Consider a convex and smoothly decreasing price function $y = G(x)$. Assume the distributions of the LT's valuations of the tokens $(\hat{\mu}_A, \hat{\mu}_B)$ have bounded support such that a trade that exhausts one token never happens under the price function $G(x)$. Then there exists $\tau = \hat{\tau} \in (0, 1)$ such that the new price function $(1 - \hat{\tau})\, y + \hat{\tau} e_B = G((1 - \hat{\tau})\, x + \hat{\tau} e_A)$ is less convex at (e_A, e_B), the LP's optimal deposit is the same at $\tau = \hat{\tau}$ as at $\tau = 0$, and $\tau = \hat{\tau}$ increases both the LP's and the LT's expected returns proportionally by $\frac{\tau}{1-\tau}$.

In particular, if $G(x)$ is the CPMM function and if $\left[\underline{\mu_i}, \overline{\mu_i}\right]$ is the support of the distribution of $\hat{\mu}_i$, then the result of Proposition 4 hold for all $\tau \leqslant \bar{\tau} = \min\left\{\sqrt{\frac{\underline{\underline{\mu_B}} e_B}{\overline{\mu_A} e_A}}, \sqrt{\frac{\underline{\underline{\mu_A}} e_A}{\overline{\overline{\mu_B}} e_B}}\right\}$ with $\bar{\tau} > 0$.

We see that with bounded beliefs, convexity hurts the LP's expected returns. In fact, with some additional conditions, the optimal price function for the LP is the linear price function:

$$\begin{cases} p_l x_A + x_B = p_l e_A + e_B, & x \geqslant e_A \\ p_h x_A + x_B = p_h e_A + e_B, & x < e_A \end{cases} \tag{17}$$

where again e_i are the LP's deposit and x_i are the tokens left in the pool after the LT's trading. Similar to the results in Milionis, Moallemi and Roughgarden (2023), we have the following proposition (proved in Appendix C).

Proposition 5: LP's Optimal Pricing Function Assume the distributions of the values of the tokens have bounded support and the LT has a budget limit on at least one token, i.e. x or y can't go to infinite. Given the LP's deposit (e_A, e_B), the optimal pricing formula is the linear pricing formula is one of the following conditions is satisfied:

1. All trades are uninformed trading, i.e. $\pi = 1$;
2. The LT's value $(\hat{\mu}_A, \hat{\mu}_B)$ follows the same distribution for both informed and uninformed trading. And one of the two tokens is a stablecoin. In the case of token A is stable, it implies $\hat{\mu}_A = \mu_A$ for sure. Also, there exists some uninformed trading, i.e. $\pi \neq 0$.

5 Conclusion

Blockchain technology has spawned a very large variety of cryptocurrency tokens. Given the large disagreement about their speculative value and heterogeneity about any utility of the tokens, trading the tokens is important. Over the past decade, a large number of new centralized exchanges have been successful (and unsuccessful) at both generating large volumes and innovating. The perpetual futures contract is one example of innovation (Soska et al. (2021), Christin et al. (2023)). Similarly, Automated Market Makers (AMM) have innovated trade by designing smart contracts (automated code on the blockchain) to conduct trade directly on a blockchain.

In this paper, we have explored the key design characteristic of AMM technology, the pricing curve. Specifically, we look at two aspects related to the pricing curve, G. First, what is the optimal ratio for deposits? Contrary to conventional AMM wisdom, depositing tokens in equal value (measured through the lens of the liquidity provider) is not optimal. Second, we explore the convexity of G and its impact on the liquidity provider profits. The tradeoff is subtle since convexity impacts the profits from trading with both informed and uninformed liquidity takers.

There are, of course, several important areas we have left for future research. Our model treats the G function as given. This, along with the "deep pockets" assumption for the liquidity takers, means the liquidity provider's decision can be made in isolation (i.e., atomistic with respect to liquidity takers). In practice, there are multiple AMM exchanges. So, thinking about competition across the design of the G function is interesting. Second, our model takes a simplified view of the timing of transactions – first, the LP posts and then the LT trades. Again, in practice, the timing of transactions in a decentralized blockchain is complicated and potentially strategic.

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A Proof of Pareto Improvement

Let $y = G(x)$ be a convex and smoothly decreasing price function where $e_B = G(e_A)$. Consider a uniform stretch of the function around the initial deposit point (e_A, e_B): $(1-\tau)y + \tau e_B = G((1-\tau)x + \tau e_A)$ where $\tau \in (0,1)$. Then the second order derivatives is $\frac{d^2y}{dx^2} = (1-\tau)^2 G''((1-\tau)x + \tau e_A)$. Therefore, the transformation is less convex around the initial deposit point (e_A, e_B) as τ increases.

Now we can write the LT's problem as:

$$\begin{aligned} \max_{q_A, q_B} \; & \hat{\mu}_A(e_A - x) + \hat{\mu}_B(e_B - y) \\ \text{s.t. } & (1-\tau)y + \tau e_B = G((1-\tau)x + \tau e_A) \end{aligned}$$

Assume the distributions of the LT's values of the tokens ($\hat{\mu}_A$, $\hat{\mu}_B$) have bounded support such that a trade that exhausts one token never happens. Then the first order condition becomes $G'((1-\tau)x + \tau e_A) = -\frac{\hat{\mu}_A}{\hat{\mu}_B}$. Similar to the CPMM case, the LP's post-trade portfolio satisfies

$$\begin{aligned} (1-\tau)y + \tau e_B &= G((1-\tau)x + \tau e_A) \\ G'((1-\tau)x + \tau e_A) &= -\frac{\hat{\mu}_A}{\hat{\mu}_B} \end{aligned}$$

Let (x_0, y_0) be the post-trade portfolio for the original function, i.e., when $\tau = 0$. Let (x_τ, y_τ) be the portfolio for some $\tau \in (0,1)$. Then given $\frac{\hat{\mu}_A}{\hat{\mu}_B}$, the ex post portfolios satisfies

$$\begin{aligned} (1-\tau)x_\tau + \tau e_A &= x_0 \\ (1-\tau)y_\tau + \tau e_A &= y_0 \end{aligned}$$

which can be written as

$$\begin{aligned} x_\tau - e_A &= \frac{1}{1-\tau}(x_0 - e_A) \\ y_\tau - e_B &= \frac{1}{1-\tau}(y_0 - e_B) \end{aligned}$$

Therefore, the trading volume is proportionally increased by $1 - \frac{1}{1-\tau} = \frac{\tau}{1-\tau}$ for every ex post scenario.

Given the probability of uninformed trading π, the LP's expected return with the transformed price function is

$$\begin{aligned} R_\tau =& \mathbb{E}[(\pi\mu_A + (1-\pi)\hat{\mu}_A)(x_\tau - e_A) + (\pi\mu_B + (1-\pi)\hat{\mu}_B)(y_\tau - e_B)] \\ =& \frac{1}{1-\tau}\mathbb{E}[(\pi\mu_A + (1-\pi)\hat{\mu}_A)(x_0 - e_A) + (\pi\mu_B + (1-\pi)\hat{\mu}_B)(y_0 - e_B)] \end{aligned}$$

Since the objective is just scaled up by a constant, the optimal deposit decision (e_A^*, e_B^*) shouldn't change as well.

B Cost of Convexity

Again let token B represent a stable coin and set $\hat{\mu}_B = \mu_B = 1$ and $h_B(\hat{\mu}_B) = 1$ if $\hat{\mu}_B = 1$. Denote $r_A = \hat{\mu}_A/\mu_A$. Assume r_A follows a distribution with CDF $F(r_A)$. For simplicity, assume $\frac{\mu_A e_A}{\mu_B e_B} = 1$. The results still go through when $\frac{\mu_A e_A}{\mu_B e_B}$ equals to some constant other than 1.

B.1 Piece-wise Linear

Consider the piece-wise linear prices 17. The region of belief where a trade happens with price p_h is when $r_A \geqslant p_h$. From the LP's perspective, the trading volume in this region is $-e_A$ for token A and $p_h e_A$ for token B. The expected return of the LP from uninformed trading is

$$\int_{p_h}^{\infty} (p_h - 1)dF(r_A)\mu_A e_A$$

with derivative as $[1 - F(p_h) - (p_h - 1)f(p_h)]\mu_A e_A$. The first term represents the increased gains to trade for all beliefs where the LT continues to trade. The second term represents the reduced trading volume on the margin.

On the other hand, the expected return (negative) of the LP from informed trading is

$$\int_{p_h}^{\infty}(p_h - r_A)dF(r_A)\mu_A e_A$$

with derivative as $(1 - F(p_h))\mu_A e_A$. Since on the marginal informed LT is just indifferent between trading and not, the second term in the case of uninformed trades is not here.

Given the proportion of uninformed trades π, the marginal benefits of increasing p_h (increasing convexity) is

$$[1 - F(p_h) - \pi(p_h - 1)f(p_h)]\mu_A e_A$$

which has finite number of roots. It implies that some degree of convexity is desirable.

B.2 Continuously Differentiable Price

Now consider the continuously differentiable price function in 15. Similarly, the region of belief where a trade happens with price p_h is when $r_A \geqslant \frac{1}{\tau^2}$. From the LP's perspective, the trading volume in this region is $-e_A$ for token A and $\frac{1}{\tau}e_B$ for token B. Denote $c = \frac{1}{\tau} \in (1, \infty)$. So, increasing c increases the local convexity. The expected return of the LP from uninformed trading is

$$\int_{c^2}^{\infty}(c - 1)dF(r_A)\mu_A e_A$$

with derivative as $[1 - F(c^2) - (c - 1)f(c^2)]\mu_A e_A$. Again the first term represents the increased gains to trade for all beliefs where the LT continues to trade. The second term represents the reduced trading volume on the margin.

On the other hand, the expected return (negative) of the LP from informed trading is

$$\int_{c^2}^{\infty}(c - r_A)dF(r_A)\mu_A e_A$$

with derivative as $[1 - F(c^2) + c(c - 1)f(c^2)]\mu_A e_A$. Since $c > 1$ there is an additional gain for the LP from reducing the trading volume further.

Given the proportion of uninformed trades π, the marginal benefits of increasing p_h (increasing convexity) is

$$[1 - F(c^2) + (c-1)((1-\pi)c - \pi)f(c^2)]\mu_A e_A$$

which is always positive for $c \geqslant \frac{\pi}{1-\pi}$. In these cases, increasing (local) convexity is always beneficial for trades induced by extremal beliefs. However, it reduces the trading volume and the returns from mild beliefs.

C Proof of Optimal Pricing Function

We can consider the optimal design problem as the LP post the ending position of the pool given the new valuation of the LT $(\hat{\mu}_A, \hat{\mu}_B)$ such that the LT is willing to participate (Individual Rational) and truthfully report the values (Incentive Compatible).

Assume the LT's value $(\hat{\mu}_A, \hat{\mu}_B)$ follows the same distribution for both informed and uninformed trading. Also, assume the LT has at most l_B token B to trade in.

Let $t_A = e_A - x$ and $t_B = e_B - y$ be the net amount of token the LP loses by trading. With the percentage of uninformed trading π, the problem can be written as:

$$\begin{aligned}
\max_{x,y} \ & \mathbb{E}_{\{\hat{\mu}_A, \hat{\mu}_B\}} \left[-\left(\pi\mu_A + (1-\pi)\,\hat{\mu}_A\right) t_A\left(\hat{\mu}_A, \hat{\mu}_B\right) - \left(\pi\mu_B + (1-\pi)\,\hat{\mu}_B\right) t_B\left(\hat{\mu}_A, \hat{\mu}_B\right)\right] \\
\text{s.t. } & \hat{\mu}_A t_A\left(\hat{\mu}_A, \hat{\mu}_B\right) + \hat{\mu}_B t_B\left(\hat{\mu}_A, \hat{\mu}_B\right) \geqslant \hat{\mu}_A t_A\left(\hat{\mu}'_A, \hat{\mu}'_B\right) + \hat{\mu}_B t_B\left(\hat{\mu}'_A, \hat{\mu}'_B\right) \\
& \hat{\mu}_A t_A\left(\hat{\mu}_A, \hat{\mu}_B\right) + \hat{\mu}_B t_B\left(\hat{\mu}_A, \hat{\mu}_B\right) \geqslant 0 \\
& t_A\left(\hat{\mu}_A, \hat{\mu}_B\right) \leqslant e_A, \ -l_B \leqslant t_B\left(\hat{\mu}_A, \hat{\mu}_B\right) \leqslant e_B
\end{aligned}$$

Since only $p = \frac{\hat{\mu}_B}{\hat{\mu}_A}\frac{e_B}{e_A}$ matters in the constraints, the problem can be written as

$$\begin{aligned}
&\max_{t_A,t_B} \mathbb{E}_p \left[\left(-\frac{t_A(p)}{e_A} - \frac{(\pi\mu_B + (1-\pi)\hat{\mu}_B)}{(\pi\mu_A + (1-\pi)\hat{\mu}_A)}\frac{e_B}{e_A}\frac{t_B(p)}{e_B}\right)(\pi\mu_A + (1-\pi)\hat{\mu}_A)\right] e_A \\
&\text{s.t. } \frac{t_A(p)}{e_A} + p\frac{t_B(p)}{e_B} \geqslant \frac{t_B(\hat{p})}{e_B} + p\frac{t_B(\hat{p})}{e_B} \\
&\quad \frac{t_A(p)}{e_A} + p\frac{t_B(p)}{e_B} \geqslant 0 \\
&\quad \frac{t_A(p)}{e_A} \leqslant 1, -\frac{l_B}{e_B} \leqslant \frac{t_B(p)}{e_B} \leqslant 1
\end{aligned}$$

Under one of the two conditions, i.e. $\pi = 0$ or $\hat{\mu}_A = \mu_A$ for sure, we know $\pi\mu_A + (1-\pi)\hat{\mu}_A$ is a constant. So the objective can be simplified. Let $-\frac{t_A(p)}{e_A} + 1 = y(p)$, $\frac{t_B(p)}{e_B} = x(p)$ and $\frac{(\pi\mu_B+(1-\pi)\hat{\mu}_B)}{(\pi\mu_A+(1-\pi)\hat{\mu}_A)}\frac{e_B}{e_A} = \pi(p_0, p)$. The problem then has the same expression as Milionis, Moallemi and Roughgarden (2023).

$$\begin{aligned}
&\max_{x,y} \mathbb{E}_p \left[y(p) - \pi(p_0, p)\, x(p)\right] \\
&\text{s.t. } px(p) - y(p) \geqslant px(\hat{p}) - y(\hat{p}) \\
&\quad px(p) - y(p) \geqslant 0 \\
&\quad y(p) \geqslant 0, -c \leqslant x(p) \leqslant 1
\end{aligned}$$