Optimal Tick Size*

Giuliano Graziani[†] Barbara Rindi[‡]

June 15, 2023

[CLICK FOR LATEST VERSION]]

Abstract

We use a model of a limit order book to determine the optimal tick size set by a social planner who maximizes the welfare of market participants. Our results show that when investors arrive sequentially and supply liquidity by undercutting or queuing behind existing orders, the optimal tick size is a positive function of the asset value and a negative function of stock liquidity. Intuitively, the tick size is a strategic tool a social planner uses to optimally affect investors' choice between liquidity demand and supply, thus mitigating the inefficiencies created by excessive undercutting and queuing. The policy implication of such findings is that both the European tick size regime and the 2022 SEC proposal dominate Reg. NMS Rule 612 that formalizes the tick size regime for the U.S. markets. Using data from the U.S. and the European markets we test our model's predictions.

Keywords: Limit Order Book, Tick Size, Social Planner, Undercutting, Queuing.

*We thank Jean Edouard Colliard, Thierry Foucault, Philippe Guillot, Bjorn Hagströmer, Stefano Lovo, Phil Mackintosh, Lars Nordén, Marco Ottaviani, Eugenio Piazza, Roberto Ricco', Ioanid Rosu, Nicole Torskiy, and Bart Z. Yueshen for very useful comments and suggestions. We also thank seminar participants to the Stockholm Business School and to HEC Paris. We thank Federico Chinello for excellent research assistance. We are grateful to the Nasdaq's Economic Research Team for sharing data.

[†]Bocconi University, giuliano.graziani@phd.unibocconi.it

[‡]Bocconi University, IGIER and Baffi-Carefin, barbara.rindi@unibocconi.it

"Many of the issues afflicting the market today can be traced back to the current tick size regime drawing the ire of both investors and issuers." (Nasdaq, 2019)

Modern limit order books (LOBs) work as double auction markets with discrete prices, governed by two fundamental rules - price and time priority. On a LOB prices are discrete and the combination of all possible prices at which traders can post their orders forms the so-called price grid, which is based on the minimum distance between two consecutive prices, known as the tick size. The tick size is generally set by regulators and sits right at the top of their agenda, all around the world: it is the crucial feature of a LOB, as it impacts the effects that fundamental priority rules have on the order submission strategies of investors willing to supply and demand liquidity.

While there exists a vast empirical literature on the tick size, there only exist a few theoretical contributions that show the effects of a tick size change on market quality and on the welfare of market participants, and there is no theoretical contribution aiming to set the OTS in order to maximize the welfare of market participants. The aim of this paper is to fill this gap by providing a theoretical framework for a LOB where a social planner (SP) determines the optimal tick size (OTS) that maximizes the welfare of market participants.

The equilibrium dynamics of a LOB depend on how the demand and the supply of liquidity change over time. To demand liquidity, investors use market orders, whereas to supply liquidity they use limit orders. Therefore, the equilibrium dynamics of a LOB crucially depend on how the investors' choice between market and limit orders changes over time. When choosing whether to take liquidity and maximize the execution probability of their order by using a market order, or to wait and maximize the price improvement of their order by supplying liquidity via a limit order - either queuing or undercutting existing orders - investors have to take into account the value of the tick size. Since the tick size determines both the minimum price improvement and the cost of undercutting, it crucially affects the trader's choice between queuing and undercutting. The novelty of our model, compared to the existing literature, is that it allows us to determine OTS by taking into account both queuing and undercutting.

There are two central related questions regulators seek to answer in relation to the tick size. The first one is about the optimal dimension of the tick size, namely whether it should be equal to zero or whether instead it should be set at a positive value. Is the tick size a rent for liquidity providers that should be normalized to zero in competitive markets?[1] Or, are there any relevant transmission channels supporting the existence of a positive tick size that maximizes the welfare of all market participants? The second research question that regulators seek to answer is about how the tick size should be optimally set across different securities.

Answering the first of the two research questions mentioned above, this paper shows that while the tick size is a friction in markets where there is no competition in the provision of liquidity, it is not a friction in a standard limit order book market where each market participant can choose to either supply or demand liquidity. The contribution of this paper is precisely to show that in a standard LOB model a SP does not set the OTS at zero, but at a value that optimizes the strategic interaction of liquidity demand and liquidity supply, thus maximizing the total welfare of market participants.

In relation to the second research question, intuitively investors crossing the spread to demand liquidity may benefit from a smaller tick size resulting in a smaller bid-ask spread. Instead, investors supplying liquidity may benefit - depending on the state of the book - either from a smaller or from a wider tick size: a smaller tick size allows investors to cheaply undercut long queues at the best prices in liquid markets (tick size constrained stocks), while a wider tick size reduces aggressive undercutting thus incentivizing investors to post limit orders in markets characterized by a large spread at the best bid offer (tick size unconstrained stocks). This paper shows that the OTS should differ across securities with different share prices and trading activity.

Over the past twenty years, exchanges implemented several tick size changes extensively documented by a substantial empirical literature that discusses the effects of the tick size changes on the quality of the markets considered.[2] Lacking theoretical guidelines on how to determine

[1]Early theoretical literature on the optimal tick size (among others, Kandel and Marx (1997), Chordia and Subrahmanyam (1995) and Anshuman and Kalay (1998)) show that the tick size creates a wedge between the underlying equilibrium price and the observed price that permits competitive market makers to realize economic profits that could help recoup fixed costs.

[2]For an incomplete review see: Angel (1997), Bessembinder (2000), Bessembinder (2003), Chordia and Subrahmanyam (1995), Chung, Chuwonganant, and McCormick (2004), Goldstein and Kavajecz (2000), Harris (1991), Harris (1994) and Harris (1996), Hu, Hughes, Ritter, Vegella, and Zhang (2018), Comerton-Forde, Grégoire, and Zhong (2019), Albuquerque, Song, and Yao (2020), Chung, Lee, and Rösch (2020) and Chakrabarty, Cox, and Upson (2022). Dayri and Rosenbaum (2015) propose a statistical model to determine the effects of a change in the tick size and to propose a concept of optimal tick size based on stylized facts from empirical evidence.

the OTS, historically the tick size was progressively reduced to minimize transaction costs. For a long time the trend in both the U.S. and in the majority of existing markets was just to gradually reduce the tick size in an undifferentiated way, aiming to reduce the trading costs for investors demanding liquidity.[3] The issue of the tick size is today particularly relevant in the U.S. markets which have traditionally maintained a binary tick size regime which governs instruments ranging from large capitalization stocks trading billions of dollars of notional value daily, to small capitalization stocks trading a few lots per day, regardless of market capitalization, volume or share price.[4] These same stocks also have a very dispersed distribution of prices ranging from \$1 to more than \$2000 per share (Table 7).

A one penny tick size is too wide for low priced stocks - especially the large liquid ones - that are constrained to trade most of the time at the 1-tick spread (Bacidore (1997) and Goldstein and Kavajecz (2000)). This distortion creates long quotation queues at the best bid-offer (BBO) which slows down execution and leads investors to focus on time rather than price priority (Ye and Yao (2014)). A one penny tick size is otherwise too small for a number of high-priced stocks, especially those trading at wider spreads: when the tick size is too small relative to the average quoted spread, patient limit orders are outbid by economically insignificant amounts. When the tick size is so small that undercutting resting orders becomes inexpensive, the value of time priority decreases thus eliminating the incentive to supply liquidity by posting patient limit orders. If the incentive to post patient limit orders declines, spreads widen and liquidity worsens.[5]

Fairly recent criticisms to the current "one-size-fits-all" U.S. tick size highlight the need to consider not only the effects that the tick size may have on the demand for liquidity but also its effects on the supply of liquidity. In the U.S. while the tick size was gradually reduced from one eighth of a dollar to one cent,[6] the Securities and Exchange Commission (SEC) launched the

[3]The smaller the tick size, the finer the price grid and the smaller the minimum inside spread. This may reduce transaction costs for liquidity demanders, making it cheaper for them to operate.

[4]In the U.S. markets the tick size is equal to \$0.01 for stocks priced above \$1 and it is equal to \$0.0001 for stocks priced below \$1. One lot in the U.S. markets corresponds to 100 shares, the minimum tradable quantity.

[5]O'Hara, Saar, and Zhong (2019) show that in a tick-constrained (tick-unconstrained) environment, larger relative ticks result in greater (less) depth. Dyhrberg, Foley, and Svec (2019) show the positive effects of an increase in the tick size in the crypto currency markets hinting to the attractiveness of a wider tick size for tick size unconstrained stocks.

[6]The reduction process was gradual and spanned over two decades: the first shift was in September 1992,

U.S. Tick Size Pilot (USTSP - running from October 2016 to October 2018), aimed at studying the effects of an increase in the tick size.

The Nasdaq also commissioned an empirical analysis to a working group including representatives from buy-side, sell-side, market makers, and retail firms that shows how the current U.S. "one-size-fits-all" tick size works optimally only for a limited group of stocks, which is gradually shrinking. In 2019 the Nasdaq issued a proposal amending Rule 612 of Reg NMS to adopt an "Intelligent Ticks" regime with a schedule of tick sizes that are adjusted regularly, based on stock-specific trading conditions (Nasdaq (2019)). The proposal has not been implemented yet, however on December 14, 2022 the SEC issued a new proposal to change the tick size. The 34-96494 SEC (2022) proposal aims to set the tick size as a function of average quoted spread only for stocks with an average spread smaller than $0.04. Therefore the proposal only focuses on tick size-constrained stocks thus neglecting the issue related to high priced stock trading at one penny increment.

Consistently with our results, some major markets (e.g., Australian Stock Exchange (ASX), Toronto Stock Exchange (TSX) and Singapore Stock Exchange (SGX)) have adopted a discrete tick size grid set as a step function of the stock price (Table 1.A). Other exchanges have a more sophisticated tick size regime (e.g., Hong Kong (HKEX), Tokyo (JPX)) where the tick size is a step function of both the stock price and the traded volume. Along these same lines, in Europe in 2018 ESMA provided the European markets with a tick size model embedding precise guidelines in relation to how the tick size should be set, based on both the price and the liquidity of the instruments (ESMA (2017)). Before the release of Article 49 of MiFID II which includes the ESMA table on the new tick size regime, AMF (2013) singled out the trade-off that should govern the choice of the OTS: *[t]oo big, a tick size can discourage investors from placing orders at the best bid/offer prices as the queuing time at these limits becomes longer, which in turn increases implementation risk. A smaller tick size, [instead], increases the room to overbid, and reduces the cost of overbidding.* Following the MiFID II revision and the release of the ESMA table on the new tick size regime, AMF (2018) presented empirical evidence showing that the

moving from a $\frac{1}{8}$ to a $\frac{1}{16}$ of a dollar minimum price regime. Then, in April 2001, the SEC introduced the current decimal system.

new regime had the desired effect on order lifetime (order-to-trade ratio), transaction size and indicators of market quality. Our model's results are consistent with this empirical evidence as they not only show that the OTS cannot be set to zero, but they also show that the OTS should differ depending on the characteristics of the instrument involved.

Existing theoretical literature (Werner, Rindi, Buti, and Wen (2022)) and empirical evidence from both academia (e.g., Harris (1996), Ronen and Weaver (2001), Rindi and Werner (2019), Chung et al. (2020) and Foley, Dyhrberg, and Svec (2022)) and the industry (e.g., AMF (2018), Mackintosh (2020), Mackintosh (2022)) highlight three main effects of an increase in the tick size: the mechanical increase of the inside spread; the potential increase in queuing induced by a clustering of orders on a coarser price grid; and the potential reduction of a now more expensive undercutting.

To determine the OTS and capture all of these effects, it is necessary that the model is characterized both by discrete prices and by an unconstrained choice between market and limit orders. Therefore, the assumption of continuous prices, such as in Roşu (2009) and Bhattacharya and Saar (2021) must necessarily be relaxed. Besides, if investors choose to post a limit order they must be able to either queue behind existing orders or to undercut previously posted limit orders. We therefore also need to depart from protocols such as Foucault, Kadan, and Kandel (2005) who have discrete prices but in order to obtain an analytical stationary solution for the expected time to execution of submitted limit orders, have to assume both that traders cannot queue behind previously posted limit orders and that buyers and sellers alternate with certainty.[7] To determine the OTS, we must also necessarily depart from the protocol used by Goettler, Parlour, and Rajan (2005), for a number of reasons. First and foremost, we need a protocol where the price grid is potentially characterized by a high number of prices: Goettler et al. (2005)'s numerical solution for a steady state equilibrium, instead, limits the number of

[7] When relaxing these assumptions, Foucault et al. (2005) cannot find an analytic solution for their stationary equilibrium because in this case the expected time to execution cannot be solved recursively: it depends on the entire state of the limit order book at the time the order is placed and not simply on the inside spread. Therefore, Foucault et al. (2005) propose some numerical examples in which they conjecture and verify equilibrium order placement strategies for patient and impatient investors with the aim to calculate the expected execution time for each limit order conditional on each possible state of the book. Cordella and Foucault (1999) also have discrete prices but their model is based on a quote-driven dealership market. They find that the OTS - the one minimizing the expected trading cost - is not zero as a larger tick size facilitates the dealers' convergence toward the equilibrium-competitive price.

price levels involved as it requires a very large number of iterations. In addition, as we discuss in Section 5, their algorithm for the stationary solution to the execution probability of a limit order does not embed the strategic trade-off between queuing and undercutting, which is essential to determine the OTS.

For the reasons explained above, a model that determines the OTS and aims to capture all of the relevant transmission channels, must necessarily depart from a steady state solution, whether analytical or numerical. To obtain a closed form solution of the trading game, we consider backwardly the entire state of the book in any period of the game so that all possible paths investors can choose are taken into account.

Our model (Section 1) draws on Parlour (1998), Chao, Yao, and Ye (2018), and Riccó, Rindi, and Seppi (2021) and determines the OTS by incrementally taking into account all of the effects that a change in the tick size generates. We start from a 2-period model (Section 2) where only two agents arrive sequentially and therefore neither queuing nor undercutting may take place so that the only effect of an increase in the tick size is an increase in the inside spread.[8] In this model the investor arriving at t_1 is a monopolistic liquidity provider as the investor arriving at t_2 - the last period of the trading game - cannot choose between taking or supplying liquidity and all he can do is to take the limit order posted at t_1 or decide not to trade. The tick size is therefore only a friction that generates price discreteness thus limiting investors' choice of limit prices on the price grid at t_1. We show that, absent queuing and undercutting, the OTS is the one which minimizes price discreteness, hence it is equal to zero. This result is consistent with Li and Ye (2022)'s model where a market maker posts competitive bid and ask prices and then informed and uninformed investors hit the quotes. As in our 2-period model, in this setting there is no endogenous queuing and undercutting and therefore the tick size only mandates price discreteness.

We then add a third trading period (Section 3) hence allowing the 2^{nd} investor to undercut the 1^{st} player's limit order. Adding the effect of undercutting, we show that the OTS is no longer equal to zero as a positive tick size reduces the incentive for the 2^{nd} player to undercut the 1^{st} player's limit order, thus preserving his incentive to supply liquidity. We also show that

[8]Results for a decrease in the tick size are symmetric.

an increase in the asset value increases the gains from trade and the investors' aggressiveness. This increases their willingness to undercut existing limit orders and induces the SP to widen the OTS to compensate the negative effects of excessive undercutting. When we extend the model to include a fourth trading period (Section 4), we allow investors to submit limit orders that queue behind existing ones. Adding the effect of queuing, we show that the OTS is still positive: although the queuing effect provides an incentive for the SP to set a smaller tick size - which layers the orders eventually clustering at the best quotes - a zero tick size is still sub-optimal. Intuitively, a zero tick size would crowd out of the market the 1^{st} player supplying liquidity, thus reducing total welfare.

By sequentially adding traders coming to the market over different trading periods - up to five traders/periods - we fully explain the transmission channels in place when determining the OTS and therefore we single out all of the effects that a change in the tick size has on both liquidity demand and liquidity supply. As already discussed, in order to allow investors to choose between market and limit orders by taking into account all of the possible future states of the book, we cannot rely on a stationary equilibrium, and therefore in each period of our model the order submission probabilities must be the result of the strategic endogenous interaction of the arriving investors with the current and the expected future states of the limit order book. We solve our model with up to five traders/periods as in this way we can consider both the effects of undercutting and the effects of queuing. If we proxy the liquidity of a stock by the number of investors active in the market, our model allows us to show that the OTS should decrease as the liquidity of the instrument increases. Intuitively, additional arriving traders increase the execution probability of limit orders thus inducing liquidity suppliers to queue behind each other. For this reason, the SP reduces the OTS to counterbalance the effects of excessive queuing. In Section 5 we discuss how by increasing the number of traders/periods the intuition behind our results still holds.

To keep our model tractable, we do not include either cancellation or asymmetric information. However, since overall both cancellation and competition from informed investors lead to more aggressive order submission strategies (e.g., Bhattacharya and Saar (2021) and Riccó, Rindi, and Seppi (2022)), and since we show that an increase in investors' aggressiveness induces the SP to set a wider OTS, we speculate that adding these two features of a limit order book - that we discuss in depth in Section 5 - would strengthen our result that the OTS should not be set to zero.

In our model we do not consider trading fees. A possible concern is that the OTS should be determined jointly with trading fees. The reason for this concern is that - as for the tick size - trading fees affect the order submission strategies of both liquidity suppliers and liquidity takers. In real markets, however, the tick size is generally set by the regulator, whereas trading fees are set by the owner of the exchange. In addition, the trading fees are usually set conditional on the existing tick size which defines a natural upper bound for the size of the trading fees. Finally, there is no market where trading fees are larger than the tick size. Therefore, we envisage that the optimal trading fee is a first-order concern for the trading platform but a second-order one for the regulator setting the tick size.[9]

Our results have an important policy implication, namely that the current binary tick size regime that governs the U.S markets is sub-optimal to both the ESMA protocol governing the European markets and to the tick size regimes governing most of the exchange platforms around the world. In addition, our model has an important empirical implication showing that the best market quality metric that the SP can use to proxy the total welfare of market participants is the spread. This result allows us to positively interpret our empirical result that the introduction of the MiFID II tick size regime in 2018 had a positive effect on spread (Section 6).

1 The Model

We model a market as a finite game of t_i periods - with i=1,..,T with $T = 5$ - in which investors trade a unique asset with fundamental value, ν, publicly known. In each period t_i a risk neutral investor arrives with certainty therefore we can interpret time t_i as traders' arrival time rather than clock time. In each period the trader arrives with a private valuation of the asset, $\beta_{t_i}\nu$, where β_{t_i} is drawn from a uniform distribution, $\beta_{t_i} \overset{i.i.d.}{\sim} U[\underline{\beta}, \overline{\beta}]$, centered around the asset value ν,

[9]As examples, consider both the 2018 MiFID II and the recent SEC (2022) proposal. In both cases, the regulator felt the need to accommodate the upper bound of the access fees when proposing a reduction in the tick size for the treated stocks.

where $\underline{\beta} = 1 - b$ and $\overline{\beta} = 1 + b$ and $b \in (0, 1)$. Therefore, while in each period one investor arrives with certainty, all types of investors may arrive in probability. β_{t_i} defines traders' willingness to supply or take liquidity: the more the private value is close to the limits of the investors' valuation support, $\underline{\beta}\nu$ and $\overline{\beta}\nu$, the more the trader is likely to opt for aggressive orders. Conversely, a $\beta_{t_i}\nu$ close to ν is associated with a trader opting for limit orders. Hence, the larger the support of the β_{t_i} distribution, $\Gamma = 2b$, the larger are the ex-ante gains from trade of investors.

The price grid $p_k \in \{p_{-n}, .., p_{-k}, .., p_{+k}, .., p_{+n}\}$ of our limit order book is centered around the asset value ν with a tick size τ. Since τ measures the distance between two consecutive prices, we can write the price grid in a recursive way (Appendix B.1):

$$p_{+k} = \nu + \left(k - \tfrac{1}{2}\right)\tau \tag{1}$$

$$p_{-k} = \nu - \left(k - \tfrac{1}{2}\right)\tau \tag{2}$$

where $\left(k - \frac{1}{2}\right)\tau$ measures the distance between p_k and the fundamental value ν, and shows that the dimension of the tick size determines how coarse the price grid is.

Let the number of shares available at each price level p_k at time t_i be the state of the limit order book, Λ_{t_i}, and let o_{t_i} indicate the order an investor arriving at the market at time t_i chooses. Investors can choose between market orders to buy or to sell $o_{t_i} \in \{mb_{t_i}, ms_{t_i}\}$ and limit orders to buy or to sell $o_{t_i} \in \{lb_{t_i}, ls_{t_i}\}$ at any price on the price grid.[10] Investors can alternatively choose not to trade (nt_{t_i}).

Given the state of the book $\Lambda_{t_{i-1}}$ and the tick size τ, the expected payoff of an order o_{t_i} for an investor with private evaluation $\beta_{t_i}\nu$ arriving at t_i is:

$$O_{t_i}(o_{t_i}|\Lambda_{t_{i-1}}, \tau, \beta_{t_i}) = \begin{cases} (\beta_{t_i}\nu - p(o_{t_i})) \times I & mb_{t_i} \text{ or } ms_{t_i} \\ (\beta_{t_i}\nu - p(o_{t_i})) \times I \times Pr(\Psi_{o_{t_i}}|\Lambda_{t_{i-1}}, \tau) & lb_{t_i} \text{ or } ls_{t_i} \\ 0 & nt_{t_i} \end{cases} \tag{3}$$

where I is an indicator function taking value $+1$ for buy orders and -1 for sell orders; $p(o_{,t_i})$

[10]The price at which market sell (ms_{t_i}) and market buy orders (mb_{t_i}) are executed are the best prices available on the opposite side of the book.

is the price at which order o_{t_i} is either executed with probability one if $o_{t_i} \in \{mb_{t_i}, ms_{t_i}\}$, or it is executed with probability $Pr(\Psi_{o_{t_i}}|\Lambda_{t_{i-1}}, \tau)$ if $o_{t_i} \in \{lb_{t_i}, ls_{t_i}\}$; and $\Psi_{o_{t_i}}$ denotes the future states of the book in which order o_{t_i} may be executed. Limit order execution probabilities are endogenous and depend parametrically on both the valuation support $2bv$ and the tick size τ. For simplicity, we assume that investors cannot cancel or modify their orders which therefore reside on the book until execution.

An investor arriving at t_i chooses his optimal order-submission strategy, given the state of the book $\Lambda_{t_{i-1}}$ and τ, by maximizing his expected payoff in (3):

$$\max_{o_{t_i}} O_{t_i}(o_{t_i}|\Lambda_{t_{i-1}}, \tau, \beta_{t_i}) \tag{4}$$

where the optimal order submission strategy $o^{\star}_{t_i}$ maps each possible investor valuation β_{t_i} in the support $[\underline{\beta}, \overline{\beta}]$ with the order that maximizes (4) conditional on the standing book $\Lambda_{t_{i-1}}$ and the tick size τ. As the investor expected payoffs $O_{t_i}(o_{t_i}|\Lambda_{t_{i-1}}, \tau, \beta_{t_i})$ are linear in the investor valuations β_{t_i}, the discrete choice optimization problem in (4) is tractable. The upper envelope of the linear expected payoff functions maximizes the investors expected payoffs for each β_{t_i} evaluation in the support $[\underline{\beta}, \overline{\beta}]$. The intersection points of the linear payoff functions are the β_{t_i} thresholds which define a number of intervals of the β_{t_i} evaluations in correspondence of which different order submission strategies are optimal. In our model investors can choose between market and limit orders and therefore they face the fundamental trade-off between price opportunity cost (POC) and non-execution cost (NEC). POC is the cost of execution at the less favourable price they face when choosing a market order, while NEC is the cost of execution uncertainty investors face when choosing a limit order. As this fundamental trade-off is crucially influenced by the tick size, so are investors' order submission strategies.

For each period of the trading game, our model allows us to compute the probability that an investor chooses either to consume liquidity via a market order, or to supply liquidity by adding a limit order. As an investor's action may affect the best bid-offer, it may alter - in probability - the spread midpoint which is a proxy for the fundamental asset value, ν. Hence, while our model does not embed the volatility of the asset value due to incoming news, it captures the volatility due to the change in the state of the book driven by liquidity reasons. In our model there is no asymmetric information - no adverse selection costs - and we focus on the strategic interaction of investors with different gains from trade conditional on the state of the book. Hence, in our model a positive spread may only be due to the liquidity component.

2 Two-Period Model

The most parsimonious model we consider has only two periods, t_1 and t_2. As discussed in Riccó et al. (2021), in a 2-period model each player has a specific and unique role. At the beginning of the trading game the book opens empty. If the investor arriving at t_1 decides to trade, he can only post a limit order, l_{t_1}, and he is forced to act as a monopolistic liquidity supplier. If the investor arriving at t_2 decides to trade, he is forced to act as a liquidity taker, as he can only take the limit order posted by the investor at t_1 (m_{t_1}), or refrain from trading.

As neither undercutting nor queuing is an attainable trading strategy in this framework, the relevant transmission channel of a variation in the tick size is only the mechanical change of the inside spread. Hence, this framework is particularly suitable to discuss how a change in the tick size affects investors' order submission strategies via a change in transaction costs proxied by the inside spread.

The game is solved by backward induction starting from the last round of trading, t_2, when investors can only post market orders. We can therefore determine the probabilities of market order buy and sell orders, mb_{t_2} and ms_{t_2}, which are respectively the probabilities of execution of limit sell and buy orders, ls_{t_1} or lb_{t_1}, at t_1. As shown in Lemma (1) (Appendix B.2), if the book is symmetric at time t_i, then investors with $\beta_{t_i} > 1$, hence $\beta_{t_i}\nu > \nu$, are potential buyers at time t_i. Similarly, investors with $\beta_{t_i}\nu < \nu$ are potential sellers. We can therefore consider the order submission strategies on the sell side of the book at t_2, the buy side being symmetric.

An investor arriving at t_2 will market sell at p_k if his payoff is strictly positive, $p_k - \beta_{t_2}\nu > 0$. Given that β_{t_i} is uniformly distributed over the support Γ and $\underline{\beta} = 1 - b$, the probability of a market sell at t_2 is equal to:

$$Pr\left(ms_{k,t_2}|\Lambda_{t_1},\tau\right) = \frac{1}{\Gamma}\left(\frac{p_k}{\nu} - (1-b)\right) = Pr(\Psi_{lb_{k,t_1}}|\Lambda_{t_0},\tau) \tag{5}$$

which, in turn, is equal to the execution probability of a limit buy at t_1.

We can therefore derive the probability of the optimal limit buy order at t_1. This is the probability that the investor chooses a limit order with a positive payoff, $(\beta_{t_1}\nu - p_k)\,Pr(\Psi_{lb_{k,t_1}}|\Lambda_{t_0},\tau)$, which must be greater than the payoff associated with a limit buy order at t_1 posted at any other feasible price, $p_{\sim k}$.

Definition 1. *A feasible price, p_k^f, is a limit price for which there exists a positive probability of execution.*[11]

The submission probability of the optimal limit buy order at t_1 is the solution to the following conditions:

$$\begin{aligned}
&Pr\left(lb_{k,t_1}|\Lambda_{t_0},\tau\right) = \\
&Pr\Big[(\beta_{t_1}\nu - p_k)\,Pr(\Psi_{lb_{k,t_1}}|\Lambda_{t_0},\tau) > 0, \\
&\quad (\beta_{t_1}\nu - p_k)\,Pr(\Psi_{lb_{k,t_1}}|\Lambda_{t_0},\tau) > (\beta_{t_1}\nu - p_{\sim k})\,Pr(\Psi_{lb_{\sim k,t_1}}|\Lambda_{t_0},\tau), \forall \sim k\Big]
\end{aligned} \tag{6}$$

Given a value of τ, the 2-period model has the equilibrium solution presented in Proposition 1.

Proposition 1. *For any bv, if $\beta_{t_1}\nu > \nu$ - hence a buyer arrives at t_1 - the optimal set of p_k and the optimal order submission probabilities are:*

t_1 : *The set of optimal prices is $p_{-k} \in \left[p_{-\frac{bv}{2\tau}}, p_{-1}\right]$. The equilibrium order submission probability for the optimal p_{-k} is $Pr\left(lb_{-k,t_1}|\Lambda_{t_0},\tau\right) = \frac{\tau}{b\nu}$ $\forall \tau \in (0, \tau^{\max})$*

t_2 : *Defined in Lemma 1.3 (Appendix B.2).*

[11] For a limit price to be *feasible*, the two terms of the limit order payoff in (3) must be positive. This means that an investor will choose to post a limit order at p_k only if $(\beta_{t_i}\nu - p_k) \times I > 0$, and $Pr(\Psi_{l_{k,t_i}}|\Lambda_{ti-1},\tau) > 0$. Note that the probability of execution $Pr(\Psi_{l_{k,t_i}}|\Lambda_{t_{i-1}},\tau)$ is positive only if the private valuation of a potential buyer (seller) hitting p_k is smaller (greater) than the upper (lower) bound of the valuation support $2b\nu$. We derive in Appendix B.2 the set of *feasible* prices associated with the set of *feasible* τ: $p_k^f \in (\underline{\beta}\nu, \overline{\beta}\nu)$. Without loss of generality, we do not consider price levels on the grid which have zero probability of execution.

Symmetric results apply if $\beta_{t_1}\nu < \nu$, hence a seller arrives at t_1.

The proof of Proposition (1) is in Appendix (C.1). In a 2-period trading game the 1^{st} player knows that he is a monopolist in liquidity provision and therefore he never submits a limit buy order at a price, p_{+k}, higher than the fundamental asset value ν. In addition, the equilibrium order submission probability of the 1^{st} limit buy (sell) order is constant across the optimal p_k prices and equal to $\frac{\tau}{b\nu}$: the 1^{st} buyer (seller) is ex-ante indifferent (before his β_{t_1} is drawn) to submitting a limit buy (sell) order at any of the optimal prices $p_{-k} \in \left[p_{-\frac{bv}{2\tau}}, p_{-1}\right]$ ($p_{+k} \in \left[p_{+1}, p_{+\frac{bv}{2\tau}}\right]$). Being the order submission probability constant across the optimal prices, the price opportunity cost an investor bears to submit, for example, a limit buy at p_{-k-1} as opposed to p_{-k}, is just equal to the increase in the non execution cost that buying at lower price, p_{-k-1}, entails.

In a 2-period model the probability of execution of a limit order at t_1 is equal to the probability of submission of a market order at t_2. As we show that the order submission probabilities of the last player of our trading game can be written recursively, in a 2-period model also the order submission probability of the 1^{st} player can be written recursively as a function of the tick size.[12]

The SP chooses the tick size that maximizes the total welfare $\Omega(\tau)$ of market participants:

$$\max_{\tau\in(0,\tau^{max})} \Omega(\tau) = \omega_{t_1}(lb_{t_1}\ |\tau) + \omega_{t_2}(ms_{t_2}|\tau) \tag{7}$$

where $\omega_{t_1}(lb_{t_1}\ |\tau)$ and $\omega_{t_2}(ms_{t_2}|\tau)$ are the welfare of the 1^{st} and 2^{nd} player, respectively. The tick size domain of the SP's objective function includes, without loss of generality, only *feasible* τ values. These are the tick size values consistent with at least one *feasible* price on each side of the market. The upper bound of the set of *feasible* τ, $\tau \in (0, \tau^{max})$, is the tick size that is equal to the investors' valuation support:[13]

$$\tau^{max} = 2\, b\, \nu. \tag{8}$$

Given the optimization problems solved by traders and the SP, we can define the equilibrium of our trading game:

[12] This property only characterizes the 2-period model but does not hold when we add one or more periods to our trading game, as in this case the probability of order execution of the 1^{st} player does not depend exclusively on the probability of submission of the last player.

[13] Consistently with real market practice, we set the lower bound of the feasible τ to be non negative.

Definition 2. *A sub-game Perfect Nash Equilibrium of the trading game is the set of limit order submission probabilities, $Pr(l_{k,t_1})$, that solves the optimization problem of investors at t_1, such that the equilibrium execution probabilities, $Pr(\Psi_{l_{k,t_1}}|\Lambda_{t_0},\tau^\star)$, are consistent with the optimal order submission probabilities at t_2, and with a tick size, $\tau^\star \in (0, \quad \tau^{max})$, set by the SP to maximize total welfare $\Omega(\tau)$.*

2.1 Welfare Analysis

The SP expected welfare for the investor (e.g., a buyer) arriving at t_1 depends on two components: the execution probability of each optimal limit buy order, $Pr(\Psi_{lb_{-k,t_1}}|\Lambda_{t_1},\tau)$, and the price improvement associated with that order which is the difference between the investor valuation and the transaction price p_{-k}. Without loss of generality, assume, by Proposition (1), that for the generic tick size τ, there are m prices chosen with positive probability by the investor arriving at t_1. His welfare is therefore:

$$\omega_{t_1}(lb_{t_1}\ |\tau) = \sum_{k=1}^{m} Pr(\Psi_{lb_{-k,t_1}}|\Lambda_{t_0},\tau) \times \frac{1}{\Gamma}\int_{\beta_{t_1}\in B(\tau)} (\beta_{t_1}v - p_{-k})\ d\beta \tag{9}$$

where $B(\tau)$ is the interval on the support Γ of the β_{t_1} realizations for which any limit buy order, lb_{k,t_1}, is optimal. In Appendix (C.2) we express equation (9) as a function of $\hat{\tau} > \tau$, $\omega_{t_1}(lb_{t_1}|\hat{\tau})$, and show that:

$$\Delta\omega_{t_1}(lb_{t_1}\ |\hat{\tau},\tau) = \omega_{t_1}(lb_{t_1}\ |\hat{\tau}) - \omega_{t_1}(lb_{t_1}|\tau) < 0 \tag{10}$$

$\Delta\omega_{t_1}(lb_{t_1}\ |\hat{\tau},\tau)$ is decreasing in τ, and is consistent with Panel A of Figure 1, illustrating that the welfare of the 1^{st} player is negatively related to the tick size for the parameterization $b = 0.06$ and $\nu = 10$ and $\tau \in N^+|\ \frac{b\nu}{2\tau} \in (1,\ 50)$.[14]

For the generic tick size, τ, the SP expected welfare for the seller arriving at t_2 and hitting

[14] We use a parameterization for gains from trade and stock value, $2b\nu$, which is in line with both Goettler et al. (2005) and Hollifield, Miller, Sandås, and Slive (2006). Following Goettler et al. (2005), we consider a private evaluation of 2.5% from the empirically estimates of Hollifield et al. (2006) for three stocks on the Vancouver exchange with asset value close to $10CAD$. This private evaluation characterizes the average value between 32% and 52% of all traders active on these stocks. We then compute this metric assuming a uniform distribution instead of a normal distribution, and obtain $b \approx 0.06$. Results based on different parameterizations with different investors' ex ante gains from trade do not change qualitatively.

the limit buy order submitted at t_1 is:

$$\omega_{t_2}(ms_{t_2}|\tau) = \sum_{k=1}^{m} Pr\left(lb_{-k,t_1}|\Lambda_{t_0},\tau\right) \times \frac{1}{\Gamma}\int_{(1-b)}^{\frac{p_{-k}}{v}} \left(p_{-k} - \beta_{t_2} v\right) d\beta_{t_2} \tag{11}$$

where $Pr\left(lb_{-k,t_1}|\Lambda_{t_0},\tau\right)$ is the order submission probability of a limit buy order, lb_{-k,t_1}, posted at p_{-k} at t_1. In Appendix (C.2.3) we express equation (11) as a function of $\hat{\tau} > \tau$, $\omega_{t_2}\left(ms_{t_2}|\hat{\tau}\right)$, and show that:

$$\Delta\omega_{t_2}(ms_{t_2}\ |\hat{\tau},\tau) = \omega_{t_2}(ms_{t_2}\ |\hat{\tau}) - \omega_{t_2}\left(ms_{t_2}|\tau\right) < 0 \tag{12}$$

$\Delta\omega_{t_2}(ms_{t_2}\ |\hat{\tau},\tau)$ is decreasing in τ. These results lead to our Corollary 1:

Corollary 1. *In the 2-period model, the welfare function of the investors arriving either at t_1 or at t_2 is decreasing in τ.*

Panel B of Figure 1 illustrates our general result that also the welfare of the 2^{nd} player is negatively related to the tick size for the same parameterization used in Panel A. Corollary (1) drives our result on the OTS which we summarize in Proposition (2):

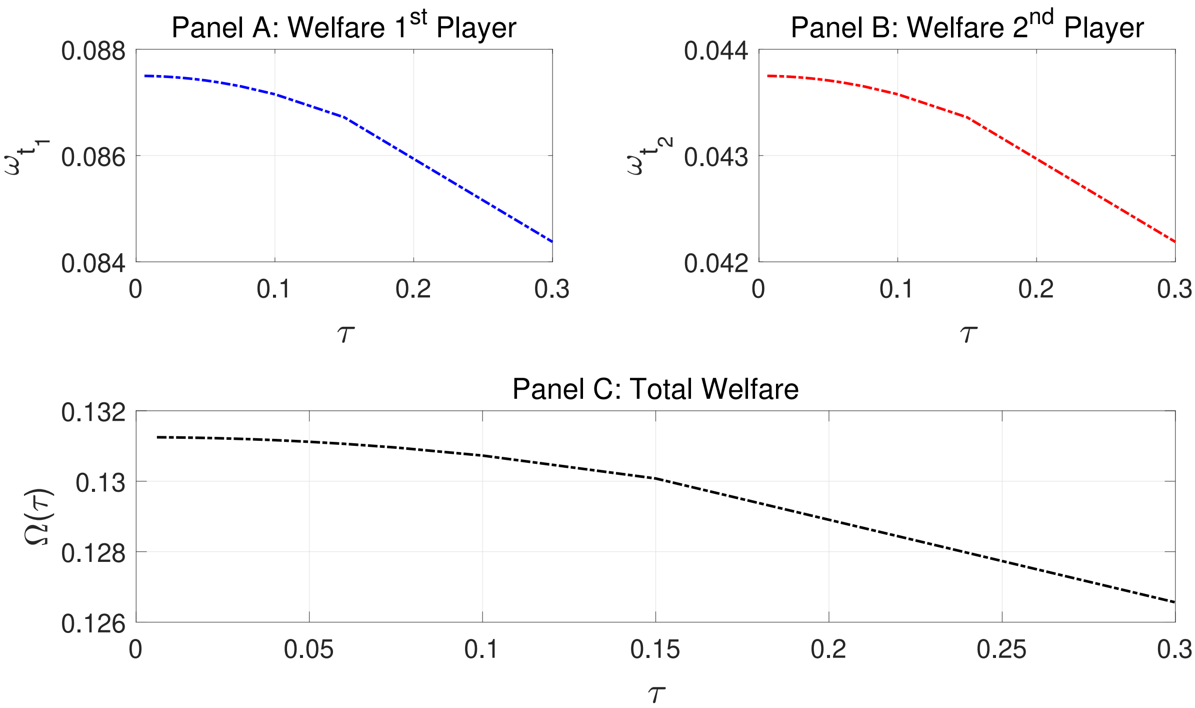
Proposition 2. *In the 2-period model, the OTS set by the SP is zero.*

Appendix (C.2.4) provides an analytical proof of Proposition (2). Panel C of Figure 1 illustrates that the total welfare of market participants is a decreasing function of the tick size, for the same parameterization used in Panel A and B.

In the 2-period model the tick size is a friction that constraints market participants to use a limited set of prices. In particular, the tick size constraints the t_1 liquidity supplier to post his order at a price which is not necessarily equal to his private valuation. Therefore, by reducing the tick size, the negative welfare effects induced by the tick size discretization decrease and the welfare of the 1^{st} player increases. Appendix C.2.2 provides a simple example showing that players with extreme evaluation can maximize their welfare only when $\tau = 0$, as they can post a limit order at the fundamental value, ν.

The intuition behind the welfare of the 2^{nd} player being decreasing in τ lies in how the 1^{st} player submits his limit order. The 2^{nd} player can only execute the order posted at t_1 in a

Figure 1: **Welfare Analysis Two Period Model**
The figure reports the welfare of each market participant in a 2-period game and the total welfare: Panel A shows the welfare of the 1^{st} player ($\omega_{t_1}(\tau)$, blue dotted line), Panel B the welfare of the 2^{nd} player ($\omega_{t_2}(\tau)$, red dotted line) and Panel C the total welfare of market participants ($\Omega(\tau)$, black dotted line) for $b = 0.06$, $\nu = 10$ and for a set of tick size values that define up to 50 equilibrium price levels $\left\{\tau \in N^{+} | \ \frac{b\nu}{2\tau} \in (1,\ 50)\right\}$. Results do not change qualitatively by considering a larger set of tick size values encompassing a larger number of price levels.



take or leave fashion. By Proposition (1), the investor - e.g., a buyer - arriving at t_1 submits an order only at prices below the fundamental value of the asset, and an increasing tick size mechanically lowers these prices toward the lower bound. Hence, for an increasing tick size, the investor arriving at t_2 is automatically forced to sell at lower prices, diminishing his overall welfare. Hence, the social planner optimally sets the tick size to zero.

Our results for the 2-period model are consistent with the existing literature. Absent asymmetric information in the Glosten and Milgrom (1985) framework, the optimal bid-ask spread (tick size) is equal to zero. Our results from the 2-period model are also in line with Li and Ye (2022) model which shows that the tick size that maximizes liquidity in an extended Budish, Cramton, and Shim (2015) model is zero. In this model only market makers can undercut each other to supply liquidity ahead of the other market participants hitting their quotes.[15] For this

[15] Li and Ye (2022) assume that the informed investor can only post a one lot size order whereas the uninformed investor optimally posts an order equal to the whole firm's outstanding share (split into a series of child orders

reason, there is no endogenous choice between market and limit orders by all market participants with the result that all investors cannot queue behind existing limit orders or undercut them to gain price priority. Absent queuing and undercutting, the only transmission channel driving investors' order submission strategies reacting to a change in the tick size is the mechanical change in the inside spread.

3 Three-Period Model

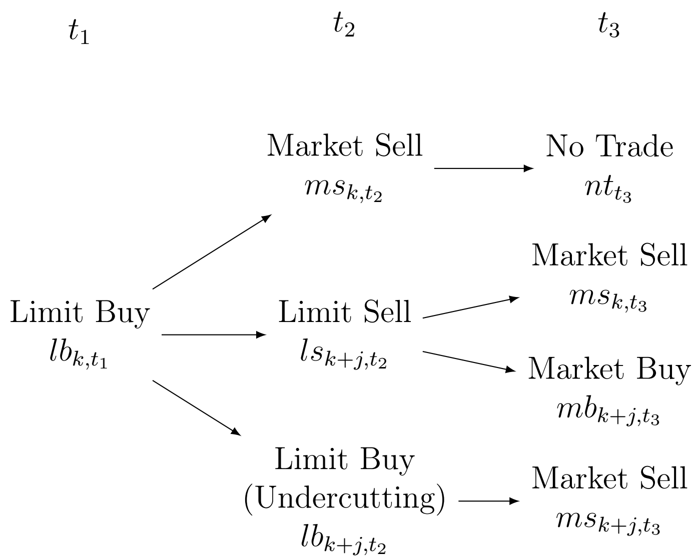
We now extend our analysis to a 3-period framework. With a new further period to trade, the investor arriving in the first period is no longer a monopolist in the provision of liquidity: the investor arriving in the second period can now both offer and take liquidity. However, although the 2^{nd} player can offer liquidity and undercut the existing limit order posted by the 1^{st} player at t_1, his actions are limited by the fact that he cannot queue behind that limit order, as at t_3 the trading game finishes. Therefore, the 3-period trading game does not include strategic queuing although it now includes strategic undercutting and therefore it is a further step towards an increasingly more realistic limit order book.

For any chosen value of $\Gamma\nu$ and therefore for any $\tau \in (0, \tau^{max})$, we can solve our 3-period model in closed-form. We report the objective function of the investors arriving in the three periods in Appendix D.1. As for the 2-period trading game, we derive the equilibrium order submission strategies starting from the investor arriving at the last period t_3, and we indicate the possible trading actions available to market participants in our 3-period trading game in Figure 2. Without loss of generality, taking advantage of Lemma 1.2 we focus on the case of a limit buy (lb_{k,t_1}) order posted by the investor arriving at t_1 at a generic price p_k. The 2^{nd} player has now three options: he can hit the standing limit buy order posted at t_1 and take liquidity via a market sell order (ms_{k,t_2}); he can instead supply liquidity at a price higher than p_k both on the sell or on the buy side of the market. If he decides to add liquidity on the sell side, the 2^{nd} player posts a limit sell order (ls_{k+j,t_2}) at p_{k+j}, otherwise he would effectively take liquidity

of one lot size each). For this reason, when a jump occurs, the informed investor can snipe the market maker by one lot only.

via a marketable limit sell order. If instead the 2^{nd} player decides to limit buy, he can only post a limit buy order at a more aggressive price level (lb_{k+j,t_2}), higher than p_k, thus undercutting the existing limit buy order. Otherwise having only one period left to trade, his limit buy order at a price $p_{k-j} \leq p_k$ would be effectively queuing behind the previously posted limit buy order and would have zero execution probability.

Figure 2: **Extensive Form of the Three Period Game.**
This figure shows the different sets of actions for market participants in each period t_i of the trading game. The book opens empty and a buyer arrives at t_1. A symmetric extensive form holds if a seller arrives at t_1.



The novelty of the 3-period game is that the 2^{nd} player has a wider range of equilibrium strategic choices. The probability that the 2^{nd} player will opt to supply liquidity crucially depends on the distribution of the investors' personal evaluation. The larger the evaluation support, $\Gamma\nu$, the greater is the probability that the 2^{nd} player will aggressively either take liquidity by market selling or undercut the existing limit order by limit buying. Investors with larger gains from trade are generally more aggressive and they favor the faster execution probability granted by either a market sell or an aggressive limit buy to the larger price improvement granted by the more patient limit sell order. Panel A of Figure 3 shows that for a given τ, undercutting is a positive function of the gains from trade $\Gamma\nu$. Jumping the queue by aggressively limit buying implies a higher cost in terms of price improvement when the tick size is larger, therefore we expect undercutting to decrease with the tick size as shown in Panel B where for a given evaluation support $\Gamma\nu$, undercutting is a negative function of the set of feasible tick sizes, $\tau \in (0, \tau^{max})$.[16]

Figure 3: **Three Period Model: Undercutting**
This figure shows the probability of undercutting of a limit buy at t_2: $Pr(lb_{k+j,t_2}) = \sum_{k=-n^f}^{+n^f-1} \sum_{j>1} Pr\left(lb_{k,t_1}|\Lambda_{t_0},\tau\right) Pr\left(lb_{k+j,t_2}|\Lambda_{t_1},\tau\right)$. Panel A shows $Pr(lb_{k+j,t_2})$ for $\tau = 0.278$ (which as we will show in Section 3.2, is the OTS for the 3-period benchmark case with $b = 0.06$ and $\nu = 10$) and the set of $2b\nu$ values defined by all possible combinations of three values of $b = \{0.045, 0.06, 0.075\}$ and $\nu = \{8, 10, 13.333\}$. $b = 0.06$ is estimated following Goettler et al. (2005) and Hollifield et al. (2006) and the two other b values are $b = 0.06(1 \pm 0.25\%)$. $\nu = 10$ is our benchmark asset value and $\nu = 8$, $\nu = 13.333$ are obtain such that $2b\nu$ is constant across the three pairs of values (b, ν). Results do not change qualitatively if we consider a different chosen τ and set of $2b\nu$. Panel B shows $Pr(lb_{k+j,t_2})$ for $b = 0.06$, $\nu = 10$ and for a set of tick sizes, defined in Appendix (D.6), that defines the price grids between 2 and 30 prices. Results do not change qualitatively if we consider price grids embedding more prices.

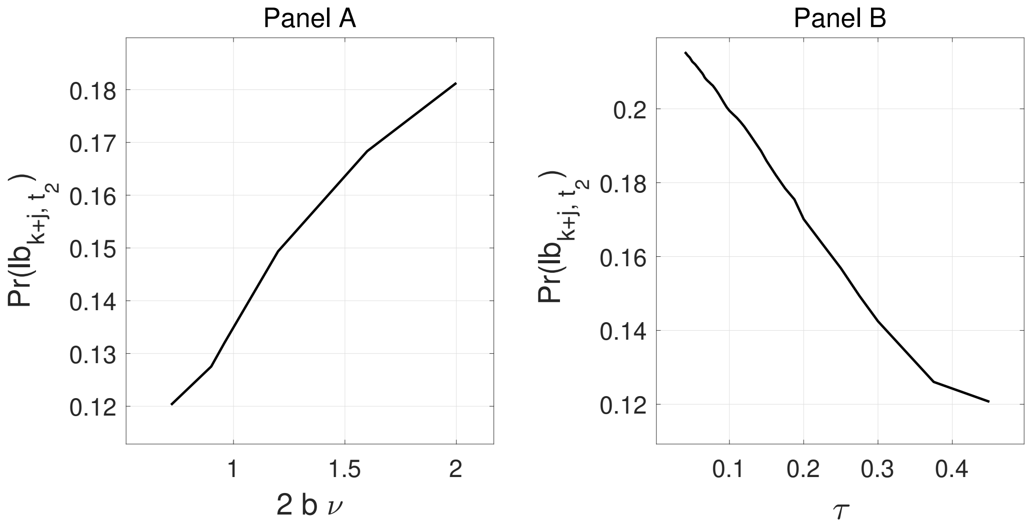
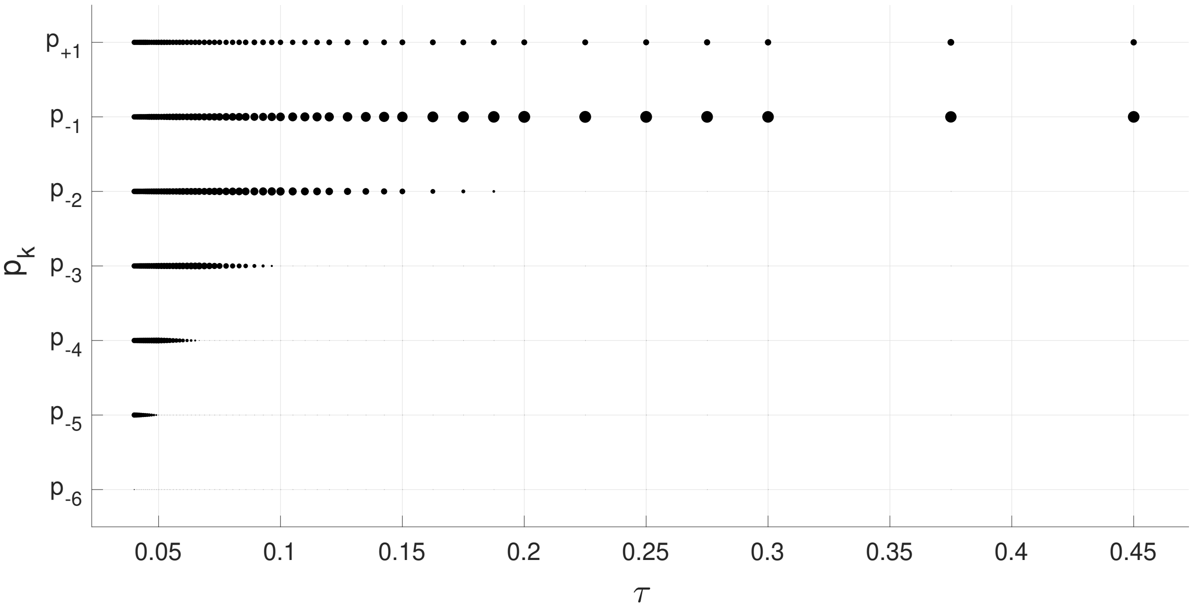


Figure 4 reports the equilibrium submission strategies of the 1^{st} player (a buyer) for the set of tick sizes (Appendix D.6) that defines the price grids between 2 and 30 prices. As expected, the 1^{st} player supplies liquidity at a wider range of price levels when the tick size is small, whereas he tends to supply liquidity at the best bid-ask prices (recall $\nu = 10$) for wider values of the tick size. In addition, in equilibrium, the 1^{st} player supplies liquidity at the best bid of his own side of the market with increasingly higher probability (despite the fact that such a best price is gradually lower as the tick size widens) because a larger tick size reduces the probability that the 2^{nd} player undercuts his quote.

To study the properties of the OTS in a 3-period model, we first show analytically (Section

[16]In Appendix D.3, we also show that given a limit buy order posted at p_k by the 1^{st} player, the probability that the 2^{nd} player will undercut at p_{k+j} increases as the tick size decreases.

Figure 4: **Three Period Model: Equilibrium Submission Strategies 1^{st} Player**
This figure shows the t_1 equilibrium probability of a limit buy order $Pr(lb_{k,t_1}|\Lambda_{t_0}, \tau)$ for $b = 0.06$, $\nu = 10$ and for a set of tick sizes (Appendix D.6) that defines the price grids between 2 and 30 prices. For each game considered (defined by the triplet (b, ν, τ), the submission probabilities have been analytically computed following Appendix D.1. The size of each dot is proportional to the submission probability $Pr(lb_{k,t_1}|\Lambda_{t_0}, \tau)$.



3.1) that the tick size is no longer a friction and therefore the OTS is not zero. We then (Section 3.2) characterize the OTS and show that it depends both on the fundamental asset value and on the investors' dispersion of beliefs. Specifically, we show that the OTS is a positive function of ν and of b.

3.1 Optimal Tick Size

As shown in Figure 4, when a third period is added to the protocol, the space of the order submission strategies increases substantially. To show analytically that the OTS is positive, for tractability in this section only we assume that if the 2^{nd} player wishes to undercut the 1^{st} player limit order posted at p_k or wishes to supply liquidity on the other side of the book, he can only do so at adjacent prices, p_{k+1}. In Appendix D.1 we relax this assumption and show that the results for the OTS qualitatively hold. In the following proposition we summarize the equilibrium properties of this 3-period game:

Proposition 3. *For any $b\,\nu$, if $\beta_{t_1}\nu > \nu$, hence a buyer arrives at t_1, the equilibrium order submission strategies $\forall \tau \in (0, \tau^{max})$ are:*

t_1: $\cdot$ $\forall \tau \in [(0,\ \tau^{max}) \mid n^f \geq 2]\ \exists\, k < +n^f | Pr\left(lb_{k,t_1}|\Lambda_{t_0}, \tau\right) > 0$ *and* $Pr\left(lb_{+n^f,t_1}|\Lambda_{t_0}, \tau\right) = 0$

$\cdot$ $\forall \tau \in \left\{(0,\ \tau^{max}) \mid n^f = 1\right\}$, $Pr\left(lb_{k,t_1}|\Lambda_{t_0}, \tau\right) > 0 \quad \forall k \in \{-1,\ 1\}$

and $\lim_{\tau \to \tau^{max}} Pr\left(lb_{+1,t_1}|\Lambda_{t_0}, \tau\right) \to 0$

t_2: $\cdot$ $Pr\left(ms_{k,t_2}|\Lambda_{t_1}, \tau\right) = max\left[0, Pr\left((1-b) < \beta_{t_2} < \frac{p_k}{\nu} - \frac{\tau}{\nu}w\right)\right]$ *with* $w = \frac{Pr\left(mb_{k+1,t_3}|\Lambda_{t_2},\tau\right)}{1-Pr\left(mb_{k+1,t_3}|\Lambda_{t_2},\tau\right)}$.

$\cdot$ $Pr\left(ls_{k+1,t_2}|\Lambda_{t_1}, \tau\right) = Pr\left(\frac{p_k}{\nu} - \frac{\tau}{\nu}w < \beta_{t_2} < \frac{p_k}{\nu} + \frac{\tau}{\nu}\right)$ *if* $Pr\left(ms_{k,t_2}|\Lambda_{t_1}, \tau\right) > 0$

and $Pr\left(ls_{k+1,t_2}|\Lambda_{t_1}, \tau\right) = Pr\left((1-b) < \beta_{t_2} < \frac{p_k}{\nu} + \frac{\tau}{\nu}\right)$, *otherwise.*

$\cdot$ $Pr\left(lb_{k+1,t_2}|\Lambda_{t_1}, \tau\right) = Pr\left(\frac{p_k}{\nu} + \frac{\tau}{\nu} < \beta_{t_2} < (1+b)\right)$.

t_3: *Defined in Lemma 1.3 (Appedix B.2).*

Symmetric results apply if $\beta_{t_1}\nu < \nu$, hence a seller arrives at t_1.

Proposition 3, proved in Appendix D.4, characterizes the equilibrium order submission strategies of the investors arriving at each period of the trading game. The investor arriving at t_1 submits a limit buy at any price $p_k < p_{+n^f}$ with positive probability and has no incentive to lock the market when the price grid includes at least two prices on each side of the book ($n^f \geq 2$). When instead investors' gains from trade are so small relative to the tick size ($\frac{\Gamma\nu}{\tau} < 3$) that the price grid only includes one feasible price on each side of the market ($n^f = 1$), the 1^{st} player can either limit buy at p_{-1} or act as a monopolist in liquidity provision at p_{+1} thus locking the market with a positive probability. Under this extreme scenario, the equilibrium strategies at t_2 of the 3-period model are the same as the equilibrium strategies at t_2 of the 2-period model. As the tick size increases relative to the valuation support, $\frac{\Gamma\nu}{\tau} \to 1$, the 1^{st} player limit buys at p_{-1} and the 2^{nd} player in equilibrium posts a limit sell at p_{+1} with a probability that converges to 1.[17] Under this scenario, when the 3^{rd} player arrives at t_3, the book provides liquidity both on the bid side where the 1^{st} player makes the market, and on the ask side where the 2^{nd} player

[17]When $\frac{\Gamma\nu}{\tau} = 1$ there are no feasible prices and hence no trade.

makes the market. Therefore as the value of the investors' evaluation support tends to be equal to the tick size, i.e., the gains from trades are extremely small relative to the tick size, the 3-period model degenerates to an extended 2-period trading game with investors mainly acting as liquidity providers at both t_1 and t_2. This type of market making model is consistent with Li, Wang, and Ye (2021) who show that HFT dominate liquidity provision if the bid-ask spread is binding at one tick.

Point t_2 in Proposition 3 characterizes the equilibrium order submission strategies of the 2^{nd} player, which in turn depend on the aggressiveness of the limit order posted by the 1^{st} player. More specifically, the higher the chosen p_k by the 1^{st} player, the higher is the incentive for the 2^{nd} player to take liquidity $(Pr\,(ms_{k,t_2}|\Lambda_{t_1},\tau))$; while the lower is the chosen p_k by the 1^{st} player, the higher is the probability that the 2^{nd} player will supply liquidity. The 2^{nd} player will limit sell with a probability $(Pr\,(ls_{k+1,t_2}|\Lambda_{t_1},\tau))$ that is decreasing in p_k - from (22) in Appendix B.2 $Pr\,(mb_{k+1,t_3}|\Lambda_{t_2},\tau)$ decreases and w increases when p_k decreases - and will limit buy with probability $(Pr\,(lb_{k,t_1}|\Lambda_{t_0},\tau))$ which is also decreasing in p_k, thus undercutting the existing limit buy order.

Our results for the 2-period model show that the OTS that maximizes the welfare of market participants is zero. In contrast, as we increase the number of trading periods to three, we show that the OTS is different from zero as stated in the next proposition:

Proposition 4. *In a 3-period trading game, the tick size that maximizes the welfare of market participants is positive.*

This result is analytically proved in Appendix D.5 where we show that the total welfare of market participants for $\tau \rightarrow 0^+$ is smaller than the total welfare of market participants associated with a $\tau > 0$ and therefore $\tau \rightarrow 0^+$ cannot be the OTS set by the SP in a 3-period model.

Intuitively the SP will set the OTS that maximizes liquidity supply over time. This means that the OTS will maximize the liquidity provision of both the 1^{st} and the 2^{nd} player. If the tick size is very small, the 1^{st} player runs the risk of being undercut by the 2^{nd} player with the result that the 3^{rd} player can only benefit of the liquidity posted by the 2^{nd} player on one side of the market. If instead the tick size is larger, the 2^{nd} player will have less of an incentive to undercut the 1^{st} player or hit the 1^{st} player's limit order by taking liquidity (crowding out the 3^{rd} player), and he will have more of an incentive to submit a limit sell order. If the 2^{nd} player supplies liquidity on the other side of the market, the 3^{rd} player will benefit of the liquidity supply on both sides of the market, and therefore both liquidity demand and liquidity supply will be maximized. Taken together these transmission channels indicate that the OTS cannot be zero as in order to maximize liquidity supply, it must induce the 2^{nd} player to supply liquidity on the opposite side of the 1^{st} player's limit order.

Our result that the OTS cannot be zero is reminiscent of Cordella and Foucault (1999) who show that in a dealership market a zero minimum price variation never minimizes the expected trading costs. In Cordella and Foucault (1999) dealership market model, the transmission mechanism behind this result is different from our's: a larger tick size increases the speed of convergence of the dealers' selling quotes toward the competitive price and therefore it does not necessarily result in larger expected trading costs for liquidity demanders.[18]

3.2 Optimal Tick Size and Welfare of Market Participants

We now study the properties of the optimal tick size without restricting the set of actions available to the 2^{nd} player, as we did in Section 3.1. We solve the 3-period game described in Appendix D.1 in closed-form and then we determine the OTS in quasi-closed form. We can solve the 3-period model analytically for any given $\beta\,\nu$ and any associated $\tau \in (0,\ \tau^{max})$, and for each value of τ we can compute the welfare of the three investors arriving at t_1, t_2 and t_3 respectively.

The 3-period game implies a variety of actions that preclude the recursive property that characterize the 2-period trading game. In a 2-period model, the execution probability of the limit order posted at t_1 is just equal to the market order submission probability at t_2 (equation (5)) that can be written recursively. This implies that also the order submission probability of the limit order posted at t_1 can be written recursively (Proposition (1)). In the 3-period model this property no longer applies: the 1^{st} player strategically changes his trading behavior conditional on different values of $b\nu$ and τ as he now plays a strategic game with the 2^{nd} player.

[18] In Cordella and Foucault (1999) the competitive price is the first price above the dealers' reservation price, hence it is the first quoted price which cannot be profitably undercut.

In turn, the 2^{nd} player's strategies are contingent on the 1^{st} player's actions and hence change conditionally on different combinations of $b\nu$ and τ (Figure 3-4). Therefore, as in real markets, in the 3-period game each player's optimal strategy depends on the other players' reaction and all strategies crucially depend on both the evaluation support $b\ \nu$ and the tick size which defines the number of feasible prices on the grid. By changing the tick size, the price grid changes and so do the optimal reactions of all investors. As the limit order submission and execution probabilities are endogenous and depend on the state of the book and the price grid, we cannot characterize the OTS as a function of the parameter of the model, b and ν. For this reason, we solve the 3-period model for a discrete grid of tick size values and choose the tick size that maximizes the welfare of all market participants:

$$\max_{\tau \in (0,\tau^{max})} \Omega(\tau) =$$

$$\omega_{t_1}(lb_{t_1}\ |\tau) + \omega_{t_2}(ms_{t_2}\ \vee ls_{t_2}\ \vee lb_{t_2}\ |\tau) + \omega_{t_2}(ms_{t_2}\ |\tau) + \omega_{t_3}(ms_{t_3}\ \vee mb_{t_3}\ |\tau) + \omega_{t_3}(ms_{t_3}\ |\tau) \tag{13}$$

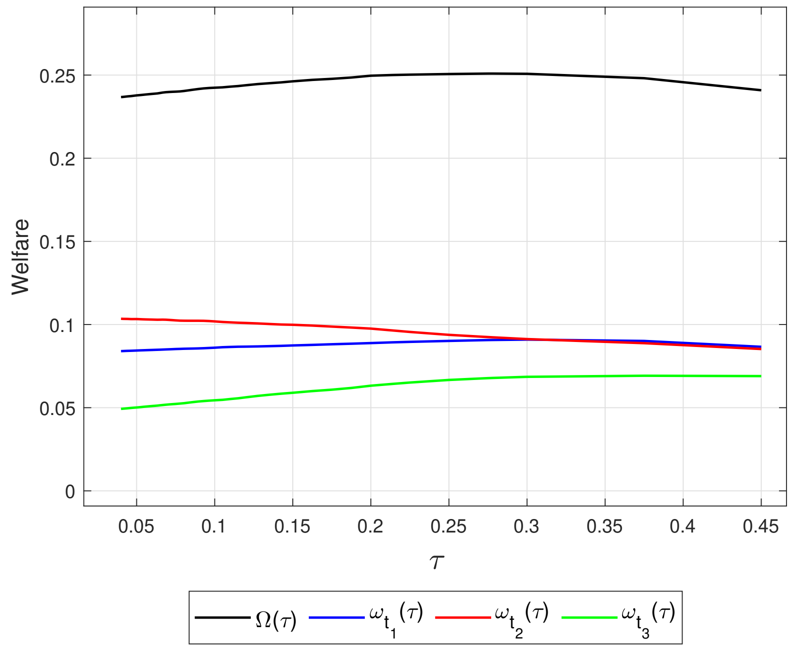
where each component of equation (13) - investors' welfare at t_i - is defined in Appendix D.2. Investors' welfare depends on their order submission probabilities which in turn depend on the number of prices on the grid. As shown in Appendix D.6, we therefore discretize the search grid by considering a set of tick sizes consistent with price grids that includes between 2 to 30 feasible prices. This upper bound is reasonable given the standard practice of investors using a limited number of prices on the price grid of real markets limit order books.[19] For each tick size in the discretization grid, we solve the equilibrium order submission strategies and the associated welfare of each market participant in closed-form. The SP then sets the OTS by choosing the tick size associated with the highest total welfare, hence we solve the OTS problem in *quasi-closed* form.

Figure 5 shows both the total welfare (black line) and the welfare of each market participant associated with all of the tick sizes in the chosen discretization grid. In the 2-period game the

[19] In the emerging crypto-currencies limit books the number of prices generally considered by market participants is larger than in markets for traditional instruments, where traders generally use around 20 price levels on each side of the book.

welfare of each market participant is a decreasing function of the tick size. Figure 5 shows instead that in the 3-period game while the welfare of the 2^{nd} player (red line) is still decreasing in τ, the welfare of the 1^{st} (blue line) and of the 3^{rd} player (green line) is a concave function of τ. Therefore, total welfare is also a concave function of the tick size, τ. This means that the choice of the OTS is no longer a straightforward problem as in the 2-period model but has to reconcile the interest of different market participants.

Figure 5: **Welfare Three Period Game**
This figure shows the welfare of the 1^{st} ($\omega_{t_1}(\tau)$, blue line), 2^{nd} ($\omega_{t_2}(\tau)$, red line), 3^{rd} player ($\omega_{t_3}(\tau)$, green line), and the total welfare of market participants ($\Omega(\tau)$, black line) for $b = 0.06$, $\nu = 10$ and for a set of tick sizes, defined in Appendix (D.6), that considers price grid between 2 and 30 prices. Results do not change qualitatively considering more prices.



To provide an intuition for the OTS, we consider how both the total welfare of market participants and the welfare of each player change with the tick size. Figure 5 shows that the welfare of the 2^{nd} player is a decreasing function of the tick size as the smaller is the tick size, the greater is the probability that the 2^{nd} player will undercut the limit order posted by the 1^{st} player, thus increasing the probability of execution of his limit order. In addition, a smaller tick size increases the space of the possible strategies available to the 2^{nd} player without increasing the probability that any of his limit order is undercut, as the 3^{rd} player can no longer be a liquidity provider. In contrast, the 1^{st} player faces the following trade-off: while - as for the 2^{nd}

player - a smaller tick size widens his choice of the feasible prices at which he can post a limit order, thus increasing his welfare, a smaller tick size increases the probability that the 2^{nd} player will undercut his limit order thus reducing his welfare. We show that the 1^{st} player's welfare is a concave function of the tick size. When the tick size is extremely small the number of feasible prices is extremely large but the probability of undercutting is also very high (Figure 3). As the tick size increases, still conditional on a sufficient number of feasible prices among which the 1^{st} player can choose, the probability of undercutting decreases and his welfare increases. However, as the tick size further increases, there will be a threshold beyond which the number of feasible prices becomes too small, the inside spread widens, and the probability of limit order execution decreases so that the 1^{st} player's welfare also decreases.

The 3^{rd} player's welfare is also a concave function of the tick size. With a small tick size, the 2^{nd} player will most likely undercut the 1^{st} player's limit order and the 3^{rd} player will be able to take liquidity only from one side of the book. As the tick size increases, the probability that the 2^{nd} player will offer liquidity on the other side of the book increases, and the 3^{rd} player will have the opportunity to take liquidity from both sides of the book. However, as the tick size further increases, due to the mechanical increase of the bid ask spread, the 3^{rd} player will take liquidity at unfavorable prices, and his welfare will deteriorate.

3.3 Optimal Tick Size, Stock Price and Market Quality

In this section we show how the effects discussed in Section 3.2 change with the gains from trade of market participants. Gains from trade may change either due to a change in the dispersion of investors' private evaluations, or due to a change in the asset price. In Table 1 we report the OTS and the total welfare of market participants. We also report the standard market quality metrics - expected volume, expected quoted semi-spread, and expected total depth defined in detail in Appendix D.7 - associated both with different values of the investors' evaluation support $2b\nu$, and with different combinations of b and of the fundamental asset value ν.

When all else equal either the dispersion of the investors' gains from trade, b, or the asset value, v, increases, investors' gains from trade also increase, and the SP sets a wider OTS. The

Table 1: Optimal Tick Size and Market Quality in the Three Period Game
The table reports the OTS and the associated total welfare (Ω), expected volume (*vol*), quoted spread (*quoted spread*) and total depth (*depth*) for each combination of $\nu = \{8, 10, 13.333\}$ and $b = \{0.045, 0.06, 0.075\}$. $b = 0.06$ is estimated following Goettler et al. (2005) and Hollifield et al. (2006) and the two other b values are $b = 0.06(1 \pm 0.25\%)$. $\nu = 10$ is our benchmark asset value and $\nu = 8$, $\nu = 13.333$ are obtain such that $2b\nu$ is constant across the three pairs of values (b, ν). The discretization grid used to derive the *quasi-closed* form solution for each (ν, b) is defined in in Appendix D.6: for each (ν, b), we consider a set of tick sizes consistent with price grids that include between 2 to 30 feasible prices. The results are rounded at the 3^{rd} decimal digit.

	ν \ b	0.045	0.06	0.075
OTS	8.000	0.167	0.223	0.278
Ω		0.151	0.201	0.251
vol		0.384	0.384	0.384
quoted spread		0.200	0.267	0.333
depth		1.186	1.186	1.186
OTS	10.000	0.209	0.278	0.348
Ω		0.188	0.251	0.314
vol		0.384	0.384	0.384
quoted spread		0.250	0.333	0.416
depth		1.186	1.186	1.186
OTS	13.333	0.278	0.371	0.464
Ω		0.251	0.335	0.418
vol		0.384	0.384	0.384
quoted spread		0.333	0.444	0.555
depth		1.186	1.186	1.186

economic intuition for this result is the following. With larger gains from trade, the trade-off that all investors face when choosing their orders changes. The 1^{st} player knows that larger gains from trade increase the execution probability of his marginal outside limit orders and therefore he submits more patient orders. Larger gains from trade coupled with a higher probability of facing patient limit orders posted by the 1^{st} player induce the 2^{nd} player's to either undercut or hit the 1^{st} player limit order with a higher probability. As a result, the 2^{nd} player offers liquidity on the opposite side of the market with a smaller probability. Hence, the 3^{rd} player's welfare decreases for two reasons. First, as discussed above, if the 2^{nd} player takes the liquidity posted by the 1^{st} player with a higher probability, the 3^{rd} player is crowded out of the market with a higher probability; second, if the 2^{nd} player offers liquidity on the other side of the market with a lower probability, the trading opportunities offered to the 3^{rd} player decrease as he is only able to trade on one side of the book. Table 1 shows these results. An increase in either the asset value or the dispersion of investors' evaluation leads to an increase in the OTS and also - due to the increased gains from trade - to an increase in total welfare. The OTS set by the SP takes into account all of these effects. The SP therefore sets a larger tick size that leads the impatient 2^{nd} player to switch from being aggressive - undercutting or executing the 1^{st} player's limit order - to behave more patiently by supplying liquidity on the other side of the market. This result is reminiscent of Foucault et al. (2005) who show that a reduction in the tick size may impair market resiliency and have an adverse effect on spread when the proportion of impatient traders increases.

When all else equal, either the asset value ν or the dispersion of investors evaluation b increases, the SP widens the OTS and as a result the equilibrium investors' order submission probabilities do not change. The intuition behind this result is that in our 3-period model the OTS associated with different values of b or ν all define a price grid with the same number of price levels: two on the ask and two on the bid side of the book. As our proxy for volume and depth are only function of the equilibrium order submission probabilities, they do not change in correspondence of different OTS. Our proxy for quoted spread instead increases reflecting the different OTS values.

Up to here we have assumed that when the asset value increases, the dispersion of investors' private evaluations does not change. If instead an increase in the asset value ν induces market participants to revise the dispersion of their personal evaluations in such a way that the overall evaluation support, $2b\nu$, remains constant, our model shows (Table 1 - grey shaded diagonal) that both total welfare, and the OTS, and our market quality metrics remain unchanged. Intuitively, if the asset value increases from 8 to 10, and the dispersion of the investors' evaluation support decreases proportionally from 0.075 to 0.06 the price grid is simply shifted upward so that the distance between the new asset value and the different price levels remain unchanged. These results lead to our Corollary 2:

Corollary 2. *When either the asset value or the dispersion of investors' gains from trade in-*

creases, the OTS set by a SP increases. Consequently, expected volume and expected depth do not change, whereas expected quoted spread increases.

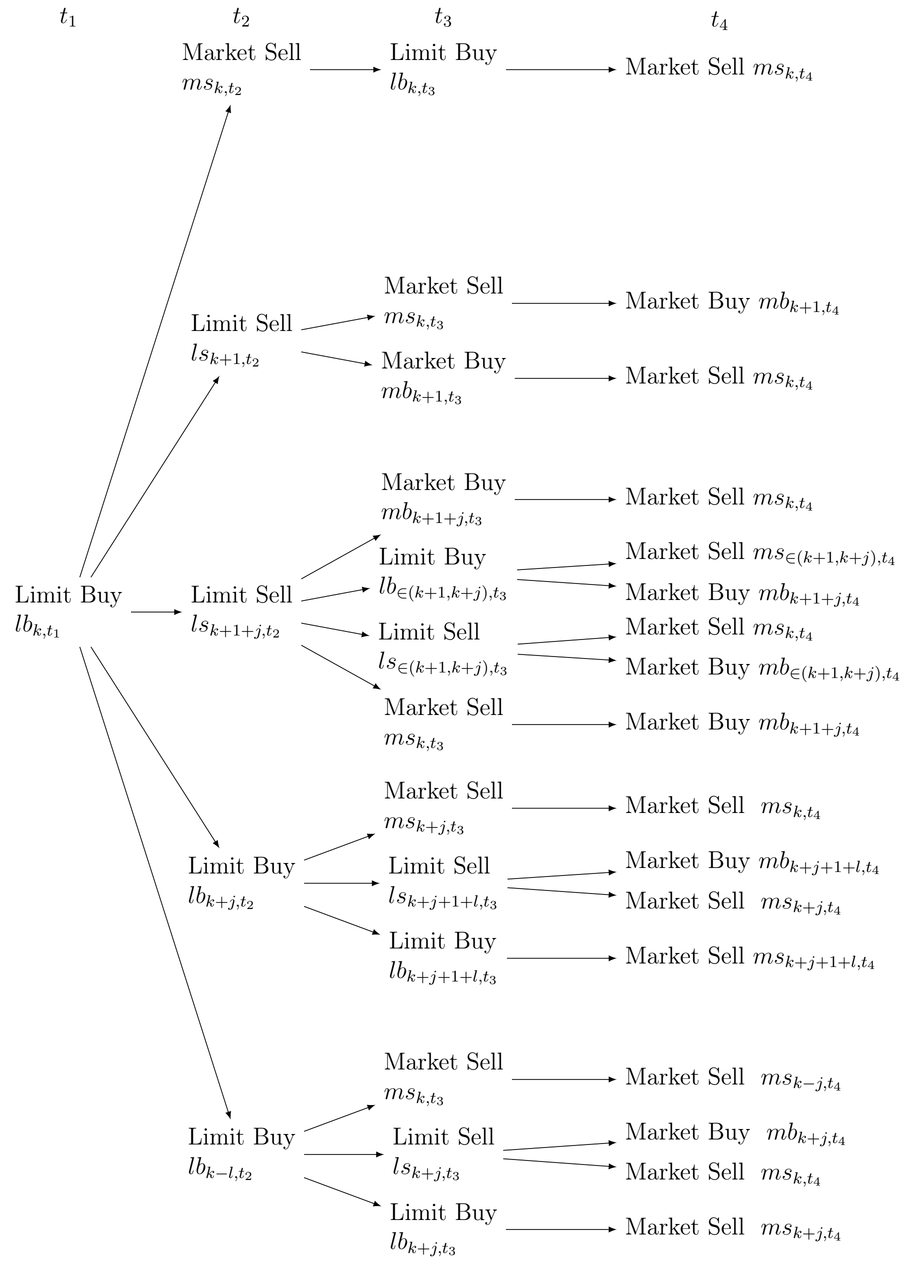
This result is consistent with the tick size schedules set by regulators in the majority of the existing trading platforms where the tick size is a step function of the asset price. Although in the U.S. markets the tick size differs for stocks priced above and below 1 USD, this binary tick size schedule only aims to differentiate the tick size for penny stocks. Our model shows that the tick size schedule should instead be a step function of the price of all stocks. Consistent with our theoretical results, we reviewed most of the existing major trading platforms and found that they have a tick size schedule with more than two bins (See Table 1.A).

4 Four-Period Model

When the trading game lasts three periods, orders cannot queue behind each other. Intuitively, in a 3-period model the 2^{nd} player never opts to queue behind the 1^{st} player's order as - due to time priority - his order would never be executed at t_3. In a 4-period model instead, orders can profitably queue behind each other, and the creation of queues can actually affect the order submission strategies of investors in future periods. Therefore with an extra period to trade, the SP has to choose a tick size that takes into account both the undercutting effect and the queuing effect.

Figure 6 presents the extensive form of the 4-period trading game. Without loss of generality (Lemma 1), we consider the case of a buyer ($\beta_{t_1} > 1$) arriving at t_1 who posts a limit buy order (lb_{k,t_1}) in the empty book. The incoming 2^{nd} player has now three options. First, he can take liquidity by posting a market sell order (ms_{k,t_2}); second, he can supply liquidity on the other side of the market by submitting either a limit sell order at the next higher price (ls_{k+1,t_1}), or a limit sell order at p_{k+1+j} where $j \geq 1$ (ls_{k+1+j,t_2}); and third, he can supply liquidity on the same side of the market either by undercutting the existing limit buy order (lb_{k+j,t_1}) at $p_{k+j} \in \{p_{k+1},\ p_{+n^f}\}$; or - new compared to the 3-period model - by queuing behind that order at $p_{k-l} \in \{p_{-n^f}\ ,\ p_k\}$ ($l \geq 0$). The 3^{rd} player can react to the 2^{nd} player's actions by either supplying or taking liquidity on the buy or on the sell side of the market. If the 2^{nd} player

Figure 6: **Extensive Form of the Four Period Game**
This Figure shows the different sets of actions for the market participants in each period t_i of the trading game. The book opens empty and a buyer arrives at t_1, $j \geq 1$ and $l \geq 0$. A symmetric extensive form of the game holds if a seller arrives at t_1.



submits a limit sell order at p_{k+1} (ls_{k+1,t_1}), he locks the market and the only option available to the 3^{rd} player is taking liquidity: there are no price levels available between the best limit buy and the best limit sell order and the execution probability of a limit order queuing behind the existing limit orders is zero when there is only one period ahead before the end of the game.

If instead the 2^{nd} player does not lock the market, he faces three options. First, he can post a limit sell at p_{k+1+j}: in this case the 3^{rd} may either take liquidity on the buy (p_k) or on the sell (p_{k+1+j}) side of the market, or he can supply liquidity at any price between the best bid (p_k) and ask (p_{k+1+j}) posted by the 1^{st} and 2^{nd} player respectively. Second, he can undercut the existing limit order at p_{k+j}: in this case, the 3^{rd} player can market sell at p_{k+j} (ms_{k+j,t_3}), or, alternatively, he can either limit sell or limit buy at $p_{k+j+1+l}$ with $l \geq 0$. Third, the 2^{nd} player can queue behind the existing limit order at p_{k-l}: in this last case the 3^{rd} player can either market sell at p_k or supply liquidity at p_{k+j} as there is already a limit buy order at p_k posted by the 1^{st} player.

The 4^{th} and last player can only take liquidity with probability defined by Lemma 1.3. It is important to notice that the last player will be able to access liquidity on both sides of the market only if: either the 2^{nd} player supplies rather than take liquidity and does not lock the market; or the 2^{nd} player undercuts or queues behind the existing limit order and the 3^{rd} player supplies liquidity on the other side of the market.

4.1 Optimal Tick Size

In this section only - as for the 3-period model in Section 3.1 - we assume that an incoming trader can react to an existing limit buy order posted at p_k by submitting a limit buy or a limit sell order only at the next feasible price p_{k+1}. He can alternatively join the queue with a limit buy order at p_k, or he can market sell hitting p_k. Proposition (5) presents the equilibrium solution to the 4-period model:

Proposition 5. *For any $b\,\nu$, if $\beta_{t_1}\nu > \nu$, hence a buyer arrives at t_1, the equilibrium order submission probabilities are:*

For a generic τ defined by $\tau \in [(0,\ \tau^{max}) \mid n^f \geq 2]$:

t_1: $\cdot$ $\forall \tau \in [(0,\ \tau^{max}) \mid n^f \geq 2]\ \exists\ k < +n^f | Pr\left(lb_{k,t_1}|\Lambda_{t_0}, \tau\right) > 0 \quad and\ Pr\left(lb_{+n^f,t_1}|\Lambda_{t_0}, \tau\right) = 0$

$\cdot$ $\forall \tau \in \left\{(0,\ \tau^{max}) \mid n^f = 1\right\}$, $Pr\left(lb_{k,t_1}|\Lambda_{t_0}, \tau\right) > 0 \ \ \forall k \in \{-1,\ 1\}$ *and* $\lim_{\tau \to \tau^{max}} Pr\left(lb_{+1,t_1}|\Lambda_{t_0}, \tau\right) \to 0$

t_2: $\cdot$ $Pr\left(ms_{k,t_2}|\Lambda_{t_1}, \tau\right) = max\left[0,\ Pr\left((1-b) < \beta_{t_2} < \frac{p_k}{\nu} - \frac{\tau}{\nu}\frac{f}{1-f}\right)\right]$

$\cdot$ $Pr\left(ls_{k+1,t_2}|\Lambda_{t_1},\tau\right) = Pr\left(\frac{p_k}{\nu} - \frac{\tau}{\nu}\frac{f}{1-f} < \beta_{t_2} < \frac{p_k}{\nu} + \frac{\tau}{\nu}\frac{f}{f+l}\right)$ *if* $Pr\left(ms_{k,t_2}|\Lambda_{t_1},\tau\right) > 0$, *and* $Pr\left(ls_{k+1,t_2}|\Lambda_{t_1},\tau\right) = Pr\left((1-b) < \beta_{t_2} < \frac{p_k}{\nu} + \frac{\tau}{\nu}\frac{f}{f+l}\right)$, *otherwise*

$\cdot$ $Pr\left(lb_{k,t_2}|\Lambda_{t_1},\tau\right) = Pr\left(\frac{p_k}{\nu} + \frac{\tau}{\nu}\frac{f}{f+l} < \beta_{t_2} < \frac{p_k}{\nu} + \frac{\tau}{\nu}\frac{g}{g-l}\right)$ *if* $\frac{p_k}{\nu} + \frac{\tau}{\nu}\frac{g}{g-l} < 1+b$ *and* $Pr\left(lb_{k,t_2}|\Lambda_{t_1},\tau\right) = Pr\left(\frac{p_k}{\nu} + \frac{\tau}{\nu}\frac{f}{f+l} < \beta_{t_2} < 1+b\right)$, *otherwise*

$\cdot$ $Pr\left(lb_{k+1,t_2}|\Lambda_{t_1},\tau\right) = Pr\left(\frac{p_k}{\nu} + \frac{\tau}{\nu}\frac{g}{g-l} < \beta_{t_2} < 1+b\right)$ *if* $\frac{p_k}{\nu} + \frac{\tau}{\nu}\frac{g}{g-l} < 1+b$ *and* $Pr\left(lb_{k+1,t_2}|\Lambda_{t_1},\tau\right) = 0$, *otherwise*

t_3: $\cdot$ *Defined in Proposition* (1) *at* t_1, *if* $\Lambda_{t_2} = \{lb_{k,t_1}, ms_{k,t_2}\}$.

$\cdot$ *Defined in Lemma 1.3, if* $\Lambda_{t_2} = \{lb_{k,t_1}, ls_{k+1,t_2}\}$.

$\cdot$ *Defined in Proposition* (3) *at* t_2, *if* $\Lambda_{t_2} = \{lb_{k,t_1}, lb_{k+1,t_2}\}$ *or* $\Lambda_{t_2} = \{lb_{k,t_1}, lb_{k,t_2}\}$.

t_4: *Defined in Lemma 1.3 (Appedix B.2).*

Symmetric results apply if $\beta_{t_1}\nu < \nu$, *hence a seller arrives at* t_1.

Where $f = Pr\left(mb_{k+1,t_3}|\Lambda_{t_2},\tau\right) + \left(1 - Pr\left(mb_{k+1,t_3}|\Lambda_{t_2},\tau\right)\right) \times Pr\left(mb_{k+1,t_4}|\Lambda_{t_3},\tau\right)$,
$l = Pr\left(ms_{k,t_3}|\Lambda_{t_2},\tau\right) \times Pr\left(ms_{k,t_4}|\Lambda_{t_3},\tau\right)$,
$g = Pr\left(ms_{k+1,t_3}|\Lambda_{t_2},\tau\right) + \left(Pr\left(nt_{k+1,t_3}|\Lambda_{t_2},\tau\right) + Pr\left(ls_{k+2,t_3}|\Lambda_{t_2},\tau\right)\right) \times Pr\left(ms_{k+1,t_4}|\Lambda_{t_3},\tau\right)$.
The proof of Proposition 5 is in Appendix (E.1).

As for the 3-period protocol, with the exception of the extreme case in which the support-to-tick ratio is $\frac{\Gamma\nu}{\tau} < 3$ and the 1^{st} player can lock the market at p_{+n^f} which may happen with a negligible positive probability, in equilibrium the 1^{st} buyer (symmetrically the 1^{st} seller) submits a limit buy order at p_k, and the probability of the 2^{nd} player choosing any of his four different optional strategies depends on the value of p_k. The higher is p_k, the higher is the probability that the 2^{nd} player will either market sell $\left(Pr\left(ms_{k,t_2}|\Lambda_{t_1},\tau\right)\right)$ or queue behind the 1^{st} player's limit buy order $\left(Pr\left(lb_{k,t_2}|\Lambda_{t_1},\tau\right)\right)$. For lower values of p_k instead, the 2^{nd} player will either limit sell $\left(Pr\left(ls_{k+1,t_2}|\Lambda_{t_1},\tau\right)\right)$ on the other side of the market, or undercut the existing limit buy order $\left(Pr\left(lb_{k+1,t_2}|\Lambda_{t_1},\tau\right)\right)$ at p_{k+1} with increasing probability. Depending on the state of the book at the end of t_2, the 3^{rd} player will choose different orders. If the book at t_3 opens empty ($\Lambda_{t_2} = \{lb_{k,t_1}, ms_{k,t_2}\}$), the 3^{rd} player will limit buy (or symmetrically limit sell) with the same submission probability and order aggressiveness as the 1^{st} player in the 2-period trading game (Proposition (1)). If instead the book opens with both a limit buy and a limit sell order ($\Lambda_{t_2} = \{lb_{k,t_1}, ls_{k+1,t_2}\}$), he will take liquidity with either a market sell or a market buy as shown in Lemma 1.3. Finally, if the book opens with two limit buy orders ($\Lambda_{t_2} = \{lb_{k,t_1}, lb_{k+1,t_2}\}$ or $\Lambda_{t_2} = \{lb_{k,t_1}, lb_{k,t_2}\}$), the 3^{rd} player will act exactly as the 2^{nd} player in the 3-period trading game (Proposition (3)): he will either market sell hitting the best limit buy order, or he will undercut it submitting a more aggressive limit buy order, or alternatively he will offer liquidity at a higher p_k. The order submission strategies of the incoming investor at t_4 will depend on the state of the book at the end of t_3 and the unconditional order submission probabilities are defined by Lemma 1.3. As for the 3-period trading game, we show that in a 4-period model the OTS is different from 0, leading to Proposition 6 :

Proposition 6. *In a 4-period trading game, the tick size that maximizes the welfare of market participants is positive.*

Following the same line of reasoning of Appendix D.5, Proposition 6 is analytically proved in Appendix E.2, and confirms Proposition 4: when market participants face the fundamental trade-off between selecting limit and market orders in a LOB model and can endogenously decide to queue behind or undercut existing limit orders, the tick size is no longer a friction. In the next Section 4.2 we study the properties of the new OTS.

4.2 Optimal Tick Size, Welfare and Market Quality

In this section, we solve the OTS problem described in Appendix E.3 and E.4 in *quasi-closed* form without any restriction on the 2^{nd} and 3^{rd} player trading strategies and characterize the properties of the OTS in the 4-period trading game. We also solve in *quasi-closed* form the OTS problem for the 5-period trading game and use the results obtained to discuss how the OTS changes with the asset value, the investor's personal evaluation and the number of trading periods (proxing average number of trade).

As for the previous trading games, the OTS in our 4-period protocol is the tick size associated with the optimal total welfare of market participants. Figure 7 reports both the welfare of each market participant arriving in the 3-period and in the 4-period trading games, and the welfare of all market participants, $\Omega(\tau)$. This allows us to discuss how the new queuing trading strategy available to the 2^{nd} player (with the addition of a new trading period) affects the choice of the OTS. Figure 7 confirms that in a 4-period trading game the OTS is positive, although smaller that in the 3-period model.

We now provide an intuition and some additional results which explain why in a 4-period trading game the OTS is smaller than in a 3-period game but has to be positive to optimally balance the interaction between liquidity demand and liquidity supply and therefore to maximize total welfare. Table 2 reports - for $b = 0.06$ and $\nu = 10$ - the equilibrium order submission strategies for both the 3-period (Panel A) and the 4-period trading game (Panel C), associated with their respective OTS.[20] Table 2 also reports the order submission strategies for the 4-period trading game associated with the 3-period OTS (Panel B). Panel D reports the unconditional order submission strategies of the 3^{rd} player in the 4-period trading game.[21] Starting from the 3-period game, the equilibrium strategies show that the 1^{st} player either aggressively limit buys at $p_k = 10.139$ above the fundamental asset value; or he limit buys at the best bid $p_k = 9.861$ below the asset value. When he buys aggressively ($Pr\,(lb_{k,t_1}|\Lambda_{t_0}, \tau) = 0.111$), the 2^{nd} player mainly market sells ($Pr\,(ms_{k,t_2}|\Lambda_{t_1}, \tau) = 0.574$) as opposed to limit sell ($Pr\,(ls_{>k,t_2}|\Lambda_{t_1}, \tau) = 0.274$) or undercutting ($Pr\,(lb_{>k,t_2}|\Lambda_{t_1}, \tau) = 0.152$), whereas when he buys more patiently ($Pr\,(lb_{k,t_1}|\Lambda_{t_0}, \tau) = 0.389$), the 2^{nd} player mainly provides liquidity either limit selling ($Pr\,(ls_{>k,t_2}|\Lambda_{t_1}, \tau) = 0.423$) or undercutting the existing limit buy order ($Pr\,(lb_{>k,t_2}|\Lambda_{t_1}, \tau) = 0.338$). Therefore the 2^{nd} player has more incentive to supply liquidity when the 1^{st} player is more patient.

Adding a fourth period - holding the OTS of the 3-period game (OTS 3P) constant - allows us to focus on the effects that queuing may have on the order submission strategies of market participants (Panel B). When the 2^{nd} player is allowed to queue behind the 1^{st} player's limit buy order, he does so substantially ($Pr\,(lb_{\leq k,t_2}|\Lambda_{t_1}, \tau) = 0.339$) when the 1^{st} player submits an aggressive limit order above the fundamental value of the asset ($Pr\,(lb_{k,t_1}|\Lambda_{t_0}, \tau) = 0.129$).

[20] For any chosen support relative to the associated OTS, the order submission strategies of the trading game are unchanged.

[21] To economize space we report the unconditional order strategies for the 3^{rd} player in the 4-period trading game.

In this case, the 2^{nd} player substitutes his liquidity provision on the buy side (where he was undercutting) or on the sell side (where he was limit selling), with queuing. Hence, overall, with an extra trading period and the OTS 3P, the 2^{nd} player offers worse liquidity to the incoming players. However, if we do not restrict the new 4-period model to the OTS 3P but solve the 4-period problem with the OTS 4P (0.214) - smaller than the OTS 3P (0.278) - results change (Table 2, Panel C). The main effect of the reduction in the OTS is to induce the 2^{nd} player to switch from queuing to undercutting as now undercutting is cheaper. This leads to our empirical prediction.

Empirical Prediction 1.

Undercutting (Queuing) is a negative (positive) function of the tick size.

The other effect of a reduction in tick size is to increase the 2^{nd} player's liquidity provision on the sell side ($Pr\left(ls_{>k,t_2}|\Lambda_{t_1},\tau\right)$) from 0.119 to 0.178, as the new finer price grid offers him more price levels where to optimally post his aggressive limit sell order rather than market sell. Therefore, the main effect of switching from OTS 3P to the smaller 4P is that it induces investors to offer better liquidity to the incoming liquidity takers. The same intuition holds when comparing the equilibrium order submission strategies of the 3^{rd} player in the 4-period OTS 3P game with the OTS 4P game, reported in Panel D: the 3^{rd} player substitutes market order with more undercutting on the buy side and reduces no trade. This means that moving from the 3-period to the 4-period trading game, the SP reduces the OTS to optimize the demand and the supply of liquidity which in turn leads to maximize total welfare of market participants.[22]

This discussion also provides an intuition for the dynamic pattern of the welfare presented in Figure 7. Specifically, Figure 7 (and Table 2, Panel E) indicates that in correspondence of the new smaller OTS 4P, the 1^{st} and 2^{nd} players are worse off, whereas that the 3^{rd} and 4^{th} players are better off. Therefore when setting the OTS the SP has to mediate the interests of different market participants. This analysis highlights two important findings. First, the OTS

[22]Replicating the comparative static analysis for the 4-period and the 5-period trading game, we obtain analogous results. Table 3.E in Appendix E compares the equilibrium order submission strategies for the 5-period model solved for the OTS of both the 4-period and the 5-period trading game and shows that the SP adjusts the 5-period OTS to induce investors arriving later in the trading game to switch from queuing to price improving liquidity provision. Table 2 confirms that moving from the 4 to the 5-period trading game the SP sets the OTS to optimally manage the liquidity of the book, and in turn to maximize the welfare of market participants.

Table 2: Comparative Analysis of Equilibrium Submission Probabilities and Welfare
Panel A, B and C summarize the submission strategies of the first two players in 3 and 4-period games. The first column report prices for which the 1^{st} player attaches a positive equilibrium submission probability $Pr(lb_{k,t_1}|\Lambda_{t_0},\tau)$. The columns $3-6$ of Panel A, B and C report the probabilities of market selling at t_2 ($Pr(ms_{k,t_2}|\Lambda_{t_1},\tau)$), limit sell ($Pr(ls_{>k,t_2}|\Lambda_{t_1},\tau)$), queuing ($Pr(lb_{\leq k,t_2}|\Lambda_{t_1},\tau)$) and undercutting ($Pr(lb_{>k,t_2}|\Lambda_{t_1},\tau)$). Panel D reports the equilibrium unconditional submission probabilities of the 3^{rd} player for the 4-period model solved for the OTS of both the 3-period (2^{nd} row) and 4-period trading game (3^{rd} row). We report the unconditional probability of market sell at t_3 ($Pr(ms_{t_3})$) (column 2), of limit sell (undercutting) ($Pr(ls_{>k,t_3})$) (column 3), of limit sell (queuing) (column 4) ($Pr(ls_{\leq k,t_3})$), of no trade ($Pr(nt_{k,t_3})$), of limit buy (queuing) ($Pr(lb_{\leq k,t_3})$), of limit, buy (undercutting) ($Pr(lb_{>k,t_3})$) and of market buy ($Pr(mb_{k,t_3})$). Panel E summarizes the welfare of each player ($\omega_{t_i}(\cdot)$), the total welfare (Ω), and the measures of market quality (Expected Volume, Quoted Spread, and Total Depth) for OTS $3P$ in the 3 and 4-period game, for OTS $4P$ in the 4 and 5-period game and for OTS $5P$ in the 5-period game. Results are reported for the baseline example ($b=0.06$ and $\nu=10$).

Panel A: 3-period game - 1^{st} and 2^{nd} player conditional order submission strategies with OTS $3P$ (0.278)

Price p_k	Limit Buy $t1$ $Pr(lb_{k,t_1}\|\Lambda_{t_0},\tau)$	Market Sell t_2 $Pr(ms_{k,t_2}\|\Lambda_{t_1},\tau)$	Limit Sell t_2 $Pr(ls_{>k,t_2}\|\Lambda_{t_1},\tau)$	Queuing t_2 $Pr(lb_{\leq k,t_2}\|\Lambda_{t_1},\tau)$	Undercutting t_2 $Pr(lb_{>k,t_2}\|\Lambda_{t_1},\tau)$
10.139	0.111	0.574	0.274	0.000	0.152
9.861	0.389	0.239	0.423	0.000	0.338

Panel B: 4-period game 1^{st} and 2^{nd} player conditional order submission strategies with OTS $3P$ (0.278)

Price p_k	Limit Buy $t1$ $Pr(lb_{k,t_1}\|\Lambda_{t_0},\tau)$	Market Sell t_2 $Pr(ms_{k,t_2}\|\Lambda_{t_1},\tau)$	Limit Sell t_2 $Pr(ls_{>k,t_2}\|\Lambda_{t_1},\tau)$	Queuing t_2 $Pr(lb_{\leq k,t_2}\|\Lambda_{t_1},\tau)$	Undercutting t_2 $Pr(lb_{>k,t_2}\|\Lambda_{t_1},\tau)$
10.139	0.129	0.525	0.119	0.339	0.017
9.861	0.371	0.005	0.615	0.029	0.351

Panel C: 4-period game 1^{st} and 2^{nd} player conditional order submission strategies with OTS $4P$ (0.214)

Price p_k	Limit Buy $t1$ $Pr(lb_{k,t_1}\|\Lambda_{t_0},\tau)$	Market Sell t_2 $Pr(ms_{k,t_2}\|\Lambda_{t_1},\tau)$	Limit Sell t_2 $Pr(ls_{>k,t_2}\|\Lambda_{t_1},\tau)$	Queuing t_2 $Pr(lb_{\leq k,t_2}\|\Lambda_{t_1},\tau)$	Undercutting t_2 $Pr(lb_{>k,t_2}\|\Lambda_{t_1},\tau)$
10.107	0.146	0.465	0.178	0.239	0.118
9.893	0.342	0.075	0.536	0.016	0.372
9.679	0.012	0.000	0.498	0.000	0.502

Panel D: 4-period game - 3^{rd} player unconditional order submission strategies

	Market Sell $Pr(ms_{k,t_3})$	Undercutting $Pr(ls_{>k,t_3})$	Queuing $Pr(ls_{\leq k,t_3})$	No Trade $Pr(nt_{k,t_3})$	Queuing $Pr(lb_{\leq k,t_3})$	Undercutting $Pr(lb_{>k,t_3})$	Market Buy $Pr(mb_{k,t_3})$
OTS 3P (0.278)	0.199	0.058	0.000	0.052	0.000	0.071	0.083
OTS 4P (0.214)	0.183	0.056	0.000	0.033	0.000	0.100	0.081

Panel E: Welfare & Market Metrics

Game	$\omega_{t_1}(\cdot)$	$\omega_{t_2}(\cdot)$	$\omega_{t_3}(\cdot)$	$\omega_{t_4}(\cdot)$	$\omega_{t_5}(\cdot)$	Ω	*vol*	*quoted spread*	*depth*
3-period game & OTS $3P$	0.091	0.092	0.068	0.000	0.000	0.251	0.384	0.333	1.186
4-period game & OTS $3P$	0.106	0.113	0.095	0.064	0.000	0.378	0.551	0.285	2.069
4-period game & OTS $4P$	0.105	0.112	0.099	0.067	0.000	0.383	0.546	0.281	2.018
5-period game & OTS $4P$	0.097	0.125	0.102	0.091	0.061	0.476	0.678	0.263	2.755
5-period game & OTS $5P$	0.097	0.123	0.104	0.096	0.060	0.479	0.675	0.261	2.774

cannot be zero in a 4-period game, confirming our 3-period model results. Second, the SP faces a trade-off when setting the OTS for a game with more trading periods and therefore with more trading opportunities: by reducing the OTS it reduces the queues at the top of the book and enhances price-improving liquidity provision thus making future players better off. However, a smaller tick size that increases the probability of undercutting also harms the 1^{st} and 2^{nd} period liquidity providers. This result is reminiscent of Goettler et al. (2005) who show that a smaller tick size (from $\frac{1}{8}$ to $\frac{1}{16}$ of a \$) is not Pareto improving.

Replicating the analysis of the 3 period model (Section 3.3) for the 4 and 5-period frameworks, our results confirm that the OTS is a function of both the asset value and the population active in the market, and also show that the OTS is a function of the number of trading periods. Table 2 (Panel E) shows that - all else equal - when an extra period is added to the trading game, - either moving from 3 to 4 periods or moving from 4 to 5 periods, the OTS decreases and market quality measured by volume, spread and depth improves. We can therefore conclude that the SP sets an OTS which is decreasing in the liquidity of the instrument. This result leads to the following corollary:

Corollary 3. *When the number of trading periods increases, the liquidity of the instrument - proxied by expected volume, expected quoted spread and expected total depth - improves, and the OTS set by the SP decreases.*

If we consider the number of trading periods - or the expected volume in the 3, 4 and 5-period trading game - as a proxy for the average number of trades, our results are consistent with the ESMA tick size table introduced in 2018, suggesting exchanges to set the tick size as a decreasing function of the liquidity of the instrument. Table 1.A shows that in Europe - as well in the UK, Japan, Hong Kong and Switzerland - the tick size is a function of both the stock price and the liquidity of the instrument. Consistently, in the cryptocurrency markets the tick size is set by the owner of the trading platform conditional on both the price and the liquidity of the instrument (Foley et al. (2022)).

The reason why we further extend the model to include a fifth period, is that - as we discussed - adding a fourth period not only makes the market more liquid but it also introduces the new queuing transmission channel. Therefore, to isolate the effects of an increase in the liquidity of the instrument (increase in the number of trading periods) on the equilibrium order submission strategies of market participants, and to control for the effects of the introduction of the new queuing channel, we add an extra fifth period.

Table 3 reports the OTS and the market quality results obtained from both the 4-period (columns 2-4) and the 5-period (columns 5-7) trading game. Table 3 also shows the equilibrium OTS for increasing values of both the investors evaluation support (b), and the asset value (ν). Table 3 confirms that for any given number of trading periods, an increase in b and/or ν leads to a larger tick size.[23] Holding instead b and ν constant and moving from 4 to 5 periods, the OTS set by the SP further decreases. As the number of trading periods increases, the SP still sets the OTS to balance liquidity demand and liquidity supply, and the transmission channels that drive this optimization process depends - as for the previous model with fewer number of periods - on the trade-off between queuing and undercutting that investors face in each trading period. Early liquidity suppliers benefit from a larger tick size that disincentives future undercutting. More specifically, when the number of periods before the end on the game is sufficiently large - hence the probability of execution is high - a larger tick size allows investors to queue behind existing orders rather then undercutting them, thus enjoying a larger price improvement. When instead the end of the game approaches, investors generally become more aggressive and willing to undercut existing orders, and therefore they may benefit from a smaller tick size. Hence, when setting the OTS to maximize the interaction between liquidity supply and liquidity demand, the SP needs to trade-off the incentives of investors to either queue or undercut existing orders. If the SP sets the OTS suboptimally, total investors' welfare will not be maximized and liquidity provision would be suboptimal. If the tick size is larger than the OTS, liquidity suppliers may benefit but investors may excessively opt for queuing with the result that liquidity provision may deteriorate. If instead the tick is smaller than the OTS, then liquidity takers may benefit but investors may excessively opt for undercutting potentially worsening liquidity provision.

So far we have considered the OTS set by a SP that maximizes the total welfare of market

[23]Note that, as in the 3-period trading game, when the overall evaluation support ($2b\nu$) remains constant, both OTS and total welfare and market quality remain unchanged (as indicated in the grey shaded columns).

Table 3: Optimal Tick Size and Market Quality for the Four and Five Period Games
The table reports the OTS and the associated total welfare (Ω), expected volume (*vol*), quoted spread (*quoted spread*) and total depth (*depth*) for each combination of $\nu = \{8, 10, 13.333\}$ and $b = \{0.045, 0.06, 0.075\}$ for the 4-period (column 3-5) and 5-period (column 6-8) respectively. $b = 0.06$ is estimated following Goettler et al. (2005) and Hollifield et al. (2006) and the two other b values are $b = 0.06(1 \pm 0.25\%)$. $\nu = 10$ is our benchmark asset value and $\nu = 8$, $\nu = 13.333$ are obtain such that $2b\nu$ is constant across the three pairs of values (b, ν). The discretization grid used to derive the *quasi-closed* form solution for each (ν, b) is defined in Appendix D.6: for each (ν, b), we consider a set of tick sizes consistent with price grids that include between 2 to 30 feasible prices. The results are rounded at the 3^{rd} decimal digit.

		4-period game			5-period game		
	ν \ b	0.045	0.06	0.075	0.045	0.06	0.075
OTS		0.128	0.171	0.214	0.096	0.128	0.160
Ω		0.230	0.306	0.383	0.287	0.383	0.479
vol	8.000	0.546	0.546	0.546	0.675	0.675	0.675
quoted spread		0.169	0.225	0.281	0.157	0.209	0.261
depth		2.018	2.018	2.018	2.774	2.774	2.774
OTS		0.161	0.214	0.268	0.120	0.160	0.199
Ω		0.287	0.383	0.479	0.359	0.479	0.598
vol	10.000	0.546	0.546	0.546	0.675	0.675	0.675
quoted spread		0.211	0.281	0.352	0.196	0.261	0.327
depth		2.018	2.018	2.018	2.774	2.774	2.774
OTS		0.214	0.285	0.357	0.160	0.213	0.266
Ω		0.383	0.510	0.638	0.479	0.638	0.798
vol	13.333	0.546	0.546	0.546	0.675	0.675	0.675
quoted spread		0.281	0.375	0.469	0.261	0.348	0.436
depth		2.018	2.018	2.018	2.774	2.774	2.774

participants. In real market however, this measure is difficult to quantify, hence we investigate which metric of market quality could be used as a second best tool to set the tick size. Table 4 shows the tick size (OT) that optimizes each of our three metrics of market quality - volume, quoted spread and total depth - for the 3, 4 and 5-period trading game respectively. It also shows the welfare loss computed as the percentage difference between the total welfare associated with the OTS ($\Omega(OTS)$) and the total welfare associated with the new OT ($\Omega(OT)$). Table 4 shows that the market metric that better proxies total welfare is quoted spread as it minimizes the average welfare loss. This result leads to our last Corollary :[24]

Corollary 4. *When the SP cannot observe investors' welfare, he minimizes the welfare loss by using spread to set the tick set. Therefore, a change in the tick size that leads to a reduction in the spread increases total welfare.*

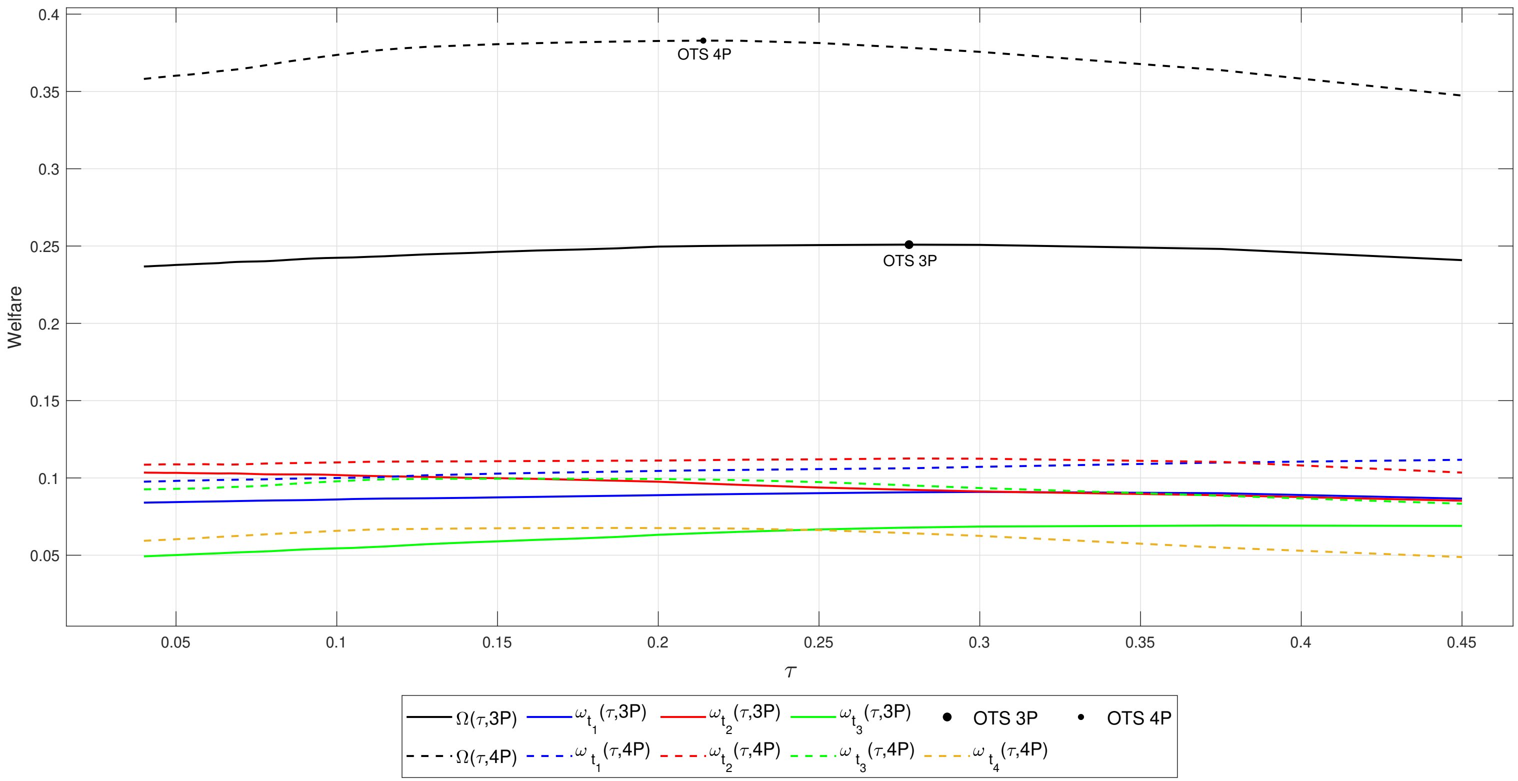
Intuitively, if the SP minimizes spread, he takes into account the optimal liquidity provision on both the ask and the bid side of the market, whereas if he maximizes volume or depth this is not necessarily the case. This result has important policy implications as it instructs regulators on the choice of the empirical metric to use when setting the tick size while taking into account both the price and the liquidity of each instrument. Our results support the "Intelligent Ticks" Nasdaq (2019) proposal setting the tick size as a function of the weighted average quoted spread of the instrument. They also partially support the recent 34-96494 SEC (2022) proposal to set the tick size as a function of the quoted spread for stocks characterized by an average spread smaller than \$0.04. However, the SEC (2022) proposal fails to address the issue of high-priced stocks, which we discuss in Session 6. Corollary 4 has also important empirical implications as it instructs researchers on how to evaluate a change in the tick size using the most popular and standard market quality metric. If a change in tick size leads to a spread improvement, then it can be considered a welfare improvement.

[24] We thank Bjorn Hagströmer for his comments on this point.

Table 4: Tick Size Optimizer for Market Qualities
This table reports the Welfare Loss and the Optimizing Tick (OT) chosen by the SP in case he sets the tick size in the 3, 4, and 5-period games by either maximizing volum or minimizing quoted spread, or maximizing depth. The Welfare Loss is measured in each period of the game as the percentage difference between the total welfare associated with OTS ($\Omega(OTS)$) and the total welfare associated with each OT ($\Omega(OT)$). The last column reports the average Welfare Loss for each market metric across the 3P, 4P and 5P trading game. We report the results for the baseline parameterization ($b = 0.06$ and $\nu = 10$). The discretization grid used to derive the *quasi-closed* form solution is defined in Appendix D.6: we consider a set of tick sizes consistent with price grids that include between 2 to 30 feasible prices. We report the OT which defines at least 2 prices on each side of the book; results do not change qualitatively by relaxing this assumption. The results are rounded at the 3^{rd} decimal digit.

		3P	4P	5P	Avg. Welfare Loss
Volume	Welfare Loss	-0.509%	-0.403%	-4.562%	-1.825%
	OT	0.2	0.25	0.375	
Quoted spread	Welfare Loss	-1.110%	-0.014%	-0.092%	-0.405%
	OT	0.375	0.225	0.1425	
Depth	Welfare Loss	-1.110%	-4.988%	-4.562%	-3.553%
	OT	0.375	0.375	0.375	

Figure 7: **Welfare Comparison of the Three and Four Period Games**
This figure shows the welfare of market participants in a 3-period and 4-period game protocols. The welfare of the 1^{st} player in a 3-period game is ($\omega_{t_1}(\tau, 3P)$, blue line), while for a 4-period game is ($\omega_{t_1}(\tau, 4P)$, blue dashed line). The welfare of the 2^{nd} player in a 3-period game is ($\omega_{t_2}(\tau, 3P)$, red line), while for a 4-period game is ($\omega_{t_2}(\tau, 4P)$, red dashed line). The welfare of the 3^{rd} player in a 3-period game is ($\omega_{t_3}(\tau, 3P)$, green line), while for a 4-period game is ($\omega_{t_3}(\tau, 4P)$, green dashed line). The welfare of the 4^{th} player in a 4-period game is ($\omega_{t_4}(\tau, 4P)$, yellow dashed line). The total welfare of market participants in a 3-period game is ($\Omega(\tau, 3P)$, black line) while for a 4-period game is ($\Omega(\tau, 4P)$, black dashed line). The OTS of a 3-period game is marked as ($OTS3P$, black dot), while for a 4-period game is ($OTS4P$, black dot). The results are presented for $b = 0.06$, $\nu = 10$ and for a set of tick sizes, defined in Appendix (D.6), that considers price grid between 2 and 30 prices. Results do not change qualitatively considering more prices.



5 Robustness

5.1 Stationary vs. non-stationary equilibrium

To determine the OTS, we need to rely on a model that embeds a number of crucial features. First of all, we need both queuing and undercutting to be fully endogenous. To accomplish this, we need discrete prices and traders to be able to post orders that queue behind the existing ones. The model of Roşu (2009) and Bhattacharya and Saar (2021) have continuous prices and the equilibrium steady state solution of Foucault et al. (2005) model implies that traders do not queue behind existing orders. Therefore, we cannot use these frameworks to determine the OTS.

In Goettler et al. (2005), investors can actually queue behind other existing limit orders but their numerical steady state solution of the probability of limit order executions requires that the initial condition on the execution probability of each limit order is sequentially updated as a weighted average of its past values with the weights depending on the frequency of execution and cancellation associated with each state of the book. More specifically, in the spirit of Pakes and McGuire (2001), the execution probability of a limit buy order posted at p_i depends on the exogenous probability of cancellation and on the net change of consensus value, as well as on the probability that a trader who obtains a positive surplus from selling at p_i will arrive, $F_\beta(p_i)$. Without cancellation this probability is equal to one and does not change over time. With a positive probability of cancellation instead the estimated probability of execution delivers conditional frequencies of buy and sell orders that generate a realistic distribution of the order book depth.[25] To determine the OTS, we need that the probability of limit orders execution is the result of a fully endogenous strategic trading game in such a way that the length of the queues affects the investors' strategic choice between market and limit orders. In practice, a fully

[25]In Goettler et al. (2005), traders' private valuation, β_t, is drawn from a continuous distribution, F_{β_t}, with support B. Initial execution probability and sequential update are respectively given by:

$$\begin{aligned} \mu_1^e(\cdot, i, \cdot, \cdot) &= \frac{(1-\underline{\delta})F_B(p^i)}{1-(1-\underline{\delta})(1-F_B(p^i))} \\ \mu_{t+1}^e(k, i, L_\tau, X_\tau) &= \frac{n}{n+1}\mu_t^e(k, i, L_\tau, X_\tau) + \frac{1}{n+1} \end{aligned} \tag{14}$$

Cancellation obtains either when an order is cancelled - which may happen with probability $\delta_t(\cdot)$ (Step 4 page 2159), or when some outside price levels with posted limit orders are cancelled following a jump in the asset value (Step 5).

endogenous limit order execution probability allows traders to strategically decide whether to queue behind existing limit orders or undercut them, thus crucially affecting the aggressiveness of their dynamic order submission strategies. To build a protocol with fully endogenous and strategic limit order submission probabilities, we need to compute the strategic optimal order submission decision of each arriving trader conditional on each possible state of the book. This is analytically very complex as the state space of the trading game drastically increases with the number of investors arriving at the market (Foucault et al. (2005)). Yet, our approach, despite relying on a limited number of arriving investors/periods, allows us to identify all of the possible transmission channels that affect the SP choice of the OTS.

In Section 4 we show that by adding an extra period to our 3-period and 4-period trading game the OTS decreases. Would the OTS converge to zero by increasing further the number of arriving investors? Our model's results suggest that the OTS should not converge to zero and the intuition is the following.

Our model shows that by increasing the number of arriving investors, competition for the provision of liquidity increases and in equilibrium the 1^{st} player submits a limit order aggressively over the fundamental asset value with increasing probability (Figure 8). This effect, coupled with a tick size that is increasingly smaller, further enhances aggressive liquidity provision (undercutting) as incoming investors lose the incentive to queue behind existing limit orders. As the number of arriving investors increases further, if the tick size were set next to zero, the 1^{st} player would react by posting a limit buy order at increasingly higher prices above the fundamental asset value to ensure an almost immediate execution - otherwise the extremely cheap undercutting would crowd his order out of the market. Under this scenario, the 2^{nd} player would have an incentive to market sell as opposed to limit buy or sell, with the result that when the next 3^{rd} player arrives, the game would start again with him supplying new liquidity aggressively above the asset value, followed by a 4^{th} investor aggressively market selling, so that the game would restart again with the 6^{th} player supplying liquidity aggressively and so on. Therefore, the dynamics of the trading game could move away from the fundamental asset value, with limit orders only posted by investors with large gains from trade, and patient liquidity providers being crowded out of the market. This outcome with a tick size next to zero would limit the provision of liquidity and we conjecture that it would not be the candidate protocol/tick size that maximizes the welfare of all market participants.

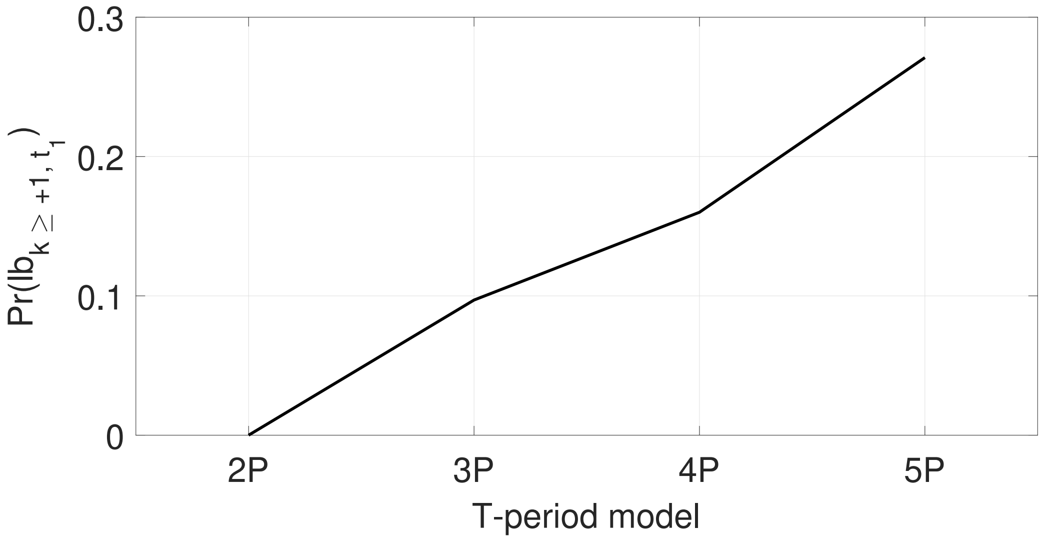
In Appendix F we present a stylized infinite-period trading game with truthful reporting to formally corroborate this intuition.[26] A zero tick size induces any of two sequentially arriving investors with different personal evaluations (e.g., consider, for the benchmark parametrization $\nu = 10$ and $b = 0.06$, $\beta_1\nu = 9.9$ and $\beta_2\nu = 10.1$) to trade immediately at any price within their evaluations, and the total gains from trade is $\beta_2\nu - \beta_1\nu = 0.2$. A non zero tick size instead does not necessarily allow for immediate execution as investors can only execute at the predetermined prices on the grid (e.g., consider for the benchmark parametrization $\nu = 10$ and $b = 0.06$, a tick size $\tau = b\nu = 0.6$ with $p_{-1} = 9.7$ and $p_{+1} = 10.3$). Therefore, the two investors from the previous example would post a limit sell order at p_{+1} and a limit buy order at p_{-1}, respectively. Since with an infinite number of arriving investors both limit orders would certainly execute, they would be matched against traders with a personal evaluation between $[(1-b)\nu = 9.4,\ p_{-1} = 9.7)$ and $(p_{+1} = 10.3,\ (1+b)\nu = 10.6]$ respectively, and consequently the expected total gains from trade (0.55) would be higher than under a zero tick size regime.

5.2 Asymmetric Information

In our model there is no asymmetric information. The existing literature shows that adding informed investors in a model of limit order book induces market participants to trade more aggressively in order to exploit their increased gains from trade. This is true both in the standard models a' la Kyle and Glosten and Milgrom (e.g., Harris (1998), Kaniel and Liu (2006) and Glosten (1994)), and in the most recent models of limit order books (e.g., Bhattacharya and Saar (2021), Riccó et al. (2022)). In both the 3-period and the 4-period trading games we have shown that when traders become more aggressive due to an exogenous increase in the support of their personal evaluation, the gains from trade increase and investors are more inclined to opt for more aggressive limit orders (undercutting) or market orders. As shown in Tables 1 and 3, the increased aggressiveness induces the SP to set a wider tick size that restores the equilibrium

[26] We thank Jean Edouard Colliard for suggesting this extension.

Figure 8: 1^{st} **Player Aggressive Order Submission Strategy**
This figure shows the 1^{st} player's equilibrium order submission probability for a limit buy order posted at a price above the fundamental value, $Pr(lb_{k\geq+1,t_1}) = \sum_{k=+1}^{+n^f} Pr\left(lb_{k,t_1}|\Lambda_{t_0},\tau\right)$, for the different trading models considered: 2-period (2P), 3-period (3P), 4-period (4P) and 5-period (5P). We report $Pr(lb_{k\geq+1,t_1})$ by solving each model for the baseline parameterization ($b = 0.06$ and $\nu = 10$) and for the 5-period OTS ($\tau = 0.160$). Results do not change qualitatively if we consider different tick sizes.



liquidity supply and demand. We therefore envisage no transmission channel showing that adding asymmetric information to our protocol would weaken our main results. In contrast, we speculate that asymmetric information would induce the SP to set a wider tick size thus strengthening our main results.

5.3 Cancellations and Resubmissions

In our model we do not allow traders to cancel their orders. Embedding this feature in a limit order book model with fully endogenous strategic choice of order submission strategies is extremely complicated.

This important feature of real financial markets trading strategies is still not fully embedded in models of limit order book as most models do not explicitly allow for cancellation and resubmission (e.g., Foucault et al. (2005), Riccó et al. (2021)). As discussed in Foucault et al. (2005), one possible approach followed by Hollifield et al. (2006) and Goettler et al. (2005) assumes that cancellation occurs exogenously at random points in time. Another more sophisticated approach followed by Roşu (2009) and Bhattacharya and Saar (2021) in continuous time stationary models without a tick size, is to assume that cancellation and resubmission is costless and instantaneous. This mechanism is the necessary tool - the Nash threat - to find a stationary equilibrium and avoid an infinite sequence of infinitesimal undercutting among market participants. With this mechanism in place, in equilibrium investors have no incentive to cancel and resubmit their orders, and therefore there is no effective cancellation and resubmission.

Allowing investors - in equilibrium - to cancel and resubmit their orders is an important enhancement of any limit order book model as in real markets most sophisticated traders actually cancel and resubmit orders (e.g. Hasbrouck and Saar (2013), Aquilina, Budish, and O'neill (2022), Biais, Foucault, et al. (2014)). There are two reasons why traders would strategically choose to cancel and resubmit their orders. First, traders may wish to cancel and resubmit their orders to avoid sniping in case of an unexpected jump in the fundamental value of the asset (Budish et al. (2015)). Second, traders may decide to cancel their orders to strategically react to previously posted limit orders. As in our model the fundamental asset value does not change, the only reason why players would cancel and resubmit their orders would be the latter. If we allowed investors to cancel and resubmit their orders starting from the 3-period trading game, in equilibrium traders would more frequently either post market orders or undercut existing limit orders. As discussed in Section 3, the SP would then set a wider OTS to incentivize liquidity provision and compensate the increased aggressiveness of liquidity demanders. As a result, we speculate that with cancellation and resubmission the wider tick size would enhance the welfare of market participants and our results would be qualitatively stronger. Foley et al. (2022) show that in the crypto currency market an increase in the tick size reduces undercutting, increases liquidity provision and quoted depth, and reduces transactions costs for all market participants. AMF (2018) also shows that a wider tick increases order lifetime, and therefore reduces both undercutting and the number of unexecuted orders, quantified by the order-to-trade ratio.

5.4 Random Investor Arrival

In our baseline model investors arrive at each period t_i with certainty. This means that over a T-period game T investors arrive with certainty. Here we show that our results hold if we assume that investors not necessarily arrive with certainty.[27] As an example, we consider a 3-period model in which the 2^{nd} player comes to the market with probability $q \in (0,1)$, whereas the 1^{st} and the last player arrive with certainty. Differently from the baseline model considered in Section 3, the 1^{st} player no longer faces competition in liquidity supply from the 2^{nd} player with certainty, but with a probability equal to q. In Table 5 we report our results showing that higher values of q are associated with higher values of OTS and total welfare $\Omega(OTS)$.

Table 5: OTS and Welfare in the Three Period Game with Random Investor Arrival
This table reports the OTS and the associated total welfare ($\Omega(OTS)$) for a 3-period model with random arrival probability q of the 2^{nd} player. We report the results for the baseline parameterization ($b = 0.06$ and $\nu = 10$) and for the following set of arrival rate $q = \{0.1, 0.25, 0.5, 0.75, 0.9\}$. The *quasi-closed* form solution of the OTS problem for each triplet (b, ν, q) follows Appendix D.6. The results are rounded at the 3^{rd} decimal digit.

	q				
	0.1	0.25	0.5	0.75	0.9
OTS	0.095	0.135	0.260	0.262	0.275
$\Omega(OTS)$	0.143	0.161	0.193	0.222	0.239

As the potential undercutting faced by the 1^{st} player increases with q, to incentivize liquidity provision on both side of the market the SP sets a wider tick size. Given that a higher q implies a higher probability to have three rather than two investors active in the market, also total welfare increases monotonically with q. It is worth noticing that even with small values of q - e.g., $q = 0.1$ - the SP sets a positive OTS. Irrespective of the value of q, when liquidity suppliers have to take into account that their order might be undercut in future periods, the SP sets a positive OTS. This confirms our results in Sections 3 and 4 where we show that the tick size is not a friction.

6 Empirical Analysis

In this section, we test our empirical prediction using U.S. data and we investigate the effects of the introduction of the new MiFID II tick size regime using European data. For the U.S. markets, we obtained from the Nasdaq's Economic Research Team, market quality data for the firms listed in the U.S. markets (3988) by 1 January 2021, during the period 1 January - 30 June

[27]We thank Stefano Lovo for suggesting this extension.

2022.[28] For the European markets, we downloaded from Refinitiv DataScope minute by minute bid and ask prices, transaction prices, volume, and number of trades for the stocks included in the main indexes of the following countries: UK, France, Germany and The Netherlands. We also downloaded for the last hour of trading Level II Refinitiv Data including the best 10 levels of the book on each side of the market. Our sample spans from 1 January 2017 to 31 December 2018 and builds around 1 January 2018 when MiFID II introduced a new tick size regime aimed at harmonizing the tick size among all of the European trading platforms.

6.1 Empirical Prediction 1

To test our Empirical Prediction 1 we need a proxy for both undercutting and queuing.[29] We proxy undercutting by the percentage use of Odd Lot Trade and Odd Lot Volume. Odd Lot Trade (Volume) (%) is the daily number of odd lots (daily number of odd lots in number of shares) over the daily average number of trades (average daily volume). This metric proxies undercutting for the following reason. When an investor wishes to supply liquidity aggressively, he posts limit orders at higher bid or lower ask prices. However, when the price of a stock is very high and trading takes place in lots of 100 shares as in the U.S. markets, liquidity suppliers may find it cheaper to outbid current best prices by trading in odd-lots, which is equivalent to undercut current best prices without paying the entire lot. Hence, the percentage use of Odd Lot Trades proxies traders' willingness to undercut existing best bids and offers. We proxy queuing by the following two metrics:

$$\begin{aligned} Queue\,(min) &= \frac{Size\ at\ NBBO}{ADV} \times \frac{23400}{60} \\ Inverted\,Share\,(\%) &= \frac{Volume\ at\ Inverted\ Venues}{U.S.\ Stocks\ Consolidated\ Volume} \end{aligned} \tag{15}$$

Where *ADV* is the Average Daily Volume, and *Size at NBBO* is the number of shares available at the NBBO. $Queue\,(min)$ measures the average time (expressed in minutes) spent in the queue

[28] To avoid dealing with penny stocks moving through the $1 tick size threshold, we follow the standard practice and remove from the initial sample stocks with an average price smaller than $3.

[29] For the U.S. markets, we obtained from the Nasdaq's Economic Research Team the proxies we use for undercutting and queuing.

at the NBBO by an order of average size.[30] The intuition behind using *Inverted Share* as a proxy for queuing is that when queues at the NBBO become longer, traders may have an incentive to move their liquidity supply to the inverted fee platforms where queues are shorter as, due to the rebate on the take fee, liquidity demanders find it cheaper to take liquidity.

Consistent with our Empirical Prediction 1, in Figure 9 we report the fitted lines of Odd-Lot Trades and Odd Lot Volume on the Relative Tick (bsp) which indicate that for the U.S. markets our proxy for undercutting is negatively related to the relative tick size. In Figure 10 we report the fitted line of our proxies for queuing (Queuing and Inverted Share) on the Relative Tick (bps) suggesting a positive relationship.

Given our Empirical Prediction 1, we expect liquidity to cluster at the minimum price increment for low priced stocks, whereas we expect high price stocks to be less likely tick size constrained. This is particularly true for the U.S. markets where the tick size is on penny for all stocks priced above \$1. To investigate this relationship, we define a tick size constrained stock (TSC) a stock that satisfies two conditions. First, it has an average number of shares at the (N)BBO greater than the 50th, 60th and 70th decile. Second, it has an average quoted spread that is less than or equal to one tick and a half (consistent with common practice at the Nasdaq's Economic Research Team). We then create three groups of stocks based on stock price terciles (T1 P, T2 P, T3 P). Table 6 shows that the percentage of U.S. TSC stocks is on average 7.75% for low priced stocks (T1 P) and decreases to 0.98% for high priced stocks (T3 P). The number of TSC stocks decreases with the stock price as - intuitively - an increase in stock price translates into a reduction of relative tick size.

These results suggest that U.S. low priced stocks exhibit a too large tick size which motivates both the Intelligent Tick Size Proposal of Nasdaq and the recent 34-96494 SEC proposal to create new buckets of smaller tick size stocks. For high priced stocks instead our model predicts that given the associated high degree of undercutting, the tick size should be wider. However, it should also reflect the liquidity of the stock with higher liquidity requiring a smaller tick size. Our empirical evidence suggests that high priced stocks are characterized by large undercutting,

[30] The ratio $\frac{23400}{60}$ is the number of seconds in a trading day (6.5 hours multiplied by seconds per hour, 3600) over the number of seconds in a minute.

Table 6: Tick Size Constrained Stocks: U.S. and E.U. comparison
This table reports the number of Tick Size Constrained (TSC) stocks in the U.S and the European markets Before and After MIDID II. Stocks are grouped in terciles of prices (T1 P, T2 P and T3 P) and they are also grouped by decile of depth at the NBBO. We define TSC a stock that satisfies two conditions. First, it has an average number of shares at the NBBO greater than the 50th, 60th and 70th decile. Second, it has an average quoted spread that is less or equal than a tick and a half.

	U.S.				E.U. BEFORE				E.U. AFTER			
	T1 P	T2 P	T3 P	AVG	T1 P	T2 P	T3 P	AVG	T1 P	T2 P	T3 P	AVG
50^{th} D	7.75%	4.14%	0.98%	4.29%	4.17%	6.55%	1.19%	3.97%	2.98%	4.76%	0.00%	2.58%
60^{th} D	7.75%	4.14%	0.98%	4.29%	4.17%	4.17%	0.60%	2.69%	2.98%	3.57%	0.00%	2.18%
70^{th} D	7.75%	4.11%	0.98%	4.28%	4.17%	2.98%	0.00%	2.38%	2.98%	2.38%	0.00%	1.79%
AVG	7.75%	4.13%	0.98%	12.85%	4.17%	4.56%	0.60%	9.33%	2.98%	3.57%	0.00%	6.55%

which according to our model's results may harm liquidity provision resulting in wider quoted spread. The distribution of the U.S. stock prices and the associated market quality metrics provide some guidelines on this issue. Table 7 shows that 548 stocks with a price larger than \$100 exhibit an average price of \$255, and 13 stocks with a price larger than \$1000 exhibit an average price of \$2184. It is therefore highly likely that a one cent tick size is not the optimal price improvement for giant stocks - e.g., GOOGLE, AMZN, BKNG, AZO - as such a negligible cost of undercutting is likely to harm their liquidity provision. Table 7 reports the average %spread(bps) associated with each price bucket of stocks. However, assessing the value of the spread of the U.S. high-priced stocks is complicated by the fact that the metric should take into account odd-lot trades which generally undercut existing best bid offers and are not reported in the consolidated tape. Yet, as a first cut, we can adjust the spread measures reported in Table 7 by using the SEC's estimate in the MDI Rule SEC (2020) of a roughly 15% reduction in spread when taking into account odd lots quoting for stocks priced $\$250 - \$1000$ (28% reduction for stocks priced $\$1000 - \$10000$).[31] This means that for stocks priced $\$250 - \$1000$ the average %spread(bps) should decrease from 29.671bps to 16.959bps and for stocks priced $> \$1000$ it should decrease from 49.729bps to 12.798bps. Considering that the relative tick size for stocks priced $\$250 - \$1000$ ($\$1000 - \$10000$) is $\frac{0.01}{625} \times 10000 = 0.16$bps ($\frac{0.01}{5500} \times 10000 = 0.018$bps), the resulting %spread(bps) would still be 100 (700) times larger than the existing relative tick size.[32]

[31] See footnote 607, page 237 of SEC (2022).

[32] As the existing spread for these stocks is larger than \$0.04, the tick in the SEC (2022) proposal would remain \$0.01. Note that the average spreads reported in the SEC (2022) proposal differ somewhat from those reported in Table 7 which are based on our Nasdaq sample. However, even with their parsimonious evaluation, the resulting %spread(bps) for stocks priced $\$250 - \$1000$ would result 30 (210 for stocks priced $\$1000 - \$10000$) times larger than the relative tick size.

It is therefore likely that taking odd-lots adjustment into account would only minimally alleviate our general concern about the SEC (2022) proposal .

Table 7: U.S. Price Distribution and Market Quality
This table reports the price distribution of the U.S. stocks considered in our analysis (3988 stocks). In column 2 we report the number of stocks belonging to the specific price bucket; in columns 3-6 we report the Average Price, Spread, % Spread (bps), and Turnover (expressed in millions) of each price bucket.

	Obs.	Avg. Price ($)	Avg. Spread ($)	Avg. % Spread (bps)	Avg. Turnover (Mill $))
$p \leq \$10$	988	5.965	0.068	122.022	8.867
$\$10 < p \leq \$50$	1833	25.137	0.148	65.142	43.200
$\$50 < p \leq \$100$	619	70.745	0.220	32.307	121.000
$\$100 < p \leq \$250$	413	153.033	0.408	25.901	355.000
$\$250 < p \leq \$1000$	122	394.292	1.247	29.671	709.000
$p > \$1000$	13	2184.435	9.777	49.729	3780.000

Due to data availability, we cannot fully test our Empirical Prediction 1 for Europe. We can only investigate the effects on the number of tick size constrained stocks of the new MiFID II tick size regime introduced in 2018. Table 6 shows that the introduction of the MiFID II regime decreases the percentage number of TSC stocks in Europe from 9.33% to 6.55% overall. It also shows that the European stocks have a non monotonic relationship with the stock price. This is probably due to the fact that the European tick size was already a positive step function of the stock price before the introduction of the MiFID II regime.

6.2 MiFID II New Tick Size Regime

Our Corollary 4 allows us to evaluate the change in tick size introduced by MiFID II in 2018. More specifically, if after the introduction of the new tick size regime market quality measured by spread improves, our model suggests that the new tick size regime is likely to have improved total welfare.

The plots in the first row of Figure 11 report the relationship between the average spread (bps) and the relative tick size (bps) for the European stocks before and after the introduction of MiFID II. Along the dashed black line the average spread (bps) is equal to the relative tick size (bps), hence the observations above this line correspond to stock-day observations with spreads greater than the minimum tick size. Consistent with the new MiFID II Tick Size regime, the

Figure 9: **Odd Lot Trade/Volume (%) vs. Relative Tick Size (bps)**
This figure reports for all the stocks listed on the U.S. markets by 1 January 2022 (3988) on the left (right) the relationship between the percentage of Odd Lot Trades (Volume) and the relative tick size (tick size over price in bps). Stocks are grouped by tercile of Average Number of Trades (T1 ANT -grey-, T2 ANT -cyan- and T3 ANT -red).

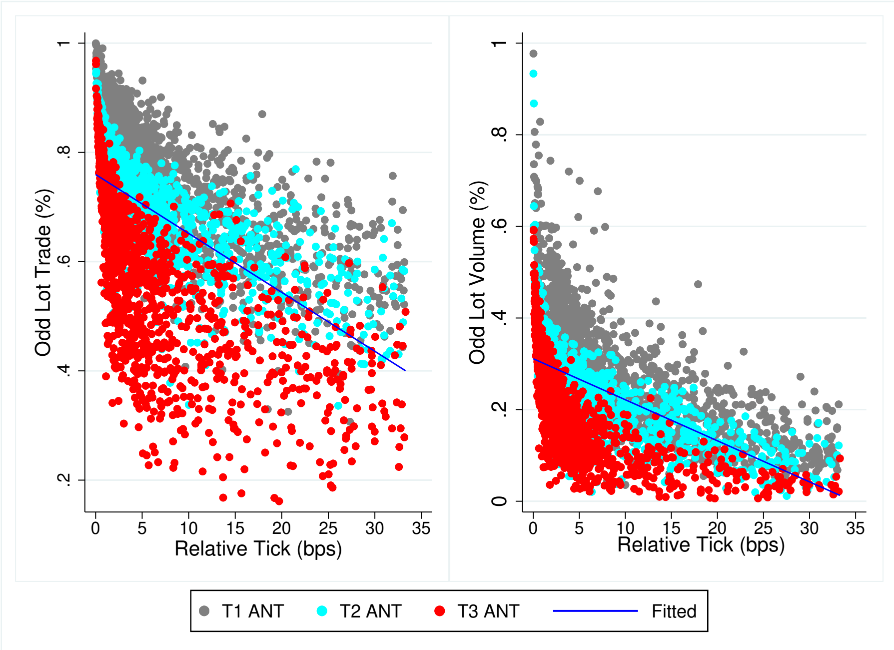
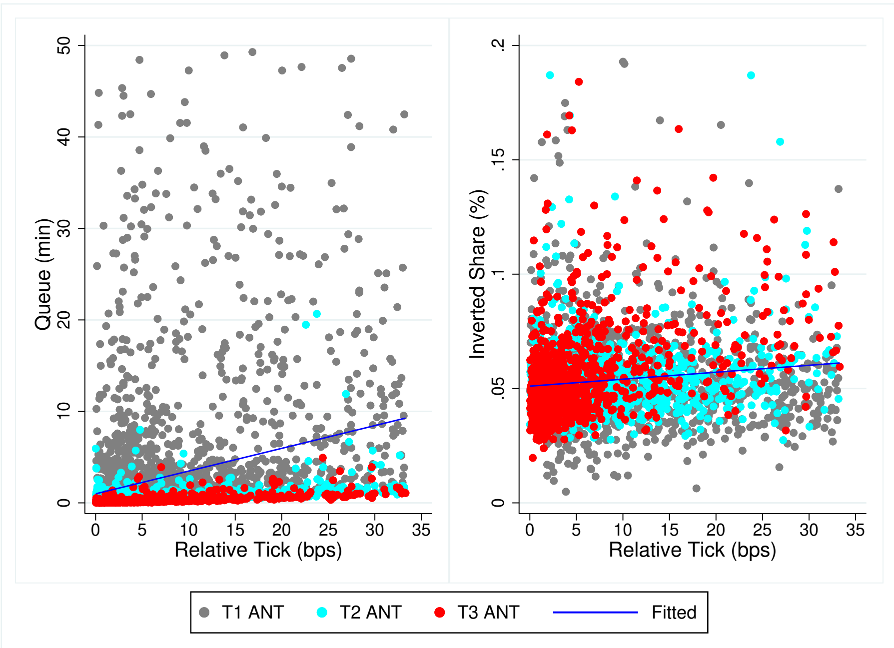


Figure 10: **Queuing - Inverted Share vs. Relative Tick Size**
This figure reports for all the stocks listed on the U.S. markets by 1 January 2022 (3988) on the left (right) the relationship between Queue (min) (Inverted Share (%)) and the relative tick size (tick size over price in bps). Stocks are grouped by tercile of Average Number of Trades (T1 ANT -grey-, T2 ANT -cyan- and T3 ANT -red).



European stocks are grouped into three terciles based on the Average Number of Trades (T1 ANT, T2 ANT and T3 ANT). The plots in the second row of Figure 11 report the fitted lines corresponding either to each group of stocks (grey, cyan and red solid lines) or to the whole sample of stocks (blue solid line). First, note that as ANT increases the fitted lines move towards the dashed black line: on average liquid stocks trade at spreads which are nearer to the minimum tick size. Second, after the introduction of MiFID II, the fitted line of the whole sample (blue line) moves substantially towards the dashed black line indicating that spreads overall improve.

To study the effects of MiFID II on market quality by using our Corollary 4, we perform a Difference in Difference (DD) analysis around the introduction of the new policy regime. We collect minute by minute data for 168 Pan-European stocks from October 2017 to March 2018. We use the following specification to evaluate the effectiveness of MiFID II:

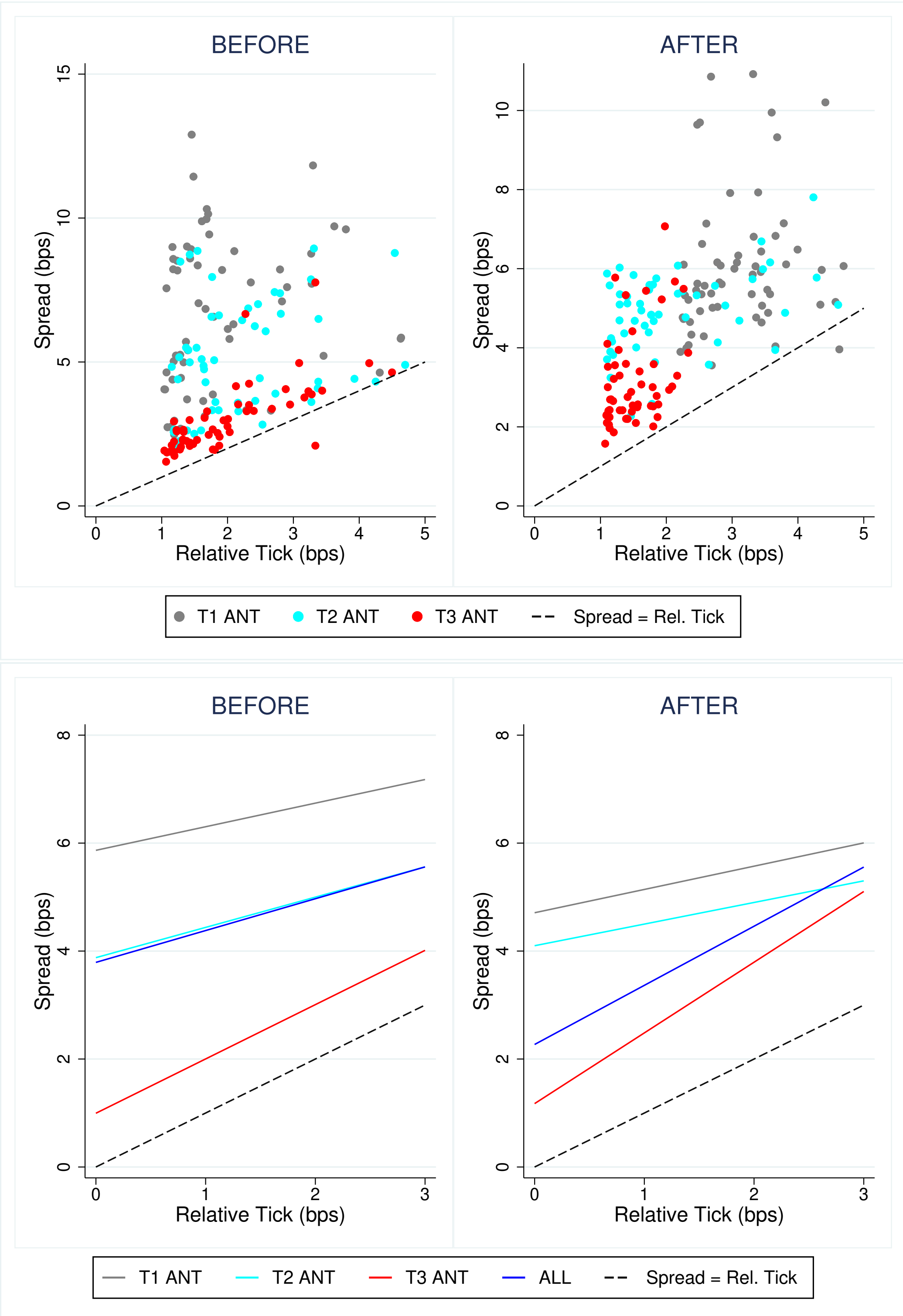
$$MQ_{i,t} = \alpha + \gamma_i + \delta_t + \phi_1 \tau_{i,t} + \beta_1 (\mathbb{I}_{inc} \times AFTER) + \beta_2 (\mathbb{I}_{dec} \times AFTER) + \phi_2 Volat_{i,t} + \phi_3 EUVIX_t + \epsilon_{i,t} \tag{16}$$

where $MQ_{i,t}$ is a market quality metric - quoted spread, %spread, depth at BBO, or volume - aggregated at daily level; $\tau_{i,t}$ is the daily tick size;[33] $AFTER$ is an indicator variable equal to 1 after 1 January 2018, and 0 otherwise; $\mathbb{I}_{inc}$ is an indicator variable equal to 1 if the tick associated to stock i increases after MiFID II, and 0 otherwise; $\mathbb{I}_{dec}$ is an indicator variable equal to 1 if the tick associated to stock i decreases after MiFID II, and 0 otherwise; $Volat_{i,t}$ is the daily volatility at the stock level, while $EUVIX_t$ is the daily STOXX volatility index. The coefficients of interest are β_1 and β_2: they measure the impact of the change in the tick size regime. We do not include dummies on stock level ($\mathbb{I}_{inc}$ and $\mathbb{I}_{dec}$) and time level ($AFTER$) as we consider both stock, γ_i, and day, δ_t, fixed effects. The stock fixed effects capture unobserved stock characteristics, while day fixed effects capture common linear trends. Despite the fact that our sample is balanced in terms of change in tick size,[34] we do not have a fully exogenous control

[33] To determine $\tau_{i,t}$ before the introduction of MiFID II, we use the different regimes in effect at LSE, Xetra and Euronext respectively; after MiFID II we use the ESMA tick size table. As the ESMA table indicates to compute ANT over the previous year (updated every year), to estimate ANT, we collected, for each stock, the number of trades for the entire 2017 year.

[34] Out of the 168 stocks considered, 34 stocks experienced a tick size increase, 61 a tick size decrease and 73 did not experience any change in the tick size.

Figure 11: **Percentage Spread (bps) - Relative Tick Size (bps)**
This figure reports the relationship between percentage spread (bps) and relative tick size (bps). The two graphs on the first row correspond respectively to the European stocks before and after MiFID II. Stocks are grouped by tercile of average number of trades (T1 ANT -grey-, T2 ANT -cyan- and T3 ANT -red). The two graphs on the second row report both the fitted lines for the three terciles and also the fitted line for the overall sample considered in each graph (blue solid line).



group and therefore we control for both idiosyncratic and common volatility.

Table 8: Effects of MiFID II on Market Quality
This table reports the results from the Difference in Difference (DD) analysis around the introduction of the MiFID II regime. The specification is the following:

$$MQ_{i,t} = \alpha + \gamma_i + \delta_t + \phi_1 \tau_{i,t} + \beta_1(\mathbb{I}_{inc} \times AFTER) + \beta_2(\mathbb{I}_{dec} \times AFTER) + \phi_2 Volat_{i,t} + \phi_3 EUVIX_t + \epsilon_{i,t}$$

where $MQ_{i,t}$ is a market quality metric - spread, percentage spread (s-spread), depth, and volume - aggregated at daily level; $\tau_{i,t}$ is the daily tick size; $AFTER$ is an indicator variable equal to 1 after 1 January 2018, and 0 otherwise; $\mathbb{I}_{inc}$ is an indicator variable equal to 1 if the tick associated to stock i increased after MiFID II, and 0 otherwise; $\mathbb{I}_{dec}$ is an indicator variable equal to 1 if the tick associated to stock i decreased after MiFID II, and 0 otherwise; $Volat_{i,t}$ is the daily volatility at the stock level, while $EUVIX_t$ is the STOXX volatility index at daily level. We report t-stats in parentheses obtained from robust standard errors clustered by stock. $\star\star\star$: 1% significance, $\star\star$: 5% significance, and $\star$: 10% significance, respectively.

Dependent Variable	Spread	%Spread (bps)	Depth	Volume
τ	0.616***	42.502***	0.154	-4.599
	(5.744)	(3.871)	(0.498)	(−0.696)
$\mathbb{I}_{inc} \times AFTER$	−0.004***	−0.599*	-0.005	0.277
	(−2.921)	(−1.827)	(−0.463)	(0.900)
$\mathbb{I}_{dec} \times AFTER$	-0.002	−0.736***	-0.020	0.350
	(−1.150)	(−2.439)	(−1.636)	(0.861)
Volat	0.034***	17.843***	-0.014	94.102***
	(4.675)	(5.652)	(−0.391)	(5.606)
EUVIX	0.001***	0.030***	−0.001***	-0.010
	(3.768)	(3.386)	(−3.287)	(−0.527)
Stock Fixed Effects	Yes	Yes	Yes	Yes
Day Fixed Effects	Yes	Yes	Yes	Yes
Observations	20664	20664	20664	20664
N	168	168	168	168
R^2	25%	13%	12%	7%

Table 8 reports our results, indicating that after the introduction of MiFID II, %Spread significantly decreases for the treated stocks, while Spread significantly decreases only for stocks that experienced a tick size increase. Volume and Depth are not significantly impacted by the change in the tick size regime. Therefore, using our Corollary 4 we can conclude that the new MiFID II regime has likely improved total welfare of market participants.

Up to here, we have used indicators of top of the book market quality. To evaluate the effects of the new tick size regime on the overall liquidity of the European markets we run the (16) regression for each of the 10 levels of the book on each side of the market (Level II Refinitiv Data). Figure 12 reports the coefficients β_1 and β_2 with the associated confidence intervals at 95% for both the case of a tick size increase (Panel A) and the case of a tick size decrease (Panel B). Table 1.G in the Appendix reports the exact values of the coefficients and the associated t-statistics. Taken together these results confirm our analysis of the top of the book market quality as our Spread measures tend to improve (grey-shaded coefficients) whereas the effects on Depth are negligible.[35]

7 Conclusions and Policy Implications

This paper shows that in a limit order book where traders can endogenously choose between taking and supplying liquidity, and therefore are allowed to undercut or queue behind existing limit orders, the optimal tick size cannot be zero. The optimal tick size set by a social planner optimally manages the dynamic interaction between liquidity supply and liquidity demand as it needs to trade-off the incentives of investors to either queue or undercut existing orders. If the tick is set too large, liquidity suppliers may benefit but investors may excessively opt for queuing with the result that liquidity provision may deteriorate. If instead the tick is set too small, then liquidity takers may benefit but investors may excessively opt for undercutting potentially worsening liquidity provision.

This paper also shows that the optimal tick size should be set as a positive function of the asset value, and as a negative function of the liquidity of the instrument. The economic intuition behind these results is again linked to the trade-off between undercutting and queuing. A higher asset value increases investors' gains from trade and their aggressive undercutting, whereas larger trading activity increases the execution probability of their limit orders thus incentivizing queuing. The social planner would react by increasing the optimal tick size when the asset value increases while decreasing the optimal tick size when trading activity becomes more intense.

Finally, this paper shows that a change in the tick size that leads to a reduction in the spread is likely to increase total welfare. Therefore, our results suggest researchers to evaluate a change in the tick size by using spread, which is the most popular standard metric of market quality. We therefore show that the new tick size regime proposed by ESMA within the MiFID II revision

[35]Note that %Spread decreases significantly for all levels of the book beyond the first one, whereas Spread significantly improves only following a reduction in the tick size. The difference in the result reported in Table 8 is due to the fact that for Level II data we focused on the last hour of the trading day only.

in January 2018 has overall improved market quality, measured by spread, and therefore it has probably been beneficial for market participants.

Our results support the proposal of an "Intelligent Ticks" regime suggested by the Nasdaq on December 2019. The Nasdaq proposal aimed to renew the U.S. tick size regime formalized by the Regulation National Market System Rule 612 in 2007. Our theoretical results also partially support the recent SEC (2022) proposal to modify Rule 612 thus setting the tick size - only for instruments with an average spread smaller than \$0.04 - as a function of the average quoted spread of that instrument. We illustrate that for the U.S markets low priced stocks are tick size constrained at the best bid offers whereas high priced stocks tend to be less tick size constrained. We also illustrate that when the price of the U.S. stocks is relatively low (high), queues at the best bid-offer are relatively long (short). When the relative tick size (tick-to-price ratio) is too wide and stocks are tick size constrained, traders cannot undercut existing best quotes thus creating long queues at the best bid-offers. When instead the relative tick size is too small as the stock price is too high relative to the one cent tick size, undercutting becomes extremely cheap and may induce investors to refrain from offering liquidity at the best bid offer. The 34-96494 SEC (2022) proposal only addresses the issue of low-priced tick size constrained stock, but it does not address the issue of high priced stocks trading at a too small tick size thus being subject to excessive undercutting.

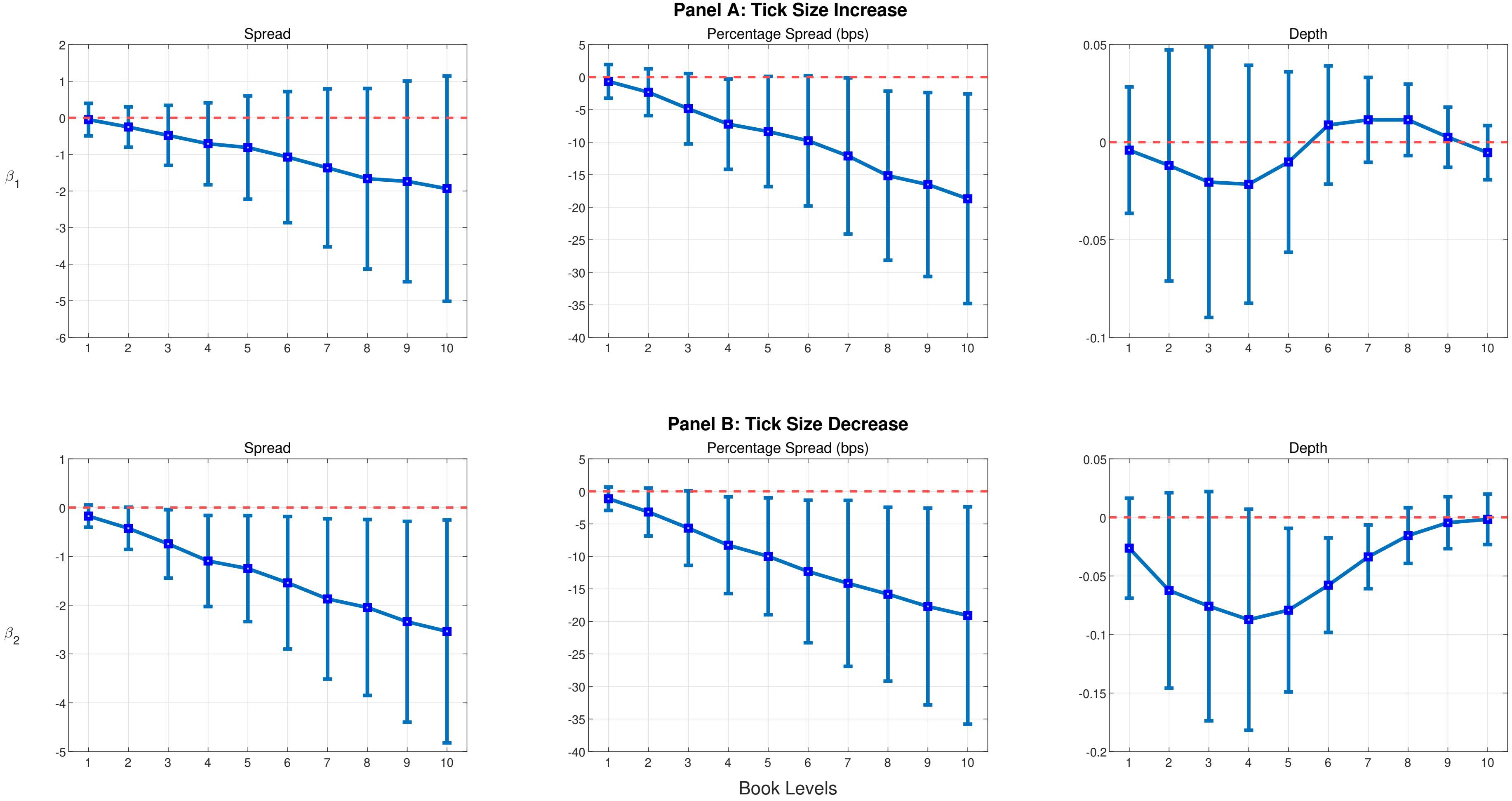
If the tick size were adjusted for the asset prices - as our model suggests - not only the smaller tick size associated with lower priced stocks would allow trading at a smaller bid-ask spread thus preventing the creation of long queues, but the larger tick size associated with higher priced stocks would increase the cost of undercutting existing quotes thus raising the incentive for liquidity suppliers to safely post limit orders (Foley et al. (2022)). Therefore considering both the liquidity of the stock and the asset value would not only solve the problem of tick size constrained stocks but would also alleviate the issue of high priced stocks trading at economically insignificant tick size. According to both the Nasdaq and the SEC proposal, the tick size should be set based only on the average quoted spread, whereas according to the MiFID tick size regime the tick size is based on both the asset price and the average number of trades. It is an empirical question which of the two regimes better capture the transmission channels that lead to the OTS.

Figure 12: **Effects of MiFID II on each Book Level**

This figure reports the coefficients and confidence interval at 95% of a tick size increase in Panel A ($\mathbb{I}_{inc} \times AFTER$) and decrease in Panel B ($\mathbb{I}_{dec} \times AFTER$) from the Difference in Difference (DD) regression analysis around the introduction of the MiFID II regime using the following specification:

$$MQ_{i,t,l} = \alpha + \gamma_i + \delta_t + \phi_1 \tau_{i,t} + \beta_1(\mathbb{I}_{inc} \times AFTER) + \beta_2(\mathbb{I}_{dec} \times AFTER) + \phi_2 Volat_{i,t} + \phi_3 EUVIX_t + \epsilon_{i,t}$$

where $MQ_{i,t,l}$ is a market quality metric - Spread, %-Spread (bps) and Depth - for stock i, day t and level l of the book with $1 \leq l \leq 10$; $\tau_{i,t}$ is the daily tick size; $AFTER$ is an indicator variable equal to 1 after January the 1^{st} 2018 and 0 otherwise; $\mathbb{I}_{inc}$ is an indicator variable equal to 1 if the tick associated to stock i increased after MiFID II and 0 otherwise; $\mathbb{I}_{dec}$ overleaf is an indicator variable is equal to 1 if the tick associated to stock i decreased after MiFID II and 0 otherwise; $Volat_{i,t}$ is the daily volatility at the stock level, while $EUVIX_t$ is the STOXX volatility index at daily level.



References

Albuquerque, R., Song, S., Yao, C., 2020. The price effects of liquidity shocks: A study of the sec's tick size experiment. Journal of Financial Economics 138, 700–724.

AMF, 2013. Tick size: the "nouveau régime" https://www.sec.gov/comments/4-657/4657-8.pdf.

AMF, 2018. Impact du nouveau regime de pas de cotation https://www.amf-france.org/fr/actualites-publications/publications/rapports-etudes-et-analyses/mif-2-impact-du-nouveau-regime-de-pas-de-cotation.

Angel, J. J., 1997. Tick size, share prices, and stock splits. The Journal of Finance 52, 655–681.

Anshuman, V. R., Kalay, A., 1998. Market making with discrete prices. The Review of Financial Studies 11, 81–109.

Aquilina, M., Budish, E., O'neill, P., 2022. Quantifying the high-frequency trading "arms race". The Quarterly Journal of Economics 137, 493–564.

Bacidore, J. M., 1997. The impact of decimalization on market quality: An empirical investigation of the toronto stock exchange. Journal of Financial Intermediation 6, 92–120.

Bessembinder, H., 2000. Tick size, spreads, and liquidity: An analysis of nasdaq securities trading near ten dollars. Journal of Financial Intermediation 9, 213–239.

Bessembinder, H., 2003. Trade execution costs and market quality after decimalization. Journal of Financial and Quantitative Analysis 38, 747–777.

Bhattacharya, A., Saar, G., 2021. Limit order markets under asymmetric information. Available at SSRN 3688473 .

Biais, B., Foucault, T., et al., 2014. Hft and market quality. Bankers, Markets & Investors pp. 5–19.

Budish, E., Cramton, P., Shim, J., 2015. The high-frequency trading arms race: Frequent batch auctions as a market design response. The Quarterly Journal of Economics 130, 1547–1621.

Chakrabarty, B., Cox, J., Upson, J. E., 2022. Tick size pilot program and price discovery in us stock markets. Journal of Financial Markets 59, 100658.

Chao, Y., Yao, C., Ye, M., 2018. Why Discrete Price Fragments U.S. Stock Exchanges and Disperses Their Fee Structures. The Review of Financial Studies 32, 1068–1101.

Chordia, T., Subrahmanyam, A., 1995. Market making, the tick size, and payment-for-order flow: theory and evidence. Journal of Business pp. 543–575.

Chung, K. H., Chuwonganant, C., McCormick, D. T., 2004. Order preferencing and market quality on nasdaq before and after decimalization. Journal of Financial Economics 71, 581–612.

Chung, K. H., Lee, A. J., Rösch, D., 2020. Tick size, liquidity for small and large orders, and price informativeness: Evidence from the tick size pilot program. Journal of Financial Economics 136, 879–899.

Comerton-Forde, C., Grégoire, V., Zhong, Z., 2019. Inverted fee structures, tick size, and market quality. Journal of Financial Economics .

Cordella, T., Foucault, T., 1999. Minimum price variations, time priority, and quote dynamics. Journal of Financial Intermediation 8, 141–173.

Dayri, K., Rosenbaum, M., 2015. Large tick assets: implicit spread and optimal tick size. Market Microstructure and Liquidity 1, 1550003.

Dyhrberg, A. H., Foley, S., Svec, J., 2019. When bigger is better: The impact of a tiny tick size on undercutting behavior. Available at SSRN 3194932 .

ESMA, 2017. Commission delegated regulation 2017/588. Official Journal of the European Union "https://eur-lex.europa.eu/legal-content/EN/TXT/PDF/?uri=CELEX:32017R0588&from=EN".

Foley, S., Dyhrberg, A., Svec, J., 2022. When bigger is better: the impact of a tiny tick size on undercutting behavior. Journal of Financial and Quantitative Analysis .

Foucault, T., Kadan, O., Kandel, E., 2005. Limit order book as a market for liquidity. Review of Financial Studies 18, 1171–1217.

Glosten, L. R., 1994. Is the electronic open limit order book inevitable? The Journal of Finance 49, 1127–1161.

Glosten, L. R., Milgrom, P. R., 1985. Bid, ask and transaction prices in a specialist market with heterogeneously informed traders. Journal of financial economics 14, 71–100.

Goettler, R. L., Parlour, C. A., Rajan, U., 2005. Equilibrium in a dynamic limit order market. The Journal of Finance 60, 2149–2192.

Goldstein, M. A., Kavajecz, K. A., 2000. Eighths, sixteenths, and market depth: changes in tick size and liquidity provision on the nyse. Journal of financial economics 56, 125–149.

Harris, L., 1991. Stock price clustering and discreteness. The Review of Financial Studies 4, 389–415.

Harris, L., 1996. Does a large minimum price variation encourage order exposure?, vol. 96. New York Stock Exchange.

Harris, L., 1998. Optimal dynamic order submission strategies in some stylized trading problems. Financial Markets, Institutions & Instruments 7, 1–76.

Harris, L. E., 1994. Minimum price variations, discrete bid–ask spreads, and quotation sizes. The Review of Financial Studies 7, 149–178.

Hasbrouck, J., Saar, G., 2013. Low-latency trading. Journal of Financial Markets 16, 646–679.

Hollifield, B., Miller, R. A., Sandås, P., Slive, J., 2006. Estimating the gains from trade in limit-order markets. The Journal of Finance 61, 2753–2804.

Hu, E., Hughes, P., Ritter, J., Vegella, P., Zhang, H., 2018. Tick size pilot plan and market quality. Working Parer .

Kandel, E., Marx, L. M., 1997. Nasdaq market structure and spread patterns. Journal of Financial Economics 45, 61–89.

Kaniel, R., Liu, H., 2006. So what orders do informed traders use? The Journal of Business 79, 1867–1913.

Li, S., Wang, X., Ye, M., 2021. Who provides liquidity, and when? Journal of financial economics 141, 968–980.

Li, S., Ye, M., 2022. The optimal price of a stock: A tale of two discretenesses. Available at SSRN .

Mackintosh, P., 2020. Why intelligent ticks make sense https://www.nasdaq.com/articles/why-intelligent-ticks-make-sense-2020-01-09.

Mackintosh, P., 2022. Why ticks matter https://www.nasdaq.com/articles/why-ticks-matter.

Nasdaq, 2019. Intelligent ticks https://www.sec.gov/rules/petitions/2019/petn4-756.pdf.

O'Hara, M., Saar, G., Zhong, Z., 2019. Relative tick size and the trading environment. The Review of Asset Pricing Studies 9, 47–90.

Pakes, A., McGuire, P., 2001. Stochastic algorithms, symmetric markov perfect equilibrium, and the 'curse'of dimensionality. Econometrica 69, 1261–1281.

Parlour, C. A., 1998. Price dynamics in limit order markets. Review of Financial Studies 11, 789–816.

Riccó, R., Rindi, B., Seppi, D. J., 2022. Information, Liquidity, and Dynamic Limit Order Markets, working Paper Carnegie Mellon University.

Riccó, R., Rindi, B., Seppi, D. J., 2021. Optimal Market Asset Pricing. Working Papers 675, IGIER (Innocenzo Gasparini Institute for Economic Research), Bocconi University.

Rindi, B., Werner, I. M., 2019. Us tick size pilot .

Ronen, T., Weaver, D. G., 2001. 'teenies' anyone? Journal of Financial Markets 4, 231–260.

Roşu, I., 2009. A dynamic model of the limit order book. The Review of Financial Studies 22, 4601–4641.

SEC, 2020. Securities exchange act release no. 90610 (dec. 9, 2020), 86 fr 18596 (apr. 9, 2021) ("mdi adopting release") https://www.sec.gov/rules/final/2020/34-90610.pdf.

SEC, 2022. Regulation nms: Minimum pricing increments, access fees, and transparency of better https://www.sec.gov/rules/proposed/2022/34-96494.pdf.

Werner, I. M., Rindi, B., Buti, S., Wen, Y., 2022. Tick size, trading strategies and market quality. Forthcoming Management Science .

Ye, M., Yao, C., 2014. Tick size constraints, market structure, and liquidity. WBS Finance Group Research Paper .

Appendices

A Miscellaneous

Table 1.A: Tick Size Regimes

This table presents an overview of the tick regimes used in most of the major trading venues.

Trading Venue	**Binary Tick Size**
USA	USD 0.01 for stocks with price $\geq$ USD 1, and USD 0.0001 for stocks with price $<$ USD 1. https://www.sec.gov/divisions/marketreg/subpenny612faq.htm
CANADA (CSE)	CAD 0.01 for stocks with price $\geq$ CAD 0.5, and CAD 0.005 for stocks with price $<$ CAD 0.5. https://www.thecse.com/en/support/dealers/order-types-and-functionality
SHANGHAI (SSE)	RMB 0.01 for A-share stocks and USD 0.001 for B-share stocks. http://english.sse.com.cn/start/trading/mechanism/
	Discrete Tick Size
AUSTRALIA (ASX)	AUD 0.01 for stock with price $>$ AUD1.995 and AUD 0.005 for stocks with price $\in$ [AUD 0.1, AUD 1.995], AUD 0.001 for stocks with price $\leq$ AUD 0.099 https://www.asx.com.au/documents/resources/australian_cash_equity_market.pdf
CANADA (TSX)	CAD 0.125 for stock with price $>$ CAD1000, and CAD 0.1 for stocks with price $\in$ [CAD 0.5, CAD 1000], CAD 0.005 for stocks with price $<$ CAD 0.5 . https://www.tsx.com/resource/en/133
SINGAPORE (SGX)	SGD 0.01 for stock with price $>$ SGD0.995, and SGD 0.005 for stocks with price $\in$ [SGD 0.2, SGD 0.995], SGD 0.001 for stocks priced $\leq$ SGD 0.2. http://rulebook.sgx.com/rulebook/833-0
	Volume Adjusted Tick Size: step function of stock price and average number of trades (ANT)
EU	'MiFID II / MiFIR' directions: 19 stock price buckets and 6 ANT buckets. https://www.esma.europa.eu/system/files_force/library/2015/11/2015-esma-1464_annex_i_-_draft_rts_and_its_on_mifid_ii_and_mifir.pdf
SWITZERLAND (SIX)	19 stock price buckets and 2 ANT buckets. https://www.ser-ag.com/dam/downloads/regulation/trading/directives/sdx-dir03-en.pdf
ENGLAND (LSE)	'MiFID II / MiFIR' directions: 19 stock price buckets and 6 ANT buckets. https://www.londonstockexchange.com/trade/equity-trading
HONG KONG (HKEX)	11 stock price buckets and 6 ANT buckets. https://www.hkex.com.hk/-/media/HKEX-Market/Services/Rules-and-Forms-and-Fees/Rules/SEHK/Stock-Options/Rule-UpdateOperational-Trading-Procedures-for-Options-Trading-Exchange-Participants-of-the-Stock/14-13-OTP-StockOptionsRevamp_e.pdf?la=en#:~:text=The%20tick%20size%20for%20HK,001.
JAPAN (JPX)	11 stock price buckets and 2 ANT buckets; distinguished for TOPIX 100 Constituents (finer grid) and Other Issuers (corser grid). https://www.jpx.co.jp/english/equities/trading/domestic/07.html

B Appendix: T-period Model

B.1 Proof of Equation (1)

Prices on the price grid are symmetric around ν and the distance between two consecutive prices is τ, hence the first price levels are:

$$\begin{aligned} p_{+1} &= \nu + \frac{1}{2}\tau \\ p_{-1} &= \nu - \frac{1}{2}\tau \end{aligned} \tag{17}$$

p_{+1} and p_{-1} are the initial term of an arithmetic progression with the common difference of two successive members set at τ:

$$\begin{aligned} p_{+k} &= p_1 + (k-1)\tau = \nu + (k - \frac{1}{2})\tau \\ p_{-k} &= p_{-1} - (k-1)\tau = \nu - (k - \frac{1}{2})\tau \end{aligned} \tag{18}$$

B.2 Properties and definitions of the T-period Model

To determine the set of *feasible* prices associated with the set of *feasible* τ, we first equate p_{-k} and p_{+k} to the upper and lower bound of the investors' valuation support, respectively:

$$\begin{aligned} p_{-k}^{\tau^{max}} &= (1-b)\,\nu \\ p_{+k}^{\tau^{max}} &= (1+b)\,\nu \end{aligned} \tag{19}$$

$\forall\, \tau \in (0,\ \tau^{\max})$. To determine the number of *feasible* prices $+n^f$ $(-n^f)$ on the sell (buy) side of the price grid we equate the largest (smallest) valuation a trader may have, $\overline{\beta}\,\nu$ $(\underline{\beta}\,\nu)$, to the highest (lowest) price level, p_{+n} (p_{-n}). Using (1):

$$\begin{aligned} (1+b)\nu &= \nu + \left(n - \frac{1}{2}\right)\tau \\ (1-b)\nu &= \nu - \left(n - \frac{1}{2}\right)\tau \end{aligned} \tag{20}$$

and solving (20) for n, we obtain $+n^f$ $(-n^f)$:

$$
\begin{aligned}
+n^f &= +floor\left(\frac{b\nu}{\tau} + 0.5\right) \\
-n^f &= -floor\left(\frac{b\nu}{\tau} + 0.5\right)
\end{aligned} \tag{21}
$$

Lemma 1 summarizes the properties of the price grid:

Lemma 1.

1. *For any given bv symmetric around v, there exists a set of feasible tick sizes, $\tau \in (0\ ,\ \tau^{max})$, and an associated set of feasible prices $p_k^f \in (\underline{\beta}\nu,\ \overline{\beta}\nu)$.*

2. *For any symmetric state of the book, investors with $\beta_{t_i} > 1$ are buyers and investors with $\beta_{t_i} < 1$ are sellers. For $\beta_{t_i} = 1$, the investors are indifferent between buying and selling.*[36]

3. *For the last player of an T-period game, the submission probability of a market order is:*

$$
\begin{aligned}
Pr\left(ms_{k,t_T}|\Lambda_{t_{T-1}}, \tau\right) &= \frac{1}{\Gamma}\left(\frac{p_k}{\nu} - (1-b)\right) \\
Pr\left(mb_{k,t_T}|\Lambda_{t_{T-1}}, \tau\right) &= \frac{1}{\Gamma}\left((1+b) - \frac{p_k}{\nu}\right)
\end{aligned} \tag{22}
$$

   and does not depend on the state of the other side of the book.

The proof of Lemma 1 is in Appendix B.2.1.

B.2.1 Proof of Lemma 1

1.1 A feasible price is a limit price associated with a positive probability of execution. In order to guarantee that the SP chooses an OTS that is associated with positive probability of execution, we need to define a set of feasible tick sizes which includes tick sizes associated with at least one feasible price on each side of the market. Given a price p_{+k} such that $p_{+k} \geq \overline{\beta}\nu$, the gains from trade associated with a buy order (limit or market) is determined by equation (3) for any β_{t_i} and are non positive with probability 1:

$$
\beta_{t_i}\nu - p_{+k} \leq 0 \ \ \forall\ p_{+k} \geq \overline{\beta}\nu \tag{23}
$$

[36] As β_{t_i} has a continuous distribution, the probability of $\beta_{t_i} = 1$ is zero

Hence $\forall p_{+k} > \overline{\beta}\nu$ an investor never selects a buy order at p_{+k} and therefore any $p_{+k} > \overline{\beta}\nu$ is not a feasible price. For $p_{+k} = \overline{\beta}\nu$, the only β_{t_i} extracted - with probability 0 - that makes equation (23) non negative is $\beta_{t_i} = \overline{\beta}$. For $\beta_{t_i} = \overline{\beta}$ the investor's payoff would be equal to zero, and assuming an investor with zero payoff chooses not to trade (nt), even $p_{+k} = \overline{\beta}\nu$ is not a feasible price. Symmetrically, any $p_{+k} \leq \underline{\beta}\nu$ is not a feasible price. It follows that any tick size such that $\tau \geq \tau^{max}$ is not feasible as it only defines non feasible prices: using (1) and (8), for $\tau = \tau^{\max}$ we obtain:

$$\begin{aligned} p_{+1} &= \nu + (1 - \frac{1}{2})\tau^{\max} \\ &= \nu + (1 - \frac{1}{2})2b\nu \\ &= \nu + b\nu = \nu(1+b) = \overline{\beta}\nu \end{aligned} \tag{24}$$

As p_{+1} is not feasible, $p_{\sim k} > p_{+1}$ are also not feasible.

1.2 · If the state of the book is symmetric and the arriving investor has $\beta_{t_i} > 1$, he will neither market sell, nor limit sell:

1. Market order: If $\beta_{t_i} > 1$, a market buy at p_{-k} (p_{+k}) dominates a market sell at p_{+k} (p_{-k}):

$$\begin{aligned} (\beta_{t_i} v - p_{-k}) &> (p_{+k} - \beta_{t_i} v) \\ (\beta_{t_i} v - p_{+k}) &> (p_{-k} - \beta_{t_i} v) \end{aligned} \tag{25}$$

   By symmetry, both terms in equation (25) are satisfied if $\beta_{t_i} > \frac{p_{+k}+p_{-k}}{2v}$, and using equation (1) this condition is satisfied for $\beta_{t_i} > 1$.

2. Limit Order: For a symmetric state of the book, the execution probability of a limit buy order at p_{+k} (p_{-k}) is equal to the execution probability of a limit sell

at p_{-k} (p_{+k}):[37]

$$Pr(\Psi_{lb_{k,t_i}}|\Lambda_{t_{i-1}},\tau) = Pr(\Psi_{ls_{-k,t_i}}|\Lambda_{t_{i-1}},\tau) > 0 \tag{26}$$

Hence the payoff of a limit buy order at p_{+k} (p_{-k}) dominates the payoff of a limit sell order at p_{-k} (p_{+k}):

$$(\beta_{t_i}v - p_{-k})Pr(\Psi_{lb_{-k,t_i}}|\Lambda_{t_{i-1}},\tau) > (p_{+k} - \beta_{t_i}v)Pr(\Psi_{ls_{k,t_i}}|\Lambda_{t_{i-1}},\tau) \tag{27}$$

· By the previous point, necessary conditions for no-trading (nt_{t_i}) to be a dominated strategy are:

$$\begin{aligned} &(\beta_{t_i}\nu - p_{-k})Pr(\Psi_{lb_{-k,t_i}}|\Lambda_{t_{i-1}},\tau) > 0 \\ &\beta_{t_i} > 1 \end{aligned} \tag{28}$$

where $Pr(\Psi_{lb_{-k,t_i}}|\Lambda_{t_{i-1}},\tau)$ is the execution probability of a limit buy order posted at p_{-k}, given the t_{i-1} state of the book, $\Lambda_{t_{i-1}}$, and on τ. Conditions (28) are satisfied if $\beta_{t_i}\nu > p_{-k}$, which is always true as $p_{-k} < \nu$ and $\beta_{t_i} > 1$.

Same line on reasoning applies for $\beta_{t_i} < 1$

1.3 In the last period of a generic T-period game, the possible states of the book are:

1. Only one limit order on one side of the book.
2. Two or more limit orders on one side of the book.
3. Two or more limit orders on both sides of the book.
4. No limit order on any side of the book (empty book).

In our model with unitary trade, the first two states of the book are equivalent for the investor arriving at the T^{th}-period, as he can just focus on the best possible price. Hence,

[37] Take for example the 2-period model, the execution probability of a limit buy posted at price p_{-k} is given by $(\frac{\frac{p_{-k}}{\nu}-(1-b)}{\Gamma})$ and the execution probability of a limit sell posted at p_{+k} is $(\frac{1+b-\frac{p_{+k}}{\nu}}{\Gamma})$. The two probabilities are both equal to $(\frac{b-\frac{(2k-1)}{2\nu}\tau}{\Gamma})$.

in both states of the book the submission probability of a market sell order is:

$$Pr\left(ms_{k,t_T}|\Lambda_{t_{T-1}},\tau\right) = Pr(p_k > \beta_{t_T}\nu) = \frac{1}{\Gamma}\left(\frac{p_k}{\nu} - (1-b)\right) \tag{29}$$

The 3^{rd} state of the book implies that the best limit sell price is necessarily higher than the best limit buy price (otherwise the two would match). If p_k is the price associated with the best limit buy order, it will be lower than the price associated with any best limit sell order, e.g. p_{k+n}. The probability of submitting a market sell order at p_k can be obtained if the following conditions hold:

$$Pr\left(ms_{k,t_T}|\Lambda_{t_{T-1}},\tau\right) = Pr(p_k - \beta_{t_T}\nu > 0,\ p_k - \beta_{t_T}\nu > \beta_{t_T}\nu - p_{k+n}) \tag{30}$$

which guarantee that a market sell order dominates both no-trading (nt), and a market buy order (mb). It is possible to write equation (30) as:

$$\begin{aligned}
&Pr\ (p_k - \beta_{t_T}\nu > 0,\quad p_k - \beta_{t_T}\nu > \beta_{t_T}\nu - (p_k + n\tau))\\
&Pr\ (p_k - \beta_{t_T}\nu > 0,\quad 2p_k + n\tau > 2\beta_{t_T}\nu)\\
&Pr\ (p_k > \beta_{t_T}\nu,\quad p_k > -\frac{n\tau}{2} + \beta_{t_T}\nu)
\end{aligned} \tag{31}$$

Considering the last equation in (31) as the joint probability of $p_k > \beta_{t_T}\nu$ and $p_k > -\frac{n\tau}{2} + \beta_{t_T}\nu$, and using the definition of the conditional probability, we obtain:

$Pr\ (p_k > -\frac{n\tau}{2} + \beta_{t_T}\nu|p_k > \beta_{t_T}\nu)\ \ Pr(p_k > \beta_{t_T}\nu) = Pr\ (p_k > \beta_{t_T}\nu,\ \ p_k > -\frac{n\tau}{2} + \beta_{t_T}\nu)$.

If $p_k > \beta_{t_T}\nu \implies p_k > -\frac{n\tau}{2} + \beta_{t_T}\nu \implies Pr\ (p_k > -\frac{n\tau}{2} + \beta_{t_T}\nu \mid p_k > \beta_{t_T}\nu) = 1 \implies$

$$Pr\ (p_k > \beta_{t_T}\nu) = Pr(p_k > \beta_{t_T}\nu,\quad p_k > -\frac{n\tau}{2} + \beta_{t_T}\nu) \tag{32}$$

Note that equation (32) defines the same probability of market sell order as equation (29). Hence, the probability that a market sell order is profitable (i.e., the probability of market sell is positive) is independent of the probability that the same market sell order dominates a market buy order. This means that the opportunity to market buy offered by the state of the book on the other side of the market does not affect the equilibrium order submission probability of a market sell order.

Finally, if no limit orders are standing in the book (state of the book 4), the T^{th} player cannot submit any market order and therefore the probability of submission is zero.

Symmetric results apply for the submission probability of a market buy order.

C Appendix: Two Period Model

C.1 Proof of Proposition 1

Given Lemma (1), we present our results for an investor arriving at t_1 with $\beta_{t_1} > 1$. Results for a seller hold by symmetry.

Equation (5) shows the order submission probability of a market sell order at t_2. We now show the optimal order submission probability of a limit buy order at t_1. We start showing that for any given $\tau \in (0 \, , \, \tau^{max})$ a limit buy order posted at p_{+k} is a dominated strategy.

Limit buy at p_{+k} is a dominated strategy

Necessary and sufficient condition for a limit buy at p_{+k} to be dominated is that there exists at least one limit buy posted at $p_{\tilde{k}}$ that dominates it. Consider p_{-1}, by (6) a limit buy at p_{+k} is dominated if:

$$
\begin{aligned}
&(\beta_{t_1}\nu - p_{+k})(\frac{b}{\Gamma} + \frac{2k-1}{2\upsilon\Gamma}\tau) < (\beta_{t_1}\nu - p_{-1})(\frac{b}{\Gamma} - \frac{\tau}{2\upsilon\Gamma}) \\
&(\beta_{t_1}\nu - p_{-1} - k\tau)(\frac{b}{\Gamma} - \frac{\tau}{2\upsilon\Gamma} + \frac{k\tau}{\nu\Gamma}) < (\beta_{t_1}\nu - p_{-1})(\frac{b}{\Gamma} - \frac{\tau}{2\upsilon\Gamma}) \\
&\frac{\beta_{t_1}k\tau}{\Gamma} + \frac{k\tau^2}{2\nu\Gamma} - \frac{k\tau}{\Gamma} - \frac{k\tau b}{\Gamma} + \frac{k\tau^2}{2\nu\Gamma} - \frac{(k\tau)^2}{\nu\Gamma} < 0 \\
&(\beta_{t_1} - (1+b)) + \frac{\tau}{\nu}(1-k) < 0
\end{aligned}
\tag{33}
$$

This is always true as $\beta_{t_1} \leq (1+b)$ and $+k \geq 1$. Hence given Lemma (1) the optimal order submission strategy for a buyer is a limit order at p_{-k}.

Optimal set of $\mathbf{p_{-k}}$ and optimal $\mathbf{lb_{-k,t_i}}$ submission probability

We now derive both the optimal order submission probabilities associated with p_{-k}, and the optimal set of p_{-k} prices. Considering both p_{-k-j} such that $p_{-k-j} < p_{-k}$, and p_{-k+j} such that $p_{-k+j} > p_{-k}$ with $j \in N^{+}$, and given Lemma (1) and conditions (6), a limit buy at p_{-k}, lb_{-k,t_1}, is optimal if:

$$\begin{aligned}
(\beta_{t_1}\nu - p_{-k})Pr(\Psi_{lb_{-k,t_1}}|\Lambda_{t_0},\tau) &> (\beta_{t_1}\nu - p_{-k-j})\ (Pr(\Psi_{lb_{-k,t_1}}|\Lambda_{t_0},\tau) - \frac{j\tau}{\Gamma v}) \\
(\beta_{t_1}\nu - p_{-k})Pr(\Psi_{lb_{-k,t_1}}|\Lambda_{t_0},\tau) &> (\beta_{t_1}\nu - p_{-k+j})\ (Pr(\Psi_{lb_{-k,t_1}}|\Lambda_{t_0},\tau) + \frac{j\tau}{\Gamma v})
\end{aligned} \tag{34}$$

Equations (34) can be rearranged as:

$$\Gamma Pr(\Psi_{lb_{-k,t_1}}|\Lambda_{t_0},\tau) + \frac{p_{-k}}{\nu} - \frac{j\tau}{\nu} < \beta_{t_1} < \Gamma Pr(\Psi_{lb_{-k,t_1}}|\Lambda_{t_0},\tau) + \frac{p_{-k}}{\nu} + \frac{j\tau}{\nu} \tag{35}$$

Now if $j = 1$: Equations (34) can be rearranged as:

$$\Gamma Pr(\Psi_{lb_{-k,t_1}}|\Lambda_{t_0},\tau) + \frac{p_{-k}}{\nu} - \frac{\tau}{\nu} < \beta_{t_1} < \Gamma Pr(\Psi_{lb_{-k,t_1}}|\Lambda_{t_0},\tau) + \frac{p_{-k}}{\nu} + \frac{\tau}{\nu} \tag{36}$$

if (36) holds for $j = 1$, it also holds for any $j > 1$.

To determine the set of p_{-k} prices that satisfy (36), we first determine the prices associated with the boundary β_{t_1} values for a buyer. According to Lemma (1) a buyer arriving at t_1 has $1 < \beta_{t_1} < (1 + b)$, hence we set the boundaries of the β_{t_1} range for a generic lb_{-k,t_1} in (36) equal to $1 + b$ and 1 respectively:

$$\Gamma Pr(\Psi_{lb_{-k,t_1}}|\Lambda_{t_0},\tau) + \tfrac{p_{-k}}{\nu} + \tfrac{\tau}{\nu} = 1 + b \tag{37}$$

$$\Gamma Pr(\Psi_{lb_{-k,t_1}}|\Lambda_{t_0},\tau) + \tfrac{p_{-k}}{\nu} - \tfrac{\tau}{\nu} = 1 \tag{38}$$

- Consider first the β_{t_1} upper bound. Rearranging equation (37) and using equation (29):

$$p_{-k} = v - \frac{1}{2}\tau \tag{39}$$

Using (1):

$$
\begin{aligned}
&\upsilon - \frac{1}{2}\tau = \upsilon - k\tau + 0.5\tau \\
&\implies k = 1
\end{aligned}
\tag{40}
$$

Hence the upper bound of p_{-k} is p_{-1}.

- Consider now the β_{t_1} lower bound. Rearranging (38) and using equation (29):

$$
\begin{aligned}
p_{-k} &= \nu + \tau - \Gamma\nu\left(\frac{\frac{p_{-k}}{\nu} - (1-b)}{\Gamma}\right) \\
p_{-k} &= \nu(1 - 0.5b) + 0.5\tau
\end{aligned}
\tag{41}
$$

Using (1):

$$
\begin{aligned}
&\nu(1 - 0.5b) + 0.5\tau = \nu - k\tau + 0.5\tau \\
&\implies k = \frac{b\nu}{2\tau}
\end{aligned}
\tag{42}
$$

If $\frac{b\nu}{2\tau} \in N^+$, then the lower bound of p_{-k} is $p_{-\frac{b\nu}{2\tau}}$ (Figure 1.C).

The optimal set of prices for lb_{-k,t_1} is:

$$
p_k \in \left[p_{-\frac{b\nu}{2\tau}} \quad p_{-1}\right]
\tag{43}
$$

and the associated optimal submission probability is:

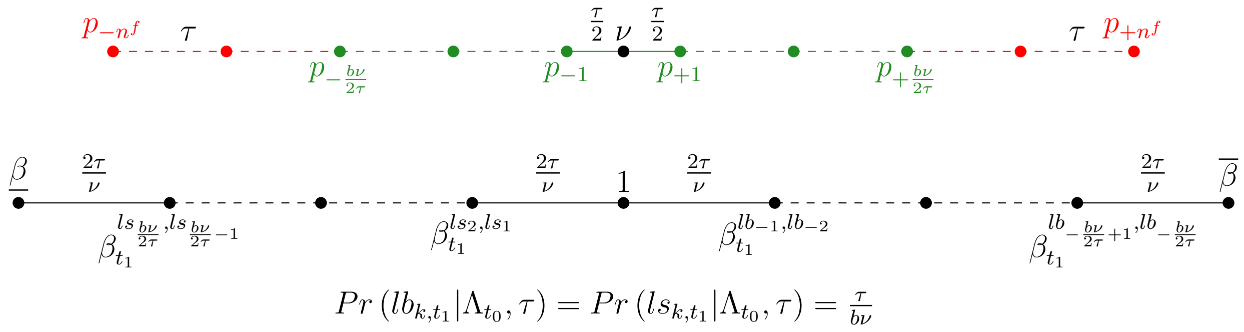
$$
\begin{aligned}
&Pr\Bigg(\Gamma Pr(\Psi_{lb_{-k,t_1}}|\Lambda_{t_0},\tau) + \frac{p_{-k}}{\nu} - \frac{\tau}{\nu} < \beta_{t_1} < \Gamma Pr(\Psi_{lb_{-k,t_1}}|\Lambda_{t_0},\tau) + \frac{p_{-k}}{\nu} + \frac{\tau}{\nu}\Bigg) \\
&= \int_{\Gamma Pr(\Psi_{lb_{-k,t_1}}|\Lambda_{t_0},\tau)+\frac{p_{-k}}{\nu}-\frac{\tau}{\nu}}^{\Gamma Pr(\Psi_{lb_{-k,t_1}}|\Lambda_{t_0},\tau)+\frac{p_{-k}}{\nu}+\frac{\tau}{\nu}} \frac{1}{\Gamma} d\beta = \frac{2\tau}{\Gamma\nu}
\end{aligned}
\tag{44}
$$

Substituting $\Gamma = 2b$ into (44):

$$
Pr\left(lb_{-k,t_1}|\Lambda_{t_0},\tau\right) = \frac{\tau}{b\nu}
\tag{45}
$$

Intuitively, as β_{t_1} is uniformly distributed with semi-support equal to b, the probability that investors use a generic p_k in (43) is equal to the ratio between the relative distance between two consecutive prices, p_{-k} and p_{-k+1} - the relative tick size $\frac{\tau}{\nu}$ - and the semi-support b.

Figure 1.C: **Feasible and Optimal Prices ($\frac{b\nu}{2\tau} \in N^+$)**
In the first line of Figure 1.C we report the feasible prices associated with a generic $b\nu$ and a feasible τ. We highlight in green the optimal set of prices for the 1^{st} player, and in red sub-optimal set of prices. The distance between two consecutive prices is τ and ν is the fundamental asset value. In the second line, we report the $\beta_{t_1}^{\cdot\cdot}$ thresholds for both a buyer $\beta_{t_1}^{lb_{-k},lb_{-k-1}}$ (between 1 and $\overline{\beta}$) and a seller $\beta_{t_1}^{ls_k,ls_{k+1}}$ (between $\underline{\beta}$ and 1). The distance between two consecutive $\beta_{t_1}^{\cdot\cdot}$ thresholds is equal to $\frac{2\tau}{\nu}$; $\overline{\beta}-\underline{\beta}=2b$, and therefore $Pr\left(lb_{k,t_1}|\Lambda_{t_0},\tau\right) = Pr\left(ls_{k,t_1}|\Lambda_{t_0},\tau\right) = \frac{\tau}{b\nu}$.



For completeness, we now also consider the case with $\frac{b\nu}{2\tau} \notin N^+$, although in Appendixes C.2.1 and C.2.3 we show that this case is irrelevant for the SP maximization problem. When $\frac{b\nu}{2\tau} \notin N^+$, the lower bound of p_k is $p_{-(floor(\frac{bv}{2\tau})+1)}$. To determine this lower bound p_k, we first show that the lower bound of the optimal β_{t_1} region where the buyer optimally chooses $p_{-floor(\frac{bv}{2\tau})}$ is strictly greater than 1:

$$
\begin{aligned}
&1 \leq \frac{p_{-floor(\frac{bv}{2\tau})}}{v} + b - 1 + \frac{p_{-floor(\frac{bv}{2\tau})}}{v} - \frac{\tau}{v} \\
&\frac{bv}{2\tau} > floor(\frac{bv}{2\tau})
\end{aligned}
\tag{46}
$$

This means that there exists a β_{t_1} region $(1, \frac{p_{-floor(\frac{bv}{2\tau})}}{v} + b - 1 + \frac{p_{-floor(\frac{bv}{2\tau})}}{v} - \frac{\tau}{v})$, such that the investor will choose $p_{-(floor(\frac{bv}{2\tau})+j)}$. As $k \in N^+$, this region may include at the most another p_k. Therefore the lowest p_k when $\frac{b\nu}{2\tau} \notin N^+$ is $p_{-(floor(\frac{bv}{2\tau})+1)}$.

The submission probabilities for $\forall p_k \in \left[p_{-floor(\frac{bv}{2\tau})}, p_{-1}\right]$ are given by (44) (and hence (45)). For $p_k = p_{-(floor(\frac{bv}{2\tau})+1)}$, (44) becomes:

$$
\int_1^{\frac{p_{-floor(\frac{bv}{2\tau})}}{v}+b-1+\frac{p_{-floor(\frac{bv}{2\tau})}}{v}-\frac{\tau}{v}} \frac{1}{\Gamma} dx = 0.5 - \frac{\tau}{b\nu} \times floor(\frac{bv}{2\tau}) \tag{47}
$$

The submission probability of a limit buy order at $p_{-(floor(\frac{bv}{2\tau})+1)}$ in (47) is the difference between the submission probability of a limit buy order that by Lemma (1) is equal to 0.5, and the cumulative probability of investors choosing a limit buy at any p_k excluding $p_{-(floor(\frac{bv}{2\tau})+1)}$ is $p_{-k} \in \left[p_{-floor(\frac{bv}{2\tau})}, p_{-1}\right]$ which is $\frac{\tau}{b\nu} \times floor(\frac{bv}{2\tau})$.

We can therefore conclude that the optimal order submission probabilities when $\frac{b\nu}{2\tau} \notin N^+$ (Case 2) are:

$$Pr\left(lb_{-k,t_1}|\Lambda_{t_0},\tau\right) = \begin{cases} \dfrac{\tau}{b\nu} & \forall p_k \in \left[p_{-floor(\frac{bv}{2\tau})}, p_{-1}\right] \\ 0.5 - \dfrac{\tau}{b\nu} \times floor(\dfrac{bv}{2\tau}) & \text{if } p_k = p_{-(floor(\frac{bv}{2\tau})+1)} \end{cases} \tag{48}$$

C.2 Welfare Analysis

Equation (9) can we rewritten as :

$$w_{t_1}\left(lb_{t_1}|\tau\right) = \sum_{k=1}^{m} Pr(\Psi_{lb_{-k,t_1}}|\Lambda_{t_0},\tau) \times gain_{-k,t_1} \tag{49}$$

where $gain_{-k,t_1} = \frac{1}{\Gamma}\int_{\beta_{t_1}\in B(\tau)}\left(\beta_{t_1}v - p_{-k}\right)\, d\beta_{t_1}$.

To show how a change in the tick size affects the welfare of the first player, we express the two components in (49), $Pr(\Psi_{lb_{-k,t_1}}|\Lambda_{t_0},\tau)$ and $gain^{t_1}_{-k}$, as a function of a generic tick $\hat{\tau} = \tau + \epsilon$, with $\hat{\tau} > \tau$:[38]

$$Pr(\Psi_{lb_{-k,t_1}}|\Lambda_{t_0},\hat{\tau}) = Pr(\Psi_{lb_{-k,t_1}}|\Lambda_{t_0},\tau) - (k - \tfrac{1}{2})\tfrac{\epsilon}{\Gamma v} \tag{50}$$

$$gain^{\hat{\tau}}_{-k,\ t_1} = gain_{-k,\ t_1} + \tfrac{\epsilon}{\Gamma v}(2bv + (1-2k)\left[\epsilon + 2\tau\right]) \tag{51}$$

Defining $j(k) = \frac{\epsilon}{\Gamma v}(2bv + (1-2k)\left[\epsilon + 2\tau\right])$ in the second term of (51) and substituting (50) and (51) into (49), we obtain the welfare of the t_1 buyer as a function of $\hat{\tau}$:

$$\omega_{t_1}(lb_{t_1}\ |\hat{\tau}) = \sum_{k=1}^{n}\left[Pr(\Psi_{lb_{-k,t_1}}|\Lambda_{t_0},\tau) - (k - \frac{1}{2})\frac{\epsilon}{\Gamma v}\right] \times \left[gain_{-k,t_1} + j(k)\right] \tag{52}$$

[38] By Proposition (1) it follows that for $\hat{\tau} > \tau$ there are $n \leq m$ prices played with positive probability by the trader arriving at t_1.

Taking the difference between (52) and (49) , we obtain:

$$\begin{aligned}\Delta\omega_{t_1}(lb_{t_1}\ |\hat{\tau},\tau) = +\sum_{k=1}^{n} Pr(\Psi_{lb_{-k,t_1}}|\Lambda_{t_0},\hat{\tau})j(k) - \sum_{k=1}^{n}(k-\frac{1}{2})\frac{\epsilon}{\Gamma v}\times gain_{-k,t_1} \\ - \sum_{k=n+1}^{m} Pr(\Psi_{lb_{-k,t_1}}|\Lambda_{t_0},\tau)\times gain_{-k,t_1}\end{aligned} \tag{53}$$

To derive equation (50) we first need to derive p_{-k} with $k \in \left[1,\ floor(\frac{b\nu}{2\tau})+1\right]$ for a generic tick $\hat{\tau}$ such that $\hat{\tau} = \tau + \epsilon$

$$\begin{aligned}p_{-k}^{\hat{\tau}} &= v - (k-\frac{1}{2})\hat{\tau} \\ &= v-(k-\frac{1}{2})(\tau+\epsilon) \\ &= p_{-k} - (k-\frac{1}{2})\epsilon\end{aligned} \tag{54}$$

Using (5), we now derive equation (50):

$$\begin{aligned}&Pr(\Psi_{lb_{-k,t_1}}|\Lambda_{t_0},\hat{\tau}) = Pr\left(p_{-k}^{\hat{\tau}} - \beta_{t_2}v > 0\right) \\ &= \frac{1}{\Gamma}\left[\frac{p_{-k}^{\hat{\tau}}}{v} - (1-b)\right] = \frac{1}{\Gamma}\left[\frac{p_{-k}^{\tau}}{v} - (k-\frac{1}{2})\frac{\epsilon}{v} - (1-b)\right] \\ &= Pr(\Psi_{lb_{-k,t_1}}|\Lambda_{t_0},\tau) - (k-\frac{1}{2})\frac{\epsilon}{v\Gamma}\end{aligned} \tag{55}$$

To derive (51), we write the $gain_{-k,t_1}$ as a function of $\hat{\tau}$, for a generic $p_{-k} \in \left[-(floor(\frac{b\nu}{2\hat{\tau}}+1),\ -1\right]$; and using condition (36) in Proposition (1) we obtain the optimal β_{t_1} region for a generic p_{-k} $\beta_{t_1} \in B(\hat{\tau}) = \left\{\beta_{t_1}^{lb_{-(k+1)},lb_{-k}}, \beta_{t_1}^{lb_{-k},lb_{-(k-1)}}\right\}$:

$$\begin{aligned}&gain_{-k,\ t_1}^{\hat{\tau}} = \left[\int_{\beta_{t_1}\in B(\hat{\tau})} \frac{\beta_{t_1}v - p_{-k}^{\hat{\tau}}}{\Gamma} d\beta_{t_1}\right] = \\ &\frac{v}{2\Gamma}\left[(\beta_{t_1}^{lb_{-k},lb_{-(k-1)}})^2 - (\beta_{t_1}^{lb_{-(k+1)},lb_{-k}})^2\right] - \frac{p_{-k}^{\hat{\tau}}}{\Gamma}\left[\beta_{t_1}^{lb_{-k},lb_{-(k-1)}} - \beta_{t_1}^{lb_{-(k+1)},lb_{-k}}\right]\end{aligned} \tag{56}$$

where the β_{t_1} thresholds corresponding to the extremes of integration are:

$$\begin{aligned}\beta_{t_1}^{lb_{-(k+1)},lb_{-k}} &:= (\beta_{t_1}v - p_{-k}^{\hat{\tau}})Pr(\Psi_{lb_{-k,t_1}}|\Lambda_{t_0},\hat{\tau}) = (\beta_{t_1}v - p_{-(k+1)}^{\hat{\tau}})Pr(\Psi_{lb_{-(k+1),t_1}}|\Lambda_{t_0},\hat{\tau}) \\ \beta_{t_1}^{lb_{-k},lb_{-(k-1)}} &:= (\beta_{t_1}v - p_{-k}^{\hat{\tau}})Pr(\Psi_{lb_{-k,t_1}}|\Lambda_{t_0},\hat{\tau}) = (\beta_{t_1}v - p_{-(k-1)}^{\hat{\tau}})Pr(\Psi_{lb_{-(k-1),t_1}}|\Lambda_{t_0},\hat{\tau})\end{aligned} \tag{57}$$

To obtain the β_{t_1} thresholds we express prices and execution probabilities in (57) as function of p_{-k}:

$$
\begin{aligned}
&p^{\hat{\tau}}_{-(k+1)} = p_{-k} - \tau - (k+1-\frac{1}{2})\epsilon \\
&Pr(\Psi_{lb_{-(k+1),t_1}}|\Lambda_{t_0},\hat{\tau}) = Pr(\Psi_{lb_{-k,t_1}}|\Lambda_{t_0},\tau) - (k+1-\frac{1}{2})\frac{\epsilon}{\Gamma v} - \frac{\tau}{\Gamma v} \\
&p^{\hat{\tau}}_{-(k-1)} = p_{-k} + \tau - (k-1-\frac{1}{2})\epsilon \\
&Pr(\Psi_{lb_{-(k-1),t_1}}|\Lambda_{t_0},\hat{\tau}) = Pr(\Psi_{lb_{-k,t_1}}|\Lambda_{t_0},\tau) - (k-1-\frac{1}{2})\frac{\epsilon}{\Gamma v} + \frac{\tau}{\Gamma v}
\end{aligned}
\tag{58}
$$

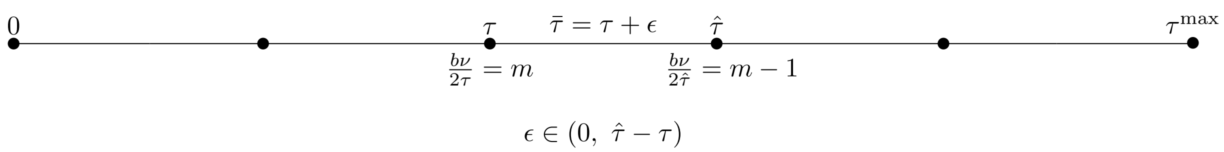
Substituting (50), (54) and (58) into (57) we obtain the β_{t_1} threshold that we substitute into (56) to derive (51).

C.2.1 Proof of Corollary 1.1

In the first part of the proof, we show that the SP can restrict its maximization problem to the set of tick sizes associated with $\frac{b\nu}{2\tau} \in N^+$ which correspond to the first part of Proposition (1). This is because in this section we prove that for any tick size such that $\frac{b\nu}{2\tau} \notin N^+$, there exists at least one tick size such that $\frac{b\nu}{2\tau} \in N^+$ with an associated greater welfare.

Without loss of generality, following Proposition (1) we consider a tick size τ such that $\frac{b\nu}{2\tau} \in N^+$ that defines m prices with equal positive submission probabilities at t_1. We then consider the next $\hat{\tau} > \tau$ that defines $m-1$ prices with equal positive submission probabilities at t_1 (Figure 2.C). As shown in the proof of Proposition (1), any tick size $\bar{\tau} = \tau + \epsilon$ with $\epsilon \in (0,\ \hat{\tau} - \tau)$ in between τ and $\hat{\tau}$, $(\bar{\tau} \mid \tau < \bar{\tau} < \hat{\tau})$, associated with $\frac{b\nu}{2\bar{\tau}} \notin N^+$, will define m prices with positive (not equal) submission probabilities. We now show that for the 1^{st} trader the welfare associated with any $\bar{\tau}$ is smaller than the welfare associated with τ and therefore the SP can restrict its maximization problem to the set of tick sizes associated with $\frac{b\nu}{2\tau} \in N^+$.

Figure 2.C: τ domain



The incremental difference of welfare between $\omega_{t_1}(lb_{t_1}\ |\tau)$ and $\omega_{t_1}(lb_{t_1}\ |\bar{\tau})$ is :

$$\begin{aligned}
&\Delta\omega_{t_1}(lb_{t_1}\ |\bar{\tau},\tau) = \omega_{t_1}(lb_{t_1}\ |\bar{\tau}) - \omega_{t_1}(lb_{t_1}\ |\tau)\\
&= \left[Pr(\Psi_{lb_{-m,t_1}}|\Lambda_{t_0},\bar{\tau}) \times gain^{\bar{\tau}}_{-m,t_1} - Pr(\Psi_{lb_{-m,t_1}}|\Lambda_{t_0},\tau) \times gain_{-m,t_1}\right] + \\
&+ \sum_{k=1}^{m-1} Pr(\Psi_{lb_{-k,t_1}}|\Lambda_{t_0},\bar{\tau})\, j(k) - \sum_{k=1}^{m-1}(k-\frac{1}{2})\frac{\epsilon}{\Gamma v} \times gain_{-k,t_1}
\end{aligned} \tag{59}$$

Considering the lower and upper optimal bounds of $gain^{\bar{\tau}}_{-m,t_1}$ used in equation (47) we obtain:

$$gain^{\bar{\tau}}_{-m,t_1} = \int_1^{2^{\frac{p^{\bar{\tau}}_{m-1}}{\nu}+b-1-\frac{\tau}{v}}} \frac{\beta_{t_1}\nu - p^{\bar{\tau}}_{-m}}{\Gamma} d\beta_{t_1} \tag{60}$$

Using the definition of derivative, we know that $\omega_{t_1}(lb_{t_1}\ |\tau)'$ in the neighborhood of $\epsilon \in (0,\ \hat{\tau}-\tau)$ is equal to

$$\omega_{t_1}(lb_{t_1}\ |\tau)' = \lim_{\epsilon\to 0} \frac{\Delta\omega_{t_1}(lb_{t_1}\ |\bar{\tau},\tau)}{\epsilon} = -O(c) \tag{61}$$

where c is a constant, hence the welfare $\omega_{t_1}(lb_{t_1}\ |\tau)$ is decreasing in τ in the interval $\epsilon \in (0,\ \hat{\tau}-\tau)$. Therefore, the subset of ticks that the SP must consider to determine the optimal welfare for the 1^{st} investor is defined by $\tau \in (0,\ \tau^{max})$ such that $\frac{b\nu}{2\tau} \in N^+$.

To show that $\Delta\omega_{t_1}(lb_{t_1}\ |\hat{\tau},\tau) < 0$ in equation(53), we choose τ and $\hat{\tau}$ such that according to Proposition (1) the buyer at t_1 chooses $m = \frac{b\nu}{2\tau}$ and $n = \frac{b\nu}{2\hat{\tau}}$ optimal p_{-k} prices respectively. Given that $\hat{\tau} = \tau + \epsilon$, setting $\epsilon = \frac{bv}{m} \implies m = 3n$. Hence we re-write (53) as:

$$\begin{aligned}
\Delta\omega_{t_1}(lb_{t_1}\ |\hat{\tau},\tau) = &- \sum_{k=n+1}^{3n} Pr(\Psi_{lb_{-k,t_1}}|\Lambda_{t_0},\tau)\, gain_{-k,t_1} + \sum_{k=1}^{n} Pr(\Psi_{lb_{-k,t_1}}|\Lambda_{t_0},\hat{\tau})\, j(k)\\
&- \sum_{k=1}^{n}(k-\frac{1}{2})\frac{2\tau}{\Gamma v} \times gain_{-k,t_1}
\end{aligned} \tag{62}$$

We consider the first line in (62).

$$
\begin{aligned}
& -\sum_{k=n+1}^{3n} Pr(\Psi_{lb_{-k,t_1}}|\Lambda_{t_0},\tau)\, gain_{-k,t_1} + \sum_{k=1}^{n} Pr(\Psi_{lb_{-k,t_1}}|\Lambda_{t_0},\hat{\tau})\, j(k) \\
& = \frac{bv}{6} - \frac{bv}{18} - \frac{bv}{6} + \frac{bv}{18} - \frac{\tau}{12} - \sum_{k=1}^{n}\left(k-\frac{1}{2}\right)\frac{\hat{\tau}}{\Gamma v} j(k) \\
& = -\frac{\tau}{12} - \sum_{k=1}^{n}\left(k-\frac{1}{2}\right)\frac{\hat{\tau}}{\Gamma v} j(k)
\end{aligned}
\tag{63}
$$

where the first term in equation (63) can be written as:

$$
\begin{aligned}
& -\sum_{k=n+1}^{3n} Pr(\Psi_{lb_{-k,t_1}}|\Lambda_{t_0},\tau) gain_{-k,t_1} \approx -\sum_{k=n+1}^{3n} \tau\left[0.5 - \left(k-\frac{1}{2}\right)\frac{\tau}{\Gamma v}\right] \\
& = -\frac{bv}{6} + \left(\frac{8n^2\tau^2}{2\Gamma v}\right) = -\frac{bv}{6} + \frac{bv}{18}
\end{aligned}
\tag{64}
$$

and using (5), the second term in equation (63) can be written as:

$$
\begin{aligned}
& \sum_{k=1}^{n} Pr(\Psi_{lb_{-k,t_1}}|\Lambda_{t_0},\hat{\tau})\, j(k) = \sum_{k=1}^{n}\left[0.5 - \left(k-\frac{1}{2}\right)\frac{\hat{\tau}}{\Gamma v}\right] j(k) \\
& = 0.5\sum_{k=1}^{n} j(k) - \sum_{k=1}^{n}\left(k-\frac{1}{2}\right)\frac{\hat{\tau}}{\Gamma v} j(k)
\end{aligned}
\tag{65}
$$

Using our previous definition of $j(k)$ we can write:

$$
\begin{aligned}
j(k) &= \frac{\epsilon}{\Gamma v}(2bv + (1-2k)\left[\epsilon + 2\tau\right]) \\
&= \frac{2\tau}{\Gamma v}(2b\nu + (1-2k)2\frac{b\nu}{m}) \\
&= \frac{2\tau}{\Gamma v}(4b\nu - 4b\nu\frac{k}{m})
\end{aligned}
\tag{66}
$$

Given that $Pr(\Psi_{lb_{-k,t_1}}|\Lambda_{t_0},\tau) > 0$, and that $j(k) > 0 \;\; \forall k \in [1,\; n]$ given that $Pr(\Psi_{lb_{-k,t_1}}|\Lambda_{t_0},\tau) > 0$, and that $j(k) > 0 \;\; \forall k \in [1,\; n]$: we can conclude that the difference between the two terms in the first line of (62) is negative.

Considering that the term in the second line of (62) is negative as $gain_{-k,t_1} > 0$ and $k > \frac{1}{2}$, we can also conclude that $\Delta\omega_{t_1}(lb_{t_1}\,|\hat{\tau},\tau) < 0$

C.2.2 Example Corollary 1

Consider first the payoff of an investor arriving at time t_1 characterized by a random personal evaluation, $\beta_{t_1}\nu > \nu$. Combining equations (3) and (5), the first player payoff from a limit buy order is equal to:

$$(\beta_{t_1}\nu - p)\left[\frac{p}{\Gamma\nu} - \frac{1-b}{\Gamma}\right] \tag{67}$$

Given (67), the investor is willing to post his limit buy order at the price, p, that maximizes his payoff. By taking the first and second order conditions of equation (67), we obtain the quoting price associated with the highest payoff for the 1^{st} player:

$$p^{\star} = \frac{\nu}{2}\left(\beta_{t_1} + 1 - b\right) \tag{68}$$

Intuitively, the smaller the tick size, the greater the probability that the 1^{st} player will be able to quote a p_k closer to $p^{\star}$. Now, given (68), there exists at least one investor - i.e., the one with the largest gains from trade $\beta_{t_1} = 1 + b$ - whose optimal price is $p^{\star} = \nu$. However, $p^{\star} = \nu$ only if $\tau = 0$, meaning that at least for one investor decreasing the tick sizes strictly increases his welfare. As for the remaining players there is no welfare loss, we can conclude that a decreasing τ is Pareto efficient for the 1^{st} player.

C.2.3 Proof of Corollary 2

Equation (11) can be rewritten as:

$$\omega_{t_2}\left(ms_{t_2}|\tau\right) = \sum_{k=1}^{m} Pr\left(lb_{-k,t_1}|\Lambda_{t_0},\tau\right) gain_{-k,t_2} \tag{69}$$

where $gain_{-k,t_2} = \int_{(1-b)}^{\frac{p_{-k}}{v}} \frac{p_{-k}-\beta_{t_2}v}{\Gamma} d\beta_{t_2}$. As for the t_1 investor, we express $gain_{-k,t_2}$ as a function of $\hat{\tau}$:

$$gain^{\hat{\tau}}_{-k,t_2} = gain_{-k,t_2} + (k - \frac{1}{2})\frac{\epsilon}{v\Gamma}\left[(1-b)v - \frac{1}{2}(p_{-k} + p^{\hat{\tau}}_{-k})\right] \tag{70}$$

The term $(k-\frac{1}{2})\frac{\epsilon}{v\Gamma}\left[(1-b)v - \frac{1}{2}(p_{-k}+p^{\hat{\tau}}_{-k})\right] = h(k)$ is negative by construction being the product of a positive term, $(k-\frac{1}{2})\frac{\epsilon}{v\Gamma}$, and a negative term, $\left[(1-b)v - \frac{1}{2}(p_{-k}+p^{\hat{\tau}}_{-k})\right]$. This last term is the difference between the valuation lower bound and the average of two p_{-k} prices obtained as a function of τ and $\hat{\tau}$ respectively. By definition, these two prices are *feasible* and therefore they are greater than $(1-b)\nu$. By substituting (70) into (69), we can therefore write (69) as a function of $\hat{\tau}$:

$$\omega_{t_2}(ms_{t_2}|\hat{\tau}) = \sum_{k=1}^{n} Pr\left(lb_{-k,t_1}|\Lambda_{t_0},\hat{\tau}\right)\left[gain_{-k,t_2} + h(k)\right] \tag{71}$$

The difference between (71) and (69) is the difference between the t_2 investor's welfare computed as a function of τ and of $\hat{\tau}$:

$$\begin{aligned}
&\Delta\omega_{t_2}(ms_{t_2}\;|\hat{\tau},\tau) \\
&= \sum_{k=1}^{n} Pr\left(lb_{-k,t_1}|\Lambda_{t_0},\hat{\tau}\right)\left[gain_{-k,t_2} + h(k)\right] - \sum_{k=1}^{m} Pr\left(lb_{-k,t_1}|\Lambda_{t_0},\tau\right) gain_{-k,t_2}
\end{aligned} \tag{72}$$

Given the *gain* expression for a generic $k \in \left[1,\ floor(\frac{b\nu}{2\hat{\tau}})+1\right]$ as a function of $\hat{\tau} = \tau + \epsilon$

$$gain^{\hat{\tau}}_{-k,t_2} = \frac{1}{\Gamma}\int_{1-b}^{\frac{p^{\hat{\tau}}_{-k}}{v}} (p^{\hat{\tau}}_{-k} - \beta_{t_2}v)d\beta_{t_2} \tag{73}$$

if we substitute (54) in (73):

$$gain^{\hat{\tau}}_{-k,t_2} = \frac{1}{\Gamma}\int_{1-b}^{\frac{p_{-k}-(k-\frac{1}{2})\epsilon}{v}} (p_{-k} - (k-\frac{1}{2})\epsilon - \beta_{t_2}v)d\beta_{t_2}, \tag{74}$$

it is possible to write the above integral as:

$$gain^{\hat{\tau}}_{-k,t_2,} = \frac{1}{\Gamma}\int_{1-b}^{\frac{p_{-k}-(k-\frac{1}{2})\epsilon}{v}} (p_{-k} - \beta_{t_2}v)d\beta_{t_2} - \frac{1}{\Gamma}\int_{1-b}^{\frac{p_{-k}-(k-\frac{1}{2})\epsilon}{v}} (k-\frac{1}{2})\epsilon\; d\beta_{t_2} \tag{75}$$

Using the integral property $\int_a^{b-c} = \int_a^b - \int_{b-c}^b$, we can re-write (75) as:

$$gain^{\hat{\tau}}_{-k,t_2} = \frac{1}{\Gamma}\int_{1-b}^{\frac{p_{-k}}{v}} (p_{-k} - \beta_{t_2} v) d\beta_{t_2} - \frac{1}{\Gamma}\int_{\frac{p_{-k}-(k-\frac{1}{2})\epsilon}{v}}^{\frac{p_{-k}}{v}} (p_{-k} - \beta_{t_2} v) d\beta_{t_2} - \frac{1}{\Gamma}\int_{1-b}^{\frac{p_{-k}-(k-\frac{1}{2})\epsilon}{v}} (k - \frac{1}{2})\epsilon \, d\beta_{t_2}$$
$$= gain_{-k,t_2} + (k-\frac{1}{2})\frac{\epsilon}{\Gamma v}\left[(1-b)\nu - \frac{1}{2}\left(p_{-k} + p^{\hat{\tau}}_{-k}\right)\right] \tag{76}$$

In the same spirit of the proof of Corollary 1.1, we now show that the SP can also restrict its maximization problem for the investor arriving at t_2 to the tick sizes such that $\frac{b\nu}{2\tau} \in N^+$. Hence, as we did in the proof of Corollary 1.1, we consider τ and $\hat{\tau}$ that define respectively m and $m-1$ prices with positive submission probabilities at t_1 (Figure 2.C). Following the case $\frac{b\nu}{2\tau} \notin N^+$ in the proof of Proposition (1), each $\bar{\tau} = \tau + \epsilon$ with $\epsilon \in (0,\ \hat{\tau} - \tau)$ also defines m prices with positive probabilities. As before we show that the incremental difference of the second player's welfare between $\omega_{t_2}(ms_{t_2}\ |\bar{\tau})$ and $\omega_{t_2}(ms_{t_2}\ |\tau)$ is :

$$\Delta\omega_{t_2}(ms_{t_2}\ |\bar{\tau},\tau) = \omega_{t_2}(ms_{t_2}\ |\bar{\tau}) - \omega_{t_2}(ms_{t_2}\ |\tau)$$
$$= \left[\left(0.5 - (m-1)\times\frac{\tau+\epsilon}{b\nu}\right)\times gain^{\bar{\tau}}_{-m,t_2} - \frac{\tau}{b\nu}\times gain_{-m,t_2}\right] \tag{77}$$
$$+ \sum_{k=1}^{m-1}\frac{\tau+\epsilon}{b\nu}h(k) + \sum_{k=1}^{m-1}\frac{\epsilon}{bv}\times gain_{-k,t_2}$$

As in the proof of Corollary 1.1, $\omega_{t_2}(ms_{t_2}\ |\tau)'$ in the neighborhood of $\epsilon \in (0,\ \hat{\tau} - \tau)$ is equal to

$$\omega_{t_2}(ms_{t_2}\ |\tau)' = \lim_{\epsilon\to 0}\frac{\Delta\omega_{t_2}(ms_{t_2}\ |\bar{\tau},\tau)}{\epsilon} = -O(c) \tag{78}$$

where c is a constant, hence the welfare $\omega_{t_2}(ms_{t_2}\ |\tau)$ is decreasing in τ in the interval $\epsilon \in (0,\ \hat{\tau}-\tau)$.

Therefore, the subset of ticks that the SP must consider to determine the optimal welfare for the second investor is defined by $\tau \in (0,\ \tau^{max})$ such that $\frac{b\nu}{2\tau} \in N^+$.

To show that $\Delta\omega_{t_2}(ms_{t_2}\ |\hat{\tau},\tau) < 0$ in (72), as before we choose τ and $\hat{\tau}$ such that according to Proposition (1) the buyer at t_1 chooses $m = \frac{b\nu}{2\tau}$ and $n = \frac{b\nu}{2\hat{\tau}}$ optimal p_{-k} prices respectively. Given that $\hat{\tau} = \tau + \epsilon$, setting $\epsilon = \frac{bv}{m} \implies m = 3n$. If we consider in Proposition (1), $\frac{b\nu}{2\tau} \in N^+$, the limit buy submission probabilities are constant and equal to:

$$\begin{aligned}Pr\left(lb_{-k,t_1}|\Lambda_{t_0},\tau\right) &= \frac{\tau}{bv} \\ Pr\left(lb_{-k,t_1}|\Lambda_{t_0},\hat{\tau}\right) &= \frac{\hat{\tau}}{bv} = \frac{3\tau}{bv}\end{aligned} \tag{79}$$

Substituting (79) into (72):

$$\begin{aligned}\Delta\omega_{t_2}(ms_{t_2}\,|\hat{\tau},\tau) &= \sum_{k=1}^{n}\frac{3\tau}{bv}\left[gain_{-k,t_2}+h(k)\right]-\sum_{k=1}^{3n}\frac{\tau}{bv}gain_{-k,t_2} \\ &= \sum_{k=1}^{n}gain_{-k,t_2}\frac{2\tau}{bv}-\sum_{k=n+1}^{3n}gain_{-k,t_2}\frac{\tau}{bv}+\sum_{k=1}^{n}h(k)\frac{3\tau}{bv}\end{aligned} \tag{80}$$

The gain expression for the investor at t_2 can be written:

$$\begin{aligned}gain_{-k,t_2} &= \frac{\left(bv-\frac{2k-1}{2}\tau\right)^2}{2\Gamma v} \\ &= \frac{(bv)^2}{2\Gamma v}+(\frac{2k-1}{2})^2\frac{\tau^2}{2\Gamma v}-bv\frac{2k-1}{2\Gamma v}\tau\end{aligned} \tag{81}$$

Hence we can now decompose each term of the right-hand side of equation (80)

$$\begin{aligned}&\sum_{k=1}^{n}gain_{-k,t_2}2\tau = \frac{(bv)^2}{\Gamma v}\tau n-\frac{1}{2}(\tau n)^2+\frac{(\tau n)^3}{3\Gamma v}-\frac{\tau^3 n}{12\Gamma v} \\ &-\sum_{k=n+1}^{3n}gain_{-k,t_2}\tau = -\frac{(bv)^2}{\Gamma v}\tau n+2(\tau n)^2-\frac{13(\tau n)^3}{3\Gamma v}+\frac{\tau^3 n}{12\Gamma v} \\ &+\sum_{k=1}^{n}h(k)3\tau = -\frac{3(\tau n)^2}{2}+\frac{4(\tau n)^3}{v\Gamma}-\frac{\tau^3 n}{\Gamma v}\end{aligned} \tag{82}$$

Substituting (82) into (80), we obtain:

$$\Delta\omega_{t_2}(ms_{t_2}\,|\hat{\tau},\tau) = -\frac{\tau^3 n}{\Gamma v} < 0 \tag{83}$$

C.2.4 Proof of Proposition 2

Thanks to Corollary (1), given $\hat{\tau} = \tau + \epsilon$ with $\epsilon > 0$, $\forall\ \hat{\tau} \in (0, \tau^{max})$ we know that:

$$\omega_{t_1}(lb_{k,t_1}|\tau) > \omega_{t_1}(lb_{k,t_1}|\hat{\tau}) \quad \wedge \quad \omega_{t_2}(ms_{k,t_2}|\tau) > \omega_{t_2}(ms_{k,t_2}|\hat{\tau}) \tag{84}$$

hence both functions in (7) are weakly decreasing in $(0, \tau^{max})$. We can therefore conclude that the *argmax* of (7) is 0.

D Appendix: Three Period Model

D.1 Model Solution

We solve the 3-period trading game by backward induction.

D.1.1 Period t_3

As for the 2-period trading game, the optimal order submission probabilities of investors arriving at t_3 are defined by Lemma (1.3).

D.1.2 Period t_2

We now derive the optimal order submission strategies at t_2. At t_1 the book opens empty. In addition, by Lemma (1) we know that at t_1 the incoming investor posts either a limit buy (if his $\beta_{t_1} > 1$) or a limit sell (if his $\beta_{t_1} < 1$) at p_k. Therefore, given that at t_2 the book symmetrically opens either with a limit buy or with a limit sell, without loss of generality we can consider a buyer arriving at t_1 so that the book opens with a limit buy at t_2. Hence, the incoming 2^{nd} player can either hit the previously posted limit buy by market selling at p_k, or limit sell at $p_j > p_k$, or he can limit buy still at $p_j > p_k$, or decide not to trade (nt).

For a generic limit buy posted by the first player at p_k, the probability that the 2^{nd} player selects a market sell is given by:

$$
\begin{aligned}
&Pr\left(ms_{k,t_2}|\Lambda_{t_1},\tau\right) = \\
&Pr\Big(p_k - \beta_{t_2}\nu > 0, \\
&p_k - \beta_{t_2}\nu > (p_j - \beta_{t_2}\nu)Pr\left(mb_{j,t_3}|\Lambda_{t_2},\tau\right), \\
&p_k - \beta_{t_2}\nu > (\beta_{t_2}\nu - p_j)Pr\left(ms_{j,t_3}|\Lambda_{t_2},\tau\right)\Big)
\end{aligned}
\tag{85}
$$

Equation (85) guarantees that market selling is more profitable than any other possible action the 2^{nd} player can take: nt, limit sell or limit buy at $p_j > p_k$. If the 1^{st} player submits a limit buy at the most aggressive price level p_{+n^f}, he locks the book in such a way that the 2^{nd} player cannot supply liquidity but only market sell at p_{+n^f}. In this special case, equation (85) reduces to:

$$
Pr\left(ms_{+n^f,t_2}|\Lambda_{t_1},\tau\right) = Pr\left(p_{+n^f} - \beta_{t_2}\nu > 0\right) \tag{86}
$$

The probability that the 2^{nd} player selects a limit sell order at a price $p_j > p_k$ is:

$$
\begin{aligned}
&Pr\left(ls_{j,t_2}|\Lambda_{t_1},\tau\right) = \\
&Pr\Big((p_j - \beta_{t_2}\nu)Pr\left(mb_{j,t_3}|\Lambda_{t_2},\tau\right) > 0, \\
&(p_j - \beta_{t_2}\nu)Pr\left(mb_{j,t_3}|\Lambda_{t_2},\tau\right) > p_k - \beta_{t_2}\nu, \\
&(p_j - \beta_{t_2}\nu)Pr\left(mb_{j,t_3}|\Lambda_{t_2},\tau\right) > (p_{\tilde{j}} - \beta_{t_2}\nu)Pr\left(mb_{\tilde{j},t_3}|\Lambda_{t_2},\tau\right), \\
&(p_j - \beta_{t_2}\nu)Pr\left(mb_{j,t_3}|\Lambda_{t_2},\tau\right) > (\beta_{t_2}\nu - p_j)Pr\left(ms_{j,t_3}|\Lambda_{t_2},\tau\right)\Big)
\end{aligned}
\tag{87}
$$

where $p_{\tilde{j}} > p_k$ is a generic price different from p_j and still greater than p_k. In the special case in which the 1^{st} player submits a limit buy at the most aggressive price level p_{+n^f}, the probability of a limit sell is zero.

The probability that the 2^{nd} player selects a limit buy order at $p_j > p_k$ thus undercutting the limit buy order posted at t_1 is:

$$\begin{aligned}
& Pr\left(lb_{j,t_2}|\Lambda_{t_1},\tau\right) = \\
& Pr\big((\beta_{t_2}\nu - p_j)Pr\left(ms_{j,t_3}|\Lambda_{t_2},\tau\right) > 0, \\
& (\beta_{t_2}\nu - p_j)Pr\left(ms_{j,t_3}|\Lambda_{t_2},\tau\right) > p_k - \beta_{t_2}\nu, \\
& (\beta_{t_2}\nu - p_j)Pr\left(ms_{j,t_3}|\Lambda_{t_2},\tau\right) > (p_j - \beta_{t_2}\nu)Pr\left(mb_{j,t_3}|\Lambda_{t_2},\tau\right), \\
& (\beta_{t_2}\nu - p_j)Pr\left(ms_{j,t_3}|\Lambda_{t_2},\tau\right) > (\beta_{t_2}\nu - p_{\tilde{j}})Pr\left(ms_{\tilde{j},t_3}|\Lambda_{t_2},\tau\right)\big)
\end{aligned} \tag{88}$$

In the special case in which the 1^{st} player locks the market and submits a limit buy at the most aggressive price level p_{+n^f}, the probability of a limit buy at t_2 is zero. Finally, if the 1^{st} player submits a limit buy at $p_k < p_{+n^f}$, the probability that the 2^{nd} player chooses nt is zero:

$$\begin{aligned}
& Pr\left(nt_{k,t_2}|\Lambda_{t_1},\tau\right) = \\
& Pr\big(0 > p_k - \beta_{t_2}\nu, \\
& 0 > (p_j - \beta_{t_2}\nu)Pr\left(mb_{j,t_3}|\Lambda_{t_2},\tau\right), \\
& 0 > (\beta_{t_2}\nu - p_j)Pr\left(ms_{j,t_3}|\Lambda_{t_2},\tau\right)\big)
\end{aligned} \tag{89}$$

Given that both $Pr\left(mb_{j,t_3}|\Lambda_{t_2},\tau\right)$ and $Pr\left(ms_{j,t_3}|\Lambda_{t_2},\tau\right)$ are positive, the second and third condition in (89) reduces to:

$$p_j > \beta_{t_2}v > p_j \tag{90}$$

which is impossible and therefore nt is a dominated strategy. If instead the 1^{st} player submits a limit buy at p_{+n^f}, the probability that the 2^{nd} player chooses $+nt$ is given by:

$$Pr\left(nt_{+n^f,t_2}|\Lambda_{t_1},\tau\right) = Pr\big(p_{+n^f} < \beta_{t_2}\nu\big) \tag{91}$$

D.1.3 Period t_1

Without loss of generality, using Lemma (1), if the 1^{st} player at t_1 is a buyer ($\beta_{t_1} > 1$), he can either limit buy at $p_k < p_{+n^f}$, or limit buy at the most aggressive price p_{+n^f}. The submission probability of a limit buy at price p_k is:

$$\begin{aligned}
&Pr\left(lb_{k,t_1}|\Lambda_{t_0},\tau\right)= \\
&Pr\Big((\beta_{t_1}\nu-p_k)\left[Pr\left(ms_{k,t_2}|\Lambda_{t_1},\tau\right)+\sum_{j>k}Pr\left(ls_{j,t_2}|\Lambda_{t_1},\tau\right)Pr\left(ms_{k,t_3}|\Lambda_{t_2},\tau\right)\right]>0, \\
&(\beta_{t_1}\nu-p_k)\left[Pr\left(ms_{k,t_2}|\Lambda_{t_1},\tau\right)+\sum_{j>k}Pr\left(ls_{j,t_2}|\Lambda_{t_1},\tau\right)Pr\left(ms_{k,t_3}|\Lambda_{t_2},\tau\right)\right]> \\
&(\beta_{t_1}\nu-p_{\tilde{k}})\left[Pr\left(ms_{\tilde{k},t_2}|\Lambda_{t_1},\tau\right)+\sum_{j>\tilde{k}}Pr\left(ls_{j,t_2}|\Lambda_{t_1},\tau\right)Pr\left(ms_{\tilde{k},t_3}|\Lambda_{t_2},\tau\right)\right], \\
&(\beta_{t_1}\nu-p_k)\left[Pr\left(ms_{k,t_2}|\Lambda_{t_1},\tau\right)+\sum_{j>k}Pr\left(ls_{j,t_2}|\Lambda_{t_1},\tau\right)Pr\left(ms_{k,t_3}|\Lambda_{t_2},\tau\right)\right]> \\
&(\beta_{t_1}\nu-p_{+n^f})\big[Pr\left(ms_{+n^f,t_2}|\Lambda_{t_1},\tau\right)+Pr\left(nt_{+n^f,t_2}|\Lambda_{t_1},\tau\right)Pr\left(ms_{+n^f,t_3}|\Lambda_{t_2},\tau\right)\big]\Big)
\end{aligned} \tag{92}$$

where $p_{\tilde{k}} < p_{+n^f}$ different from p_k, and all the expressions within square brackets are the probability of execution of a limit buy respectively at p_k, $p_{\tilde{k}}$ and p_{+n^f} (second, forth and sixth line of (184)). In the extreme case of a limit buy at p_{+n^f}, the probability of submission is:

$$\begin{aligned}
&Pr\left(lb_{k,t_1}|\Lambda_{t_0},\tau\right)= \\
&Pr\Big((\beta_{t_1}\nu-p_{+n^f})\big[Pr\left(ms_{+n^f,t_2}|\Lambda_{t_1},\tau\right)+Pr\left(nt_{+n^f,t_2}|\Lambda_{t_1},\tau\right)Pr\left(ms_{+n^f,t_3}|\Lambda_{t_2},\tau\right)\big]>0, \\
&(\beta_{t_1}\nu-p_{+n^f})\big[Pr\left(ms_{+n^f,t_2}|\Lambda_{t_1},\tau\right)+Pr\left(nt_{+n^f,t_2}|\Lambda_{t_1},\tau\right)Pr\left(ms_{+n^f,t_3}|\Lambda_{t_2},\tau\right)\big]> \\
&(\beta_{t_1}\nu-p_k)\big[Pr\left(ms_{k,t_2}|\Lambda_{t_1},\tau\right)+\sum_{j>k}Pr\left(ls_{j,t_2}|\Lambda_{t_1},\tau\right)Pr\left(ms_{k,t_3}|\Lambda_{t_2},\tau\right)\big]\Big)
\end{aligned} \tag{93}$$

D.2 Welfare Equations

In this Appendix, we report the welfare of the three players in the 3-period model. The welfare of the 1^{st} player is given by:

$$\begin{aligned}
&\omega_{t_1}(lb_{t_1}\ |\tau) = \\
&\sum_{k=-n^f}^{+n^f-1} \left[Pr\left(ms_{k,t_2}|\Lambda_{t_1},\tau\right) + Pr\left(ls_{k+1,t_2}|\Lambda_{t_1},\tau\right) Pr\left(ms_{k,t_3}|\Lambda_{t_2},\tau\right)\right] \frac{1}{\Gamma}\int_{\beta_{t_1}\in B(\tau)} \left(\beta_{t_1}v - p_k\right) d\beta_{t_1} + \\
&\left[Pr\left(ms_{+n^f,t_2}|\Lambda_{t_1},\tau\right) + Pr\left(nt_{+n^f,t_2}|\Lambda_{t_1},\tau\right) Pr\left(ms_{+n^f,t_3}|\Lambda_{t_2},\tau\right)\right] \frac{1}{\Gamma}\int_{\beta_{t_1}\in B(\tau)} \left(\beta_{t_1}v - p_{+n^f}\right) d\beta_{t_1} + \\
&\mathbb{1}_G\Big\{\sum_{k=-n^f}^{+n^f-1}\sum_{j>k+1} Pr\left(ls_{j,t_2}|\Lambda_{t_1},\tau\right) Pr\left(ms_{k,t_3}|\Lambda_{t_2},\tau\right) \frac{1}{\Gamma}\int_{\beta_{t_1}\in B(\tau)} \left(\beta_{t_1}v - p_k\right) d\beta_{t_1}\ \Big\}
\end{aligned} \tag{94}$$

where the second line of equation (94) indicates the welfare from the possible realizations paths of a limit buy order at $p_k < p_{+n^f}$, while the third line measures the welfare of a limit buy order at $p_k = p_{+n^f}$ when the 1^{st} player locks the market thus acting as a monopolist liquidity supplier. The fourth line of equations (94) indicates the welfare of the 1^{st} player when the 2^{nd} player submits a limit order to sell at prices $p_j > p_{k+1}$. By setting $\mathbb{1}_G = 0$, we obtain the welfare for the 1^{st} player in a game (Section 3.1) in which the 2^{nd} player can submit limit orders only at adjacent price, $p_j = p_{k+1}$.

As explained in Appendix D.1, the submission strategies of the 2^{nd} player depend on the submission strategies of the 1^{st} one. If the 1^{st} player posts a limit buy order at a price $p_k < p_{+n^f}$, the 2^{nd} player can both take and supply liquidity. If instead the 1^{st} player posts a limit order at the highest possible price, p_{+n^f}, the market is locked and the 2^{nd} player can only take liquidity. If the 1^{st} player posts a limit buy order at a price $p_k < p_{+n^f}$, the welfare of the 2^{nd} player is given by:

$$
\begin{aligned}
&\omega_{t_2}(ms_{t_2} \ \vee ls_{t_2} \ \vee lb_{t_2} \ |\tau) = \\
&\sum_{k=-n^f}^{+n^f-1} \Big(Pr\left(lb_{k,t_1}|\Lambda_{t_0},\tau\right) \frac{1}{\Gamma} \int_{\beta_{t_2}\in B(\tau)} (p_k - \beta_{t_2}\nu)\ d\beta_{t_2} + \\
&Pr\left(lb_{k,t_1}|\Lambda_{t_0},\tau\right) Pr\left(mb_{k+1,t_3}|\Lambda_{t_2},\tau\right) \frac{1}{\Gamma} \int_{\beta_{t_2}\in B(\tau)} (p_{k+1} - \beta_{t_2}v)\ d\beta_{t_2} + \\
&Pr\left(lb_{k,t_1}|\Lambda_{t_0},\tau\right) Pr\left(ms_{k+1,t_3}|\Lambda_{t_2},\tau\right) \frac{1}{\Gamma} \int_{\beta_{t_2}\in B(\tau)} (\beta_{t_2}v - p_{k+1})\ d\beta_{t_2} + \\
&\mathbb{1}_G \Big\{ \sum_{j>k+1} Pr\left(lb_{k,t_1}|\Lambda_{t_0},\tau\right) Pr\left(mb_{j,t_3}|\Lambda_{t_2},\tau\right) \frac{1}{\Gamma} \int_{\beta_{t_2}\in B(\tau)} (p_j - \beta_{t_2}v)\ d\beta_{t_2} + \\
&\sum_{j>k+1} Pr\left(lb_{k,t_1}|\Lambda_{t_0},\tau\right) Pr\left(ms_{j,t_3}|\Lambda_{t_2},\tau\right) \frac{1}{\Gamma} \int_{\beta_{t_2}\in B(\tau)} (\beta_{t_2}v - p_j)\ d\beta_{t_2} \Big\} \Big)
\end{aligned}
\tag{95}
$$

The second line in equation (95) indicates the expected welfare of the 2^{nd} player in case of a market sell order; the third line indicates the welfare from a limit sell order at p_{k+1}, and the fourth line indicates the welfare from a limit buy order p_{k+1}.

If the 1^{st} player locks the market by posting a limit buy order at $p_k = p_{+n^f}$, the 2^{nd} player can only take liquidity and his welfare function is the same as the 2^{nd} player in the 2-period model:

$$
\omega_{t_2}(ms_{t_2} \ |\tau) = Pr\left(lb_{+n^f,t_1}|\Lambda_{t_0},\tau\right) \frac{1}{\Gamma} \int_{(1-b)}^{\frac{p_{+n^f}}{v}} (p_{+n^f} - \beta_{t_2}v)\ d\beta_{t_2} \tag{96}
$$

The welfare of the 3^{rd} player depends on the order submission strategies of both the 1^{st} and 2^{nd}

player. If $p_k \neq p_{+n^f}$, the welfare is:

$$
\begin{aligned}
&\omega_{t_3}(ms_{t_3} \ \vee mb_{t_3} \ |\tau) = \\
&\sum_{k=-n^f}^{+n^f-1} \Big(Pr\left(lb_{k,t_1}|\Lambda_{t_0},\tau\right) Pr\left(ls_{k+1,t_2}|\Lambda_{t_1},\tau\right) \frac{1}{\Gamma} \big(\int_{(1-b)}^{\frac{p_k}{v}} (p_k - \beta_{t_3} v) \ d\beta_{t_3} + \int_{\frac{p_{k+1}}{v}}^{(1+b)} (\beta_{t_3} v - p_{k+1}) \ d\beta_{t_3} \big) + \\
&Pr\left(lb_{k,t_1}|\Lambda_{t_0},\tau\right) Pr\left(lb_{k+1,t_2}|\Lambda_{t_1},\tau\right) \frac{1}{\Gamma} \int_{(1-b)}^{\frac{p_{k+1}}{v}} (p_{k+1} - \beta_{t_3} v) \ d\beta_{t_3} + \\
&\mathbb{1}_G \Big\{ \sum_{j>k+1} Pr\left(lb_{k,t_1}|\Lambda_{t_0},\tau\right) Pr\left(ls_{j,t_2}|\Lambda_{t_1},\tau\right) \frac{1}{\Gamma} \big(\int_{(1-b)}^{\frac{p_k}{v}} (p_k - \beta_{t_3} v) \ d\beta_{t_3} + \int_{\frac{p_j}{v}}^{(1+b)} (\beta_{t_3} v - p_j) \ d\beta_{t_3} \big) + \\
&Pr\left(lb_{k,t_1}|\Lambda_{t_0},\tau\right) Pr\left(lb_{j,t_2}|\Lambda_{t_1},\tau\right) \frac{1}{\Gamma} \int_{(1-b)}^{\frac{p_j}{v}} (p_j - \beta_{t_3} v) \ d\beta_{t_3} \Big\} \Big)
\end{aligned}
\tag{97}
$$

If instead $p_k = p_{+n^f}$, the welfare of the 3^{rd} player is:

$$
\omega_{t_3}(ms_{t_3} \ |\tau) = Pr\left(lb_{+n^f,t_1}|\Lambda_{t_0},\tau\right) Pr\left(nt_{+n^f,t_2}|\Lambda_{t_1},\tau\right) \frac{1}{\Gamma} \int_{(1-b)}^{\frac{p_{+n^f}}{v}} (p_{+n^f} - \beta_{t_3} v) \ d\beta_{t_3} \tag{98}
$$

It is important to note that in equilibrium the 1^{st} player locks the market with a very small probability. The events of locked markets are very rare as they happen only when the tick size is so large that the price grid is composed of two prices only. As proved in Appendix D.4, when the tick size increases so that - given the support - the price grid includes only two price levels, the probability that the 2^{nd} player undercuts the 1^{st} player limit order is very small but still positive, until when the tick size becomes so large that the probability of undercutting tends to zero. In the rare event that the probability of undercutting is very small but still positive, the 1^{st} player has an incentive to lock the market to prevent the 2^{nd} player from undercutting his limit order, which would crowd him out of the market. This explains why in equilibrium the probability that the market is locked by the 1^{st} player is positive but negligible, and therefore the relevant case is when the 1^{st} player posts a limit buy order at a price which is smaller than the highest possible price, $p_k < p_{+n^f}$.

We are now in the position to define the total welfare of market participants, $\Omega(\tau)$, as the sum of the welfare of the three investors arriving respectively at time t_1, t_2 and t_3 of the 3-period trading game. The SP will choose the tick size that maximizes $\Omega(\tau)$:

$$\max_{\tau\in(0,\tau^{max})} \Omega(\tau) =$$

$$\omega_{t_1}(lb_{t_1}\,|\,\tau) + \omega_{t_2}(ms_{t_2}\ \vee\ ls_{t_2}\ \vee lb_{t_2}\,|\,\tau) + \omega_{t_2}(ms_{t_2}\,|\,\tau) + \omega_{t_3}(ms_{t_3}\ \vee\ mb_{t_3}\,|\,\tau) + \omega_{t_3}(ms_{t_3}\,|\,\tau) \tag{99}$$

Given the optimization problems solved by traders and the SP, we can define the equilibrium of our trading game:

Definition 3. *A sub-game Perfect Nash Equilibrium of the trading game is the set of limit order submission probabilities and their respective execution probabilities (defined in Appendix D.1) that solve the optimization problem of investors at t_1, t_2, and t_3, and that are consistent with the tick size, $\tau^{\star} \in (0, \ \ \tau^{max})$, set by the SP to maximize total welfare $\Omega(\tau)$.*

D.3 Undercutting decreases in τ

In this appendix,we show, trough an example, that given a limit buy order posted at p_k by the 1^{st} player, the probability that the 2^{nd} player will undercut at p_{k+j} increases as the tick size decreases. More specifically Table 1.D Panel A reports the equilibrium submission strategies of a 3-period model with $b = 0.06$, $\nu = 10$ and $\tau = 0.45$, Panel B the equilibrium submission strategies of a 3-period model with $b = 0.06$, $\nu = 10$ and $\tau = 0.15$ and finally Panel B the equilibrium submission strategies of a 3-period model with $b = 0.06$, $\nu = 10$ and $\tau = 0.05$. If we focus either on $p_k = 9.925$ or $p_k = 9.775$, (prices shaded in grey in Table 1.D), we can observe that the probability of undercutting is a negative function of τ.

Table 1.D: Comparative Analysis of the 1^{st} and 2^{nd} player's equilibrium submission probabilities

Panel A, B and C summarize the submission strategies of the first two players in a 3-period game for different values of τ. The first column reports prices associated with the 1^{st} player equilibrium order submission probabilities, $Pr\left(lb_{k,t_1}|\Lambda_{t_0},\tau\right)$ reported in column 2. The columns $3-6$ of Panel A, B and C report respectively the probability at t_2 of market selling ($Pr\left(ms_{k,t_2}|\Lambda_{t_1},\tau\right)$), of limit selling ($Pr\left(lb_{\leq k,t_2}|\Lambda_{t_1},\tau\right)$), of no trade ($Pr\left(nt_{k,t_2}|\Lambda_{t_1},\tau\right)$) and of undercutting ($Pr\left(lb_{>k,t_2}|\Lambda_{t_1},\tau\right)$). Our standard parameterization applies ($\nu = 10$ and $b = 0.06$). Highlighted in grey are the prices which are common to the three panels and the associated probabilities of undercutting.

Panel A: 3-period game with $\tau = 0.45$

p_k	$Pr\left(lb_{k,t_1}\|\Lambda_{t_0},\tau\right)$	$Pr\left(ms_{k,t_2}\|\Lambda_{t_1},\tau\right)$	$Pr\left(ls_{>k,t_2}\|\Lambda_{t_1},\tau\right)$	$Pr\left(nt_{k,t_2}\|\Lambda_{t_1},\tau\right)$	$Pr\left(lb_{>k,t_2}\|\Lambda_{t_1},\tau\right)$
10.225	0.114	0.688	0.000	0.312	0.000
9.775	0.386	0.142	0.546	0.000	0.312

Panel B: 3-period game with $\tau = 0.15$

p_k	$Pr\left(lb_{k,t_1}\|\Lambda_{t_0},\tau\right)$	$Pr\left(ms_{k,t_2}\|\Lambda_{t_1},\tau\right)$	$Pr\left(ls_{>k,t_2}\|\Lambda_{t_1},\tau\right)$	$Pr\left(nt_{k,t_2}\|\Lambda_{t_1},\tau\right)$	$Pr\left(lb_{>k,t_2}\|\Lambda_{t_1},\tau\right)$
10.075	0.097	0.505	0.209	0.000	0.286
9.925	0.310	0.324	0.301	0.000	0.375
9.775	0.094	0.118	0.432	0.000	0.450

Panel C: 3-period game with $\tau = 0.05$

p_k	$Pr\left(lb_{k,t_1}\|\Lambda_{t_0},\tau\right)$	$Pr\left(ms_{k,t_2}\|\Lambda_{t_1},\tau\right)$	$Pr\left(ls_{>k,t_2}\|\Lambda_{t_1},\tau\right)$	$Pr\left(nt_{k,t_2}\|\Lambda_{t_1},\tau\right)$	$Pr\left(lb_{>k,t_2}\|\Lambda_{t_1},\tau\right)$
10.025	0.083	0.443	0.182	0.000	0.375
9.975	0.104	0.384	0.214	0.000	0.402
9.925	0.105	0.323	0.250	0.000	0.426
9.875	0.105	0.259	0.293	0.000	0.448
9.825	0.101	0.190	0.343	0.000	0.467
9.775	0.001	0.118	0.400	0.000	0.482

D.4 Proof of Proposition (3)

At t_3, the order submission probabilities are defined by Lemma (1.3)

- $Pr\left(ms_{k,t_3}|\Lambda_{t_2},\tau\right)=\frac{1}{\Gamma}\left(\frac{p_k}{v}-(1-b)\right)$
- $Pr\left(mb_{k,t_3}|\Lambda_{t_2},\tau\right)=\frac{1}{\Gamma}\left((1+b)-\frac{p_k}{v}\right)$

Notice that the alternatives options available for the 2^{nd} player are: a market sell at p_k, a limit buy or a limit sell both at p_{k+1}. At t_2, for a generic limit buy posted by the 1^{st} player at p_k, the probability that the 2^{nd} player selects a market sell is given by:

$$\begin{aligned}
&Pr\left(ms_{k,t_2}|\Lambda_{t_1},\tau\right)=\\
&Pr\Big(p_k-\beta_{t_2}\nu>0,\\
&p_k-\beta_{t_2}\nu>(p_{k+1}-\beta_{t_2}\nu)Pr\left(mb_{k+1,t_3}|\Lambda_{t_2},\tau\right),\\
&p_k-\beta_{t_2}\nu>(\beta_{t_2}\nu-p_{k+1})Pr\left(ms_{k+1,t_3}|\Lambda_{t_2},\tau\right)\Big)
\end{aligned}\tag{100}$$

Equation (100) can be simplified in the following way:

$$\begin{aligned}
Pr\left(ms_{k,t_2}|\Lambda_{t_1},\tau\right)&=max\left[0,\ Pr\left((1-b)<\beta_{t_2}<\frac{p_k}{\nu}-\frac{\tau}{\nu}\frac{Pr\left(mb_{k+1,t_3}|\Lambda_{t_2},\tau\right)}{1-Pr\left(mb_{k+1,t_3}|\Lambda_{t_2},\tau\right)}\right)\right]\\
&=max\left[0,\ \frac{1}{\Gamma}\left(\frac{p_k}{\nu}-\frac{\tau}{\nu}\frac{Pr\left(mb_{k+1,t_3}|\Lambda_{t_2},\tau\right)}{1-Pr\left(mb_{k+1,t_3}|\Lambda_{t_2},\tau\right)}-(1-b)\right)\right]
\end{aligned}\tag{101}$$

The probability that the 2^{nd} player selects a limit sell order at a price p_{k+1} is:

$$\begin{aligned}
&Pr\left(ls_{k+1,t_2}|\Lambda_{t_1},\tau\right)=\\
&Pr\Big((p_{k+1}-\beta_{t_2}\nu)Pr\left(mb_{k+1,t_3}|\Lambda_{t_2},\tau\right)>0,\\
&(p_{k+1}-\beta_{t_2}\nu)Pr\left(mb_{k+1,t_3}|\Lambda_{t_2},\tau\right)>p_k-\beta_{t_2}\nu,\\
&(p_{k+1}-\beta_{t_2}\nu)Pr\left(mb_{k+1,t_3}|\Lambda_{t_2},\tau\right)>(\beta_{t_2}\nu-p_{k+1})Pr\left(ms_{k+1,t_3}|\Lambda_{t_2},\tau\right)\Big)
\end{aligned}\tag{102}$$

Equation (102) can be simplified in the following way:

$$Pr\left(ls_{k+1,t_2}|\Lambda_{t_1},\tau\right) = \begin{cases} Pr\Big(\frac{p_k}{\nu} - \frac{\tau}{\nu}\frac{Pr\left(mb_{k+1,t_3}|\Lambda_{t_2},\tau\right)}{1-Pr\left(mb_{k+1,t_3}|\Lambda_{t_2},\tau\right)} < \beta_{t_2} < \frac{p_k}{\nu} + \frac{\tau}{\nu}\Big) & if Pr\left(ms_{k,t_2}|\Lambda_{t_1},\tau\right) > 0 \\ Pr\Big((1-b) < \beta_{t_2} < \frac{p_k}{\nu} + \frac{\tau}{\nu}\Big) & otherwise \end{cases} \tag{103}$$

The probability that the 2^{nd} player selects a limit buy order at $p_{k+1} > p_k$ thus undercutting the limit buy order posted at t_1 is:

$$\begin{aligned} & Pr\left(lb_{k+1,t_2}|\Lambda_{t_1},\tau\right) = \\ & Pr\Big((\beta_{t_2}\nu - p_{k+1})Pr\left(ms_{k+1,t_3}|\Lambda_{t_2},\tau\right) > 0, \\ & (\beta_{t_2}\nu - p_{k+1})Pr\left(ms_{k+1,t_3}|\Lambda_{t_2},\tau\right) > p_k - \beta_{t_2}\nu, \\ & (\beta_{t_2}\nu - p_{k+1})Pr\left(ms_{k+1,t_3}|\Lambda_{t_2},\tau\right) > (p_{k+1} - \beta_{t_2}\nu)Pr\left(mb_{k+1,t_3}|\Lambda_{t_2},\tau\right)\Big) \end{aligned} \tag{104}$$

Equation (104) can be simplified in the following way:

$$\begin{aligned} Pr\left(lb_{k+1,t_2}|\Lambda_{t_1},\tau\right) &= Pr\left(\frac{p_k}{\nu} + \frac{\tau}{\nu} < \beta_{t_2} < (1+b)\right) \\ &= \frac{1}{\Gamma}\left((1+b) - \frac{p_k+\tau}{\nu}\right) \end{aligned} \tag{105}$$

Without loss of generality, using Lemma (1), if the 1^{st} player at t_1 is a buyer ($\beta_{t_1} > 1$), he posts a limit buy. The submission probability of a limit buy at price p_k is:

$$\begin{aligned} & Pr\left(lb_{k,t_1}|\Lambda_{t_0},\tau\right) = \\ & Pr\Big((\beta_{t_1}\nu - p_k)\left[Pr\left(ms_{k,t_2}|\Lambda_{t_1},\tau\right) + Pr\left(ls_{k+1,t_2}|\Lambda_{t_1},\tau\right)Pr\left(ms_{k,t_3}|\Lambda_{t_2},\tau\right)\right] > 0, \\ & (\beta_{t_1}\nu - p_k)\left[Pr\left(ms_{k,t_2}|\Lambda_{t_1},\tau\right) + Pr\left(ls_{k+1,t_2}|\Lambda_{t_1},\tau\right)Pr\left(ms_{k,t_3}|\Lambda_{t_2},\tau\right)\right] > \\ & (\beta_{t_1}\nu - p_{\tilde{k}})\left[Pr\left(ms_{\tilde{k},t_2}|\Lambda_{t_1},\tau\right) + Pr\left(ls_{\tilde{k+1},t_2}|\Lambda_{t_1},\tau\right)Pr\left(ms_{\tilde{k},t_3}|\Lambda_{t_2},\tau\right)\right], \\ & (\beta_{t_1}\nu - p_k)\left[Pr\left(ms_{k,t_2}|\Lambda_{t_1},\tau\right) + Pr\left(ls_{k,t_2}|\Lambda_{t_1},\tau\right)Pr\left(ms_{k,t_3}|\Lambda_{t_2},\tau\right)\right] > \\ & (\beta_{t_1}\nu - p_{n^f})\left[Pr\left(ms_{n^f,t_2}|\Lambda_{t_1},\tau\right) + Pr\left(nt_{n^f,t_2}|\Lambda_{t_1},\tau\right)Pr\left(ms_{n^f,t_3}|\Lambda_{t_2},\tau\right)\right]\Big) \end{aligned} \tag{106}$$

The submission probabilities when the market is locked by the 1^{st} player (he submits a limit order at p_{n^f}) are:

$$\begin{aligned} & Pr\left(lb_{+n^f,t_1}|\Lambda_{t_0},\tau\right) = \\ & Pr\Big((\beta_{t_1}\nu - p_{n^f})\big[Pr\left(ms_{n^f,t_2}|\Lambda_{t_1},\tau\right) + Pr\left(nt_{n^f,t_2}|\Lambda_{t_1},\tau\right) Pr\left(ms_{n^f,t_3}|\Lambda_{t_2},\tau\right)\big] > 0, \\ & (\beta_{t_1}\nu - p_{n^f})\big[Pr\left(ms_{n^f,t_2}|\Lambda_{t_1},\tau\right) + Pr\left(nt_{n^f,t_2}|\Lambda_{t_1},\tau\right) Pr\left(ms_{n^f,t_3}|\Lambda_{t_2},\tau\right)\big] > \\ & (\beta_{t_1}\nu - p_k)\big[Pr\left(ms_{k,t_2}|\Lambda_{t_1},\tau\right) + Pr\left(ls_{k+1,t_2}|\Lambda_{t_1},\tau\right) Pr\left(ms_{k,t_3}|\Lambda_{t_2},\tau\right)\big]\Big) \end{aligned} \tag{107}$$

In this specific case, the probability of a market sell at t_2 hitting a limit buy posted at p_{n^f} is $Pr\left(ms_{n^f,t_2}|\Lambda_{t_1},\tau\right) = Pr\left(ms_{n^f,t_3}|\Lambda_{t_2},\tau\right)$, and the probability of no-trading is $Pr\left(nt_{n^f,t_2}|\Lambda_{t_1},\tau\right) = 1 - Pr\left(ms_{n^f,t_2}|\Lambda_{t_1},\tau\right)$.

We now show that when there exist at least two price levels on each side of the book ($n_f \geq 2$), the 1^{st} player never locks the market, $Pr\left(lb_{+n^f,t_1}|\Lambda_{t_0},\tau\right) = 0$, Whereas, when $n_f = 1$, the 1^{st} player has an incentive to lock the market, $Pr\left(lb_{+n^f,t_1}|\Lambda_{t_0},\tau\right) \geq 0$.

We show that for any $\tau \in \left\{(0,\tau^{max})\,|n^f \geq 2\right\}$, at t_1 there exists at least one limit buy order, e.g., at p_{+1}, that dominates a limit buy at p_{+n^f}. To derive the payoff of a limit buy at p_{+1} we need to compute the execution probability and therefore we have to consider all the trading options the 2^{nd} player has at t_2: a market sell, a limit sell and a limit buy. However, if the 2^{nd} player submits a limit buy order - thus undercutting the 1^{st} player's limit buy order - the probability of execution of this limit buy order is zero. The payoff of the 1^{st} player submitting a limit buy at p_{+1} is:

$$O_{t_1}(lb_{+1,t_1}|\lambda_{t_0},\tau) = (\beta_{t_1}\nu - p_{+1}) \times \left[Pr\left(ms_{+1,t_2}|\Lambda_{t_1},\tau\right) + Pr\left(ls_{+2,t_2}|\Lambda_{t_1},\tau\right) Pr\left(ms_{+1,t_3}|\Lambda_{t_2},\tau\right)\right] \tag{108}$$

with the submission probabilities defined in equations (100) and (102). The payoff of the 1^{st} player posting a limit buy at p_{+n^f} is:

$$O_{t_1}(lb_{+n^f,t_1}|\lambda_{t_0},\tau) = (\beta_{t_1}\nu - p_{+n^f}) \times \left[Pr\left(ms_{n^f,t_2}|\Lambda_{t_1},\tau\right) + Pr\left(nt_{n^f,t_2}|\Lambda_{t_1},\tau\right) \times Pr\left(ms_{n^f,t_3}|\Lambda_{t_2},\tau\right)\right] \tag{109}$$

$n^f \geq 2$ only if, by equation (21), $2b\nu \geq 3\tau$. We can show that under this condition the payoff of the 1^{st} player submitting a limit buy at p_{+1} (equation (108)) is strictly greater than the payoff from posting a limit buy at p_{n^f}: $O_{t_1}(lb_{+1,t_1}|\lambda_{t_0},\tau) > O_{t_1}(lb_{+n^f,t_1}|\lambda_{t_0},\tau)$

Therefore, the 1^{st} player never posts a limit buy at p_{+n^f}, which is a dominated strategy.

In the remaining part of the proof, we consider a generic τ such that $\tau \in \left[(0,\tau^{max}) \; |n^f = 1\right]$. Given only two prices, p_{-1} and p_{+1}, if the trader arriving at t_1 chooses to submit a limit buy at p_{-1}, the submission probabilities of the 2^{nd} player defined in (101), (103) and (105) can be equivalently written as :

- Probability of a market sell: $Pr\left(ms_{-1,t_2}|\Lambda_{t_1},\tau\right) = max\left[0,\; \frac{1}{\Gamma}\left(b - \frac{t}{2v} - \frac{t}{v} \times \frac{0.5-\frac{\tau}{4bv}}{0.5+\frac{\tau}{4bv}}\right)\right]$
- Probability of a limit sell at p_{+1}:
  - $Pr\left(ls_{+1,t_2}|\Lambda_{t_1},\tau\right) = \frac{1}{\Gamma}\left[\frac{\tau}{\nu} + \frac{\tau}{\nu}\frac{0.5-\frac{\tau}{4bv}}{0.5+\frac{\tau}{4bv}}\right]$ if $Pr\left(ms_{-1,t_2}|\Lambda_{t_1},\tau\right) > 0$
  - $Pr\left(ls_{+1,t_2}|\Lambda_{t_1},\tau\right) = \frac{1}{\Gamma}\left[b + \frac{\tau}{2\nu}\right]$ otherwise
- Probability of a limit buy at p_{+1}: $Pr\left(lb_{+1,t_2}|\Lambda_{t_1},\tau\right) = \frac{1}{\Gamma}\left[b - \frac{\tau}{2\nu}\right]$

It is worth noticing that for $\frac{\Gamma\nu}{\tau} \to 1$, both $Pr\left(ms_{-1,t_2}|\Lambda_{t_1},\tau\right)$ and $Pr\left(lb_{+1,t_2}|\Lambda_{t_1},\tau\right) \to 0$, while $Pr\left(ls_{+1,t_2}|\Lambda_{t_1},\tau\right) \to 1$.

If the investor arriving at t_1 limit buys at p_{+1}, the submission probabilities of the 2^{nd} player are the following:

- Probability of a market sell: $Pr\left(ms_{+1,t_2}|\Lambda_{t_1},\tau\right) = Pr\left(p_{+1} - \beta\nu > 0\right)$
- Probability of nt: $Pr\left(nt_{+1,t_2}|\Lambda_{t_1},\tau\right) = 1 - Pr\left(ms_{+1,t_2}|\Lambda_{t_1},\tau\right)$

Having defined the payoff from both a limit buy at p_{-1} and a limit buy at p_{+1} , we can determine the associated probability of submission by equating the payoff from the two strategies:

$$\begin{aligned}&(\beta_{t_1}\nu - p_{-1}) \times \left[Pr\left(ms_{-1,t_2}|\Lambda_{t_1},\tau\right) + Pr\left(ls_{+1,t_2}|\Lambda_{t_1},\tau\right) \times Pr\left(ms_{-1,t_3}|\Lambda_{t_2},\tau\right)\right] = \\ &(\beta_{t_1}\nu - p_{+1}) \times \left[Pr\left(ms_{+1,t_2}|\Lambda_{t_1},\tau\right) + Pr\left(nt_{+1,t_2}|\Lambda_{t_1},\tau\right) \times Pr\left(ms_{+1,t_3}|\Lambda_{t_2},\tau\right)\right]\end{aligned} \tag{110}$$

Solving (110) by β_{t_1} we show that in the τ region ensuring $n^f = 1$ which is defined by $\tau \in [(0, \tau^{max}) \,|2b\nu < 3\tau]$, equation (110) admits an internal solution $\beta^{\star}_{t_1}$ and the 1^{st} player order submission probabilities are:

- $Pr\left(lb_{-1,t_1}|\Lambda_{t_0},\tau\right) = \frac{1}{\Gamma}\left[\beta^{\star}_{t_1} - 1\right]$
- $Pr\left(lb_{+1,t_1}|\Lambda_{t_0},\tau\right) = \frac{1}{\Gamma}\left[1 + b - \beta^{\star}_{t_1}\right]$

If $\frac{\Gamma\nu}{\tau} \to 1$, then $\beta^{\star}_{t_1} \to (1+b)$ and therefore $Pr\left(lb_{-1,t_1}|\Lambda_{t_0},\tau\right) \to 0.5$, which is the order submission probability of the 1^{st} player in the 2-period trading game. Hence, for very coarse price grids, the 1^{st} player in a 3-period game has a submission schedule which is almost identical to the one of the 2-period game.

D.5 Proof of Proposition (4)

We show how the equilibrium order submission probabilities and the associated welfare of the strategic game described in Appendix (D.4) change for $\tau \to 0^+$. As τ decreases, the number of feasible prices within the investor's support, $2b\nu$, increases. Approaching a continuum of prices, we indicate a generic feasible price as p. The order submission probabilities associated with the trading strategies of the 3^{rd} player are:

- $Pr\left(ms_{t_3}|\Lambda_{t_2}\right) = \frac{1}{\Gamma}\left(\frac{p}{v} - (1-b)\right)$
- $Pr\left(mb_{t_3}|\Lambda_{t_2}\right) = \frac{1}{\Gamma}\left((1+b) - \frac{p}{v}\right)$

The order submission probabilities for the 2^{nd} player can be obtained by considering equations (101)-(103) -(105) for $\tau \to 0^+$:

$$lim_{\tau\to 0^+} Pr\left(ms_{k,t_2}|\Lambda_{t_1}\right) = Pr\left(1 - b < \beta_{t_2} < \frac{p}{\nu}\right) = \left(\frac{p}{\Gamma\nu} - \frac{1-b}{\Gamma}\right) \tag{111}$$

$$lim_{\tau\to 0^+} Pr\left(ls_{k+1,t_2}|\Lambda_{t_1}\right) = 0 \tag{112}$$

$$lim_{\tau\to 0^+} Pr\left(lb_{k,t_2}|\Lambda_{t_1}\right) \approx Pr\big(\frac{p}{\nu} < \beta_{t_2} < (1+b)\big) = \left(\frac{1+b}{\Gamma} - \frac{p}{\Gamma\nu}\right) \tag{113}$$

Considering the case of $\tau \to 0^+$, if the 2^{nd} player undercuts the 1^{st} player to the next adjacent price, he undercuts at a price $p_{t_2} = p + o(\epsilon)$ by an almost negligible quantity to gain price priority, hence $p_{t_2} \sim p$. When τ approaches 0, $Pr\left(ms_{t_2}|\Lambda_{t_1}\right)$ (equation (101)) is always greater than 0 for any p considered because $\frac{p}{v} > (1-b)$ by p being feasible. Hence, the 2^{nd} player will submit a market sell in probability. In addition, equation (103) shows that as τ approaches 0, in equilibrium the 2^{nd} player will not submit a limit order to sell: if he is a seller, the price improvement offered by a limit sell will be too small and he will rather market sell; if instead he is a buyer, he has the chance to outbid the 1^{st} player by an infinitesimal amount and he will therefore undercut the existing limit buy order.

This result is consistent with the intuition provided in Section (3). The 1^{st} player will therefore maximize his utility anticipating that the 2^{nd} player will either match or undercut his order. Hence from (111) the generic payoff from a limit buy order submitted by the 1^{st} player is:

$$(\beta_{t_1}\nu - p)\left[\frac{p}{\Gamma\nu} - \frac{1-b}{\Gamma}\right] \tag{114}$$

From first order conditions - taking the first and second order derivative $w.r.t.$ p of (114), for any $\beta_1 \in (1,\ 1+b)$ - the 1^{st} player will submit a limit buy order with probability 1 at the following price:

$$p^{\star} = \frac{\nu}{2}\left(\beta_{t_1} + 1 - b\right) \tag{115}$$

We can now compute the ex ante welfare of the players. Substituting (115) in (114) and integrating over β_{t_1}, we obtain the 1^{st} player's welfare as:

$$\omega_{t_1}(lb_{t_1}) = \int_1^{1+b}\left(\beta_{t_1}\nu - \frac{\nu}{2}\left(\beta_{t_1}+1-b\right)\right)\frac{1}{\Gamma}\left[\frac{\frac{\nu}{2}\left(\beta_{t_1}+1-b\right)}{\Gamma\nu} - \frac{1-b}{\Gamma}\right]d\beta_{t_1} = \frac{7}{48}b\nu \tag{116}$$

Using the Law of Total Expectation ("Tower Property"), we can write the 2^{nd} player's welfare from a market sell as:

$$\begin{aligned}
\omega_{t_2}(ms_{t_2}) &= \int_{1-b}^{1+b} E[\omega_{t_2}(ms_{t_2})|\beta_{t_1}]\frac{1}{\Gamma}d\beta_{t_1} \\
&= \int_{1-b}^{1} E[\omega_{t_2}(ms_{t_2})|\beta_{t_1}]\frac{1}{\Gamma}d\beta_1 + \int_{1}^{1+b} E[\omega_{t_2}(ms_{t_2})|\beta_{t_1}]\frac{1}{\Gamma}d\beta_{t_1} \\
&= \int_{1-b}^{1}\left(\int_{1-b}^{\frac{p}{\nu}} (p-\beta_{t_2}\nu)\frac{1}{\Gamma}Pr\left(lb_{p,t_1}|\Lambda_{t_0}\right)d\beta_{t_2}|\beta_{t_1}\right)\frac{1}{\Gamma}d\beta_{t_1}+ \\
&\int_{1}^{1+b}\left(\int_{1-b}^{\frac{p}{\nu}} (p-\beta_{t_2}\nu)\frac{1}{\Gamma}Pr\left(lb_{p,t_1}|\Lambda_{t_0}\right)d\beta_{t_2}|\beta_{t_1}\right)\frac{1}{\Gamma}d\beta_{t_1}
\end{aligned} \tag{117}$$

Note that the optimal $\beta_{t_2}^{\cdot;\cdot}$ threshold are obtained by considering the equilibrium submission strategies defined in equation (111). By Lemma (1), the 1^{st} player does not submit a limit buy order when $\beta_{t_1} < 1$ and submits with probability 1 a limit buy order at p when $\beta_{t_1} > 1$. Hence the welfare associated to a market sell at t_2 is

$$\begin{aligned}
\omega_{t_2}(ms_{t_2}) &= \int_{1}^{1+b}\left(\int_{1-b}^{\frac{p}{\nu}} (p-\beta_{t_2}\nu)\frac{1}{\Gamma}Pr\left(lb_{p,t_1}|\Lambda_{t_0}\right)d\beta_{t_2}|\beta_{t_1}\right)\frac{1}{\Gamma}d\beta_{t_1} \\
&= \int_{1}^{1+b}\left(\int_{1-b}^{\frac{1}{2}\left(\beta_{t_1}+1-b\right)} \left(\frac{\nu}{2}\left(\beta_{t_1}+1-b\right)-\beta_{t_2}\nu\right)\frac{1}{\Gamma}d\beta_{t_2}|\beta_{t_1}\right)\frac{1}{\Gamma}d\beta_{t_1} \\
&= \int_{1}^{1+b}\frac{\left(\frac{\nu}{2}\left(\beta_{t_1}+1-b\right)+(-1+b)\nu\right)^2}{4b\nu}\frac{1}{\Gamma}d\beta_{t_1} = \frac{7}{96}b\nu
\end{aligned} \tag{118}$$

With a similar argument, the welfare associated to a limit buy at t_2 is

$$\begin{aligned}
\omega_{t_2}(lb_{t_2}) &= \int_{1}^{1+b} E[\omega_{t_2}(lb_{t_2})|\beta_{t_1}]\frac{1}{\Gamma}d\beta_{t_1} \\
&= \int_{1}^{1+b}\left(\int_{\frac{p}{\nu}}^{1+b} (\beta_{t_2}\nu-p)Pr\left(lb_{p,t_1}|\Lambda_{t_0}\right)Pr\left(ms_{p,t_3}|\Lambda_{t_2}\right)\frac{1}{\Gamma}d\beta_{t_2}|\beta_{t_1}\right)\frac{1}{\Gamma}d\beta_{t_1} \\
&= \int_{1}^{1+b}\left(\int_{\frac{1}{2}\left(\beta_{t_1}+1-b\right)}^{1+b} \left(\beta_{t_2}\nu-\frac{\nu}{2}\left(\beta_{t_1}+1-b\right)\right)\frac{1}{\Gamma}\left(\frac{1}{2}\left(\beta_{t_1}+1-b\right)-(1-b)\right)\frac{1}{\Gamma}d\beta_{t_2}|d\beta_{t_1}\right)\frac{1}{\Gamma}d\beta_{t_1} \\
&= \int_{1}^{1+b}\frac{(\frac{\nu}{2}(1+\beta_{t_1}-b)+(-1+b)\nu)(\frac{\nu}{2}(1+\beta_{t_1}-b)-(1+b)v)^2}{(8b^2\nu^2)(2b)}d\beta_{t_1} = \frac{109}{1536}b\nu
\end{aligned} \tag{119}$$

The 3^{rd} player has an opportunity to buy if and only if the 2^{nd} player undercuts the limit buy posted at t_1. Hence the welfare of the 3^{rd} player is:

$$
\begin{aligned}
\omega_{t_3}(ms_{t_3}) &= \int_{1}^{1+b} E[\omega_{t_3}(ms_{t_3})|\beta_{t_1}]\frac{1}{\Gamma}d\beta_{t_1} \\
&= \int_{1-b}^{1+b}\left(\int_{1-b}^{\frac{p}{v}}(p-\beta_{t_3}\nu)Pr\left(lb_{p,t_1}|\Lambda_{t_0}\right)Pr\left(lb_{p,t_2}|\Lambda_{t_1}\right)\frac{1}{\Gamma}d\beta_{t_3}|\beta_{t_1}\right)\frac{1}{\Gamma}d\beta_{t_1} \\
&= \int_{1-b}^{1}\left(\int_{1-b}^{\frac{p}{v}}(p-\beta_{t_3}\nu)Pr\left(lb_{p,t_1}|\Lambda_{t_0}\right)Pr\left(lb_{p,t_2}|\Lambda_{t_1}\right)\frac{1}{\Gamma}d\beta_{t_3}|\beta_{t_1}\right)\frac{1}{\Gamma}d\beta_{t_1}+ \\
&\quad \int_{1}^{1+b}\left(\int_{1-b}^{\frac{p}{v}}(p-\beta_{t_3}\nu)Pr\left(lb_{p,t_1}|\Lambda_{t_1}\right)Pr\left(lb_{p,t_2}|\Lambda_{t_2}\right)\frac{1}{\Gamma}d\beta_{t_3}|\beta_{t_1}\right)\frac{1}{\Gamma}d\beta_{t_1}
\end{aligned}
\tag{120}
$$

By Lemma (1), the 1^{st} player does not submit a limit buy order when $\beta_{t_1} < 1$ and hence the welfare of a market sell at t_3 is given by:

$$
\begin{aligned}
\omega_{t_3}(ms_{t_3}) &= \int_{1}^{1+b}\left(\int_{1-b}^{\frac{p}{v}}(p-\beta_{t_3}\nu)Pr\left(lb_{p,t_1}|\Lambda_{t_1}\right)Pr\left(lb_{p,t_2}|\Lambda_{t_2}\right)\frac{1}{\Gamma}d\beta_{t_3}|\beta_{t_1}\right)\frac{1}{\Gamma}d\beta_{t_1} \\
&= \int_{1}^{1+b}\left(\int_{1-b}^{\frac{(\beta_1+1-b)}{2}}\left(\frac{v(\beta_{t_1}+1-b)}{2}-\beta_{t_3}\nu\right)\frac{1}{\Gamma}\left(1+b-\frac{(\beta_{t_1}+1-b)}{2}\right)\frac{1}{\Gamma}d\beta_{t_3}|\beta_{t_1}\right)\frac{1}{\Gamma}d\beta_{t_1} = \frac{67}{1536}b\nu
\end{aligned}
\tag{121}
$$

When τ approaches 0 ($\tau \to 0^+$), the total welfare of market participants is:

$$
\Omega(\tau \to 0^+) = \frac{b\nu}{3} \tag{122}
$$

In order to show that $\tau \to 0^+$ is not the argmax of equation (99), we need to find a $\tau > 0$ with an associated total welfare which is greater than $\frac{b\nu}{3}$. For a generic combination of (b, ν), consider $\tau = \frac{b\nu}{2}$. The price grid and the associated submissions probabilities are:

Table 2.D: 3-period Game: Order Submission Probabilities

This table reports the order submission probabilities of the 3-period model for a generic combination of (b, ν), $\tau = \frac{b\nu}{2}$ and $p_k = \{p_{-2},\ p_{-1},\ p_{+1},\ p_{+2}\}$. Note that the equilibrium order submission strategies are those associated with $p_k = \{p_{-1}, p_{+1}\}$

p_k	$Pr\left(ms_{k,t_3}\|\Lambda_{t_2},\tau\right)$	$Pr\left(mb_{k,t_3}\|\Lambda_{t_2},\tau\right)$	$Pr\left(ms_{k,t_2}\|\Lambda_{t_1},\tau\right)$	$Pr\left(ls_{k+1,t_2}\|\Lambda_{t_1},\tau\right)$	$Pr\left(lb_{k+1,t_2}\|\Lambda_{t_1},\tau\right)$	$Pr\left(nt_{k,t_2}\|\Lambda_{t_1},\tau\right)$	$Pr\left(lb_{k,t_1}\|\Lambda_{t_0},\tau\right)$
p_{+2}	0.875	0.125	0.875	0	0	0.125	0
p_{+1}	0.625	0.375	0.589286	0.285714	0.125	0	0.136364
p_{-1}	0.375	0.625	0.225	0.4	0.375	0	0.363636
p_{-2}	0.125	0.875	0	0.375	0.625	0	0

The welfare of the 1^{st} player is given by:

$$\omega_{t_1}(lb_{t_1}\ |\tau) = \sum_{k=-n^f}^{+n^f} \left[Pr\left(ms_{k,t_2}|\Lambda_{t_1},\tau\right) + Pr\left(ls_{k+1,t_2}|\Lambda_{t_1},\tau\right) Pr\left(ms_{k,t_3}|\Lambda_{t_2},\tau\right)\right] \times \int_{\beta_{t_1}\in B(\tau)} \frac{(\beta_{t_1}v - p_k)}{\Gamma}\, d\beta_{t_1}$$

$$\omega_{t_1}(lb_{t_1}\ |\tau) = 0.375 \int_{1}^{1+0.7272b} \frac{\beta_{t_1}\nu - p_{-1}}{\Gamma} d\beta_{t_1} + 0.767857 \int_{1+0.7272b}^{1+b} \frac{\beta_{t_1}\nu - p_{+1}}{\Gamma} d\beta_{t_1} = 0.14793\, b\nu \tag{123}$$

The welfare of the 2^{nd} player is given by:

$$\omega_{t_2}(ms_{t_2}\ \vee ls_{t_2}\ \vee lb_{t_2}\ |\tau) = \sum_{k=-n^f}^{+n^f} Pr\left(lb_{k,t_1}|\Lambda_{t_0},\tau\right) \times \int_{\beta_{t_2}\in B(\tau)} \frac{(p_k - \beta_{t_2}v)}{\Gamma}\, d\beta_{t_2} +$$
$$Pr\left(lb_{k,t_1}|\Lambda_{t_0},\tau\right) \times Pr\left(mb_{k+1,t_3}|\Lambda_{t_2},\tau\right) \int_{\beta_{t_2}\in B(\tau)} \frac{(p_{k+1} - \beta_{t_2}v)}{\Gamma}\, d\beta_{t_2} +$$
$$Pr\left(lb_{k,t_1}|\Lambda_{t_0},\tau\right) \times Pr\left(ms_{k+1,t_3}|\Lambda_{t_2},\tau\right) \int_{\beta_{t_2}\in B(\tau)} \frac{(\beta_{t_2}v - p_{k+1})}{\Gamma}\, d\beta_{t_2}$$
$$\omega_{t_2}(ms_{t_2}\ \vee ls_{t_2}\ \vee lb_{t_2}\ |\tau) =$$
$$0.363636 \left[\int_{1-b}^{1-0.55b} \frac{p_{-1} - \beta_{t_2}\nu}{\Gamma} d\beta_{t_2} + 0.375 \int_{1-0.55b}^{\frac{p_1}{\nu}} \frac{p_1 - \beta_{t_2}\nu}{\Gamma} d\beta_{t_2} + 0.625 \int_{\frac{p_1}{\nu}}^{1+b} \frac{\beta_{t_2}\nu - p_1}{\Gamma} d\beta_{t_2}\right] +$$
$$0.136364 \left[\int_{1-b}^{1+0.17857b} \frac{p_1 - \beta_{t_2}\nu}{\Gamma} d\beta_{t_2} + 0.125 \int_{1+0.17857b}^{\frac{p_2}{\nu}} \frac{p_2 - \beta_{t_2}\nu}{\Gamma} d\beta_{t_2} + 0.875 \int_{\frac{p_2}{\nu}}^{1+b} \frac{\beta_{t_2}\nu - p_2}{\Gamma} d\beta_{t_2}\right] =$$
$$0.136364 \times 0.413225\, b\nu + 0.363636 \times 0.266016\, b\nu = 0.153082\, b\nu \tag{124}$$

The welfare of the 3^{rd} player is given by:

$$\omega_{t_3}(ms_{t_3} \vee mb_{t_3} |\tau) =$$
$$\sum_{k=-n^f}^{+n^f} Pr\left(lb_{k,t_1}|\Lambda_{t_0},\tau\right) Pr\left(ls_{k+1,t_2}|\Lambda_{t_1},\tau\right) \times \left(\int_{1-b}^{\frac{p_k}{\nu}} \frac{p_k-\beta_{t_3}\nu}{\Gamma} d\beta_{t_3} + \int_{\frac{p_{k+1}}{\nu}}^{1+b} \frac{\beta_{t_3}\nu - p_{k+1}}{\Gamma} d\beta_{t_3}\right) +$$
$$Pr\left(lb_{k,t_1}|\Lambda_{t_0},\tau\right) Pr\left(lb_{k+1,t_2}|\Lambda_{t_1},\tau\right) \int_{1-b}^{\frac{p_{k+1}}{\nu}} \frac{p_{k+1}-\beta_{t_3}\nu}{\Gamma} d\beta_{t_3}$$
$$\omega_{t_3}(ms_{t_3} \vee mb_{t_3} |\tau) =$$
$$0.1454544 \left(\int_{1-b}^{\frac{p_{-1}}{\nu}} \frac{p_{-1}-\beta_{t_3}\nu}{\Gamma} d\beta_{t_3} + \int_{\frac{p_1}{\nu}}^{1+b} \frac{\beta_{t_3}\nu - p_1}{\Gamma} d\beta_{t_3}\right) + 0.1363635 \int_{1-b}^{\frac{p_1}{\nu}} \frac{p_1-\beta_{t_3}\nu}{\Gamma} d\beta_{t_3}$$
$$0.038961 \left(\int_{1-b}^{\frac{p_1}{\nu}} \frac{p_1-\beta_{t_3}\nu}{\Gamma} d\beta_{t_3} + \int_{\frac{p_2}{\nu}}^{1+b} \frac{\beta_{t_3}\nu - p_2}{\Gamma} d\beta_{t_3}\right) + 0.017045 \int_{1-b}^{\frac{p_2}{\nu}} \frac{p_2-\beta_{t_3}\nu}{\Gamma} d\beta_{t_3} =$$
$$0.0409091\, b\nu + 0.053267\, b\nu + 0.0158279\, b\nu + 0.0130501\, b\nu = 0.123054\, b\nu \tag{125}$$

Hence the total welfare for a generic game defined by $\tau = \frac{b\nu}{2}$ is $0.424067\, b\nu > \frac{b\nu}{3}$. We therefore conclude that $\tau \to 0^+$ is not the argmax of equation (99). ■

D.6 Discretization Grid

In this Appendix, we characterize the discretization grid used in the main body of the text. As explained in Section 3.2, we consider the tick sizes that form books including from 2 to 30 feasible prices. From equation (21), we know that if $\frac{b\nu}{\tau} \in N^+$, there are N^+ prices on each side of the book. Therefore for any duplet $(b,\ \nu)$ studied, we define the τ discretization grid, $DG_\tau(b\nu)$, as the collection of τ:

$$DG_\tau(b\nu) = \left\{\tau_n = \frac{b\nu}{n} \mid n \in (1,\ 15)\right\} \tag{126}$$

Each $\tau \in DG_\tau(b\nu)$ defines a price grid with a different number of feasible prices. To improve the accuracy of the grid and study the welfare of market participants in games with tick sizes such that $\frac{b\nu}{\tau} \notin N^+$, we augment $DG_\tau(b\nu)$ with tick sizes that lie between two consecutive ticks $(\tau_n,\ \tau_{n-1})$ that according to (126) define $\frac{b\nu}{\tau_n} \in N^+$. We consider three different weighted averages: $0.5(\tau_n + \tau_{n-1})$, $0.75\tau_n + 0.25\tau_{n-1}$, and $0.25\tau_n + 0.75\tau_{n-1}$. To study the behavior of the trading games for $\tau \to \tau^{max}$, we further augment $DG_{\tau}(b\nu)$ with three tick sizes: $0.5(\tau_1 + \tau^{max})$, $0.75\tau_1 + 0.25\tau^{max}$, and $0.25\tau_1 + 0.75\tau^{max}$. Overall, the cardinality of $DG_{\tau}(b\nu)$ is 60, and we denote each tick size that belongs to the search grid as $\tau_{DG} \in DG_{\tau}(b\nu)$.

For each τ_{DG} we analytically derive the equilibrium order submission probabilities of market participants and the associated total welfare for the T-period game. We then select the tick size associated with the highest total welfare: $\tau^{\star}_{DG}$. To further check the robustness of our result, we run the Simulated Annealing (SA) algorithm and use $\tau^{\star}_{DG}$ as the initial condition. The tick size that according to the SA algorithm maximizes total welfare of market participants is the OTS: $\tau^{\star}_{SA} = OTS$.

D.7 Market Quality Metrics

In this appendix we define our market quality metrics. The expected volume in our 3-period model is:

$$
\begin{aligned}
vol(\tau) = &\sum_{k=-p_{nf}}^{+p_{nf\,t_1}} Pr\left(lb_{k,t_1}|\Lambda_{t_0},\tau\right)\Big(Pr\left(ms_{k,t_2}|\Lambda_{t_1},\tau\right)+\\
&\sum_{l>k} Pr\left(ls_{k+l,t_2}|\Lambda_{t_1},\tau\right)\left[\left(Pr\left(ms_{k,t_3}|\Lambda_{t_2},\tau\right)+Pr\left(mb_{k+l,t_3}|\Lambda_{t_2},\tau\right)\right]+\right.\\
&\sum_{l>k} Pr\left(lb_{k+l,t_2}|\Lambda_{t_1},\tau\right)Pr\left(ms_{k+l,t_3}|\Lambda_{t_2},\tau\right)+\\
&Pr\left(nt_k|\Lambda_{t_1},\tau\right)Pr\left(ms_{k,t_3}|\Lambda_{t_2},\tau\right)\Big)
\end{aligned}
\tag{127}
$$

Equation (127) shows that volume in our model endogenously derives from the execution of limit orders. A limit buy order submitted at t_1 with probability $Pr\left(lb_{k,t_1}|\Lambda_{t_0},\tau\right)$, can be executed either at t_2 by an investor posting a market sell order with probability $Pr\left(ms_{k,t_2}|\Lambda_{t_1},\tau\right)$, or - if the investor arriving at t_2 opts for a limit sell order with probability $\sum_{l>k} Pr\left(ls_{k+l,t_2}|\Lambda_{t_1},\tau\right)$ - it can be executed at t_3 with probability $(Pr\left(ms_{k,t_3}|\Lambda_{t_2},\tau\right)$. If instead the 2^{nd} player opts not to trade with probability $Pr\left(nt_k|\Lambda_{t_1},\tau\right)$, the limit buy order posted at t_1 can be executed at t_3 with probability $Pr\left(ms_{k,t_3}|\Lambda_{t_2},\tau\right)$. Volume may also be the result of the execution of orders submitted by the 2^{nd} player. If the 2^{nd} player posts a limit sell order with probability $\sum_{l>k} Pr\left(ls_{k+l,t_2}|\Lambda_{t_1},\tau\right)$, this order will be executed at t_3 with probability $Pr\left(mb_{k+l,t_3}|\Lambda_{t_2},\tau\right)$; if instead he undercuts the 1^{st} player by posting a more aggressive limit buy order with probability $\sum_{l>k} Pr\left(lb_{k+l,t_2}|\Lambda_{t_1},\tau\right)$, this limit order will be executed at t_3 with probability $Pr\left(ms_{k+l,t_3}|\Lambda_{t_2},\tau\right)$.

We compute our metric of expected quoted spread across all the periods in which investors may supply liquidity, excluding the last period of the trading game. The quoted spread in each period of the game is equal to:

$$
\begin{aligned}
&spread(t_1,\tau) = \sum_{k=-p_{nf}}^{+p_{nf}} Pr\left(lb_{k,t_1}|\Lambda_{t_0},\tau\right)\left(\nu(1+b)-p_k\right) \\
&spread(t_2,\tau) = \sum_{k=-p_{nf}}^{+p_{nf}} Pr\left(lb_{k,t_1}|\Lambda_{t_0},\tau\right)\Big(\\
&Pr\left(ms_{k,t_2}|\Lambda_{t_1},\tau\right)\left(\nu(1+b)-\nu(1-b)\right) + Pr\left(nt_{k,t_2}|\Lambda_{t_1},\tau\right)\left(\nu(1+b)-p_k\right)+ \\
&\sum_{l>k} Pr\left(ls_{k+l,t_2}|\Lambda_{t_1},\tau\right)\left(p_{k+l}-p_k\right) + Pr\left(lb_{k+l,t_2}|\Lambda_{t_1},\tau\right)\left(\nu(1+b)-p_{k+l}\right)\Big)
\end{aligned}
\tag{128}
$$

To quantify quoted spread, we here assume that if one side of the market is empty, the spread is the distance between the posted price and the evaluation bound on the other side of the market. The expected spread is the average of the two periods expected quoted spreads in (128):

$$
spread(\tau) = \frac{1}{2}\left(spread(t_1,\tau) + spread(t_2,\tau)\right) \tag{129}
$$

We compute our metric of expected total depth across all the periods in which investors may supply liquidity, excluding the last period of the trading game. The expected total depth is equal to the expected number of shares associated with all the equilibrium feasible price levels:

$$
\begin{aligned}
&depth(\tau) = 1\times \sum_{k=-p_{nf}}^{+p_{nf}} Pr\left(lb_{k,t_1}|\Lambda_{t_0},\tau\right) + 2\times \sum_{k=-p_{nf}}^{+p_{nf}} Pr\left(lb_{k,t_1}|\Lambda_{t_0},\tau\right)\sum_{l>k} Pr\left(ls_{k+l,t_2}|\Lambda_{t_1},\tau\right) + \\
&2\times \sum_{k=-p_{nf}}^{+p_{nf}} Pr\left(lb_{k,t_1}|\Lambda_{t_0},\tau\right)\sum_{l>k} Pr\left(lb_{k+l,t_2}|\Lambda_{t_1},\tau\right) + 1\times \sum_{k=-p_{nf}}^{+p_{nf}} Pr\left(lb_{k,t_1}|\Lambda_{t_0},\tau\right) Pr\left(nt_k|\Lambda_{t_1},\tau\right)
\end{aligned}
\tag{130}
$$

For example, the first line of equation (130) refers to the number of shares associated with a limit buy order posted at p_k with probability $Pr\left(lb_{k,t_1}|\Lambda_{t_0},\tau\right)$, followed by a limit sell order posted at p_{k+l} with probability $Pr\left(ls_{k+l,t_2}|\Lambda_{t_1},\tau\right)$.

E Appendix: Four Period Model

E.1 Proof of Proposition (5)

At t_4, the order submission probabilities are defined by Lemma 1.3:

- $Pr\left(ms_{k,t_4}|\Lambda_{t_3},\tau\right) = \frac{1}{\Gamma}\left(\frac{p_k}{v} - (1-b)\right)$
- $Pr\left(mb_{k,t_4}|\Lambda_{t_3},\tau\right) = \frac{1}{\Gamma}\left((1+b) - \frac{p_k}{v}\right)$

At t_3, there are three possible states of the book:

- The book is empty ($\Lambda_{t_2} = \{lb_{k,t_1}, ms_{k,t_2}\}$), hence the 3^{rd} player submits a limit buy order following Proposition (1).
- The book has a limit buy and a limit sell ($\Lambda_{t_2} = \{lb_{k,t_1}, ls_{k+1,t_2}\}$), hence the 3^{rd} player is a liquidity taker only and his order submission probabilities are defined by Lemma 1.3.
- The book has limit buy orders only ($\Lambda_{t_2} = \{lb_{k,t_1}, lb_{k+1,t_2}\}$ or $\Lambda_{t_2} = \{lb_{k,t_1}, lb_{k,t_2}\}$), and the order submission probabilities of the 3^{rd} player are the same as the order submission probabilities of the 2^{nd} player in the 3-period model (Proposition (3)).

At t_2, the alternative options available for the 2^{nd} player are: market sell hitting the limit buy order posted by the 1^{st} player at p_k, limit sell or limit buy at p_{k+1}, and finally limit buy at p_k, thus queuing behind the 1^{st} player's limit buy order.

The probability of a market sell at t_2 is:

$$
\begin{aligned}
&Pr\left(ms_{k,t_2}|\Lambda_{t_1},\tau\right) = \\
&Pr\Big(p_k - \beta_{t_2}\nu > 0, \\
&p_k - \beta_{t_2}\nu > (p_{k+1} - \beta_{t_2}\nu)\left[Pr\left(mb_{k+1,t_3}|\Lambda_{t_2},\tau\right) + \left(1 - Pr\left(mb_{k+1,t_3}|\Lambda_{t_2},\tau\right)\right) \times Pr\left(mb_{k+1,t_4}|\Lambda_{t_3},\tau\right)\right], \\
&p_k - \beta_{t_2}\nu > (\beta_{t_2}\nu - p_k)Pr\left(ms_{k,t_3}|\Lambda_{t_2},\tau\right) \times Pr\left(ms_{k,t_4}|\Lambda_{t_3},\tau\right), \\
&p_k - \beta_{t_2}\nu > \\
&(\beta_{t_2}\nu - p_{k+1})\left[Pr\left(ms_{k+1,t_3}|\Lambda_{t_2},\tau\right) + \left(Pr\left(nt_{k+1,t_3}|\Lambda_{t_2},\tau\right) + Pr\left(ls_{k+2,t_3}|\Lambda_{t_2},\tau\right)\right) \times Pr\left(ms_{k+1,t_4}|\Lambda_{t_3},\tau\right)\right]\Big)
\end{aligned}
\tag{131}
$$

To simplify the notation, we define:

- $f = Pr\left(mb_{k+1,t_3}|\Lambda_{t_2},\tau\right) + \left(1 - Pr\left(mb_{k+1,t_3}|\Lambda_{t_2},\tau\right)\right) \times Pr\left(mb_{k+1,t_4}|\Lambda_{t_3},\tau\right)$
- $l = Pr\left(ms_{k,t_3}|\Lambda_{t_2},\tau\right) \times Pr\left(ms_{k,t_4}|\Lambda_{t_3},\tau\right)$
- $g = \left[Pr\left(ms_{k+1,t_3}|\Lambda_{t_2},\tau\right) + \left(Pr\left(nt_{k+1,t_3}|\Lambda_{t_2},\tau\right) + Pr\left(ls_{k+2,t_3}|\Lambda_{t_2},\tau\right)\right) \times Pr\left(ms_{k+1,t_4}|\Lambda_{t_3},\tau\right)\right]$

Equation (131) can be simplified as follows:

$$
Pr\left(ms_{k,t_2}|\Lambda_{t_1},\tau\right) = max\left[0,\ Pr\left((1-b) < \beta_{t_2} < \frac{p_k}{\nu} - \frac{\tau}{\nu}\frac{f}{1-f}\right)\right] \tag{132}
$$

The probability of a limit sell at t_2 is:

$$
\begin{aligned}
&Pr\left(ls_{k+1,t_2}|\Lambda_{t_1},\tau\right) = \\
&Pr\Big((p_{k+1} - \beta_{t_2}\nu)\left[Pr\left(mb_{k+1,t_4}|\Lambda_{t_3},\tau\right) + \left(1 - Pr\left(mb_{k+1,t_4}|\Lambda_{t_3},\tau\right)\right) \times Pr\left(mb_{k+1,t_4}|\Lambda_{t_3},\tau\right)\right] > 0, \\
&(p_{k+1} - \beta_{t_2}\nu)\left[Pr\left(mb_{k+1,t_4}|\Lambda_{t_3},\tau\right) + \left(1 - Pr\left(mb_{k+1,t_4}|\Lambda_{t_3},\tau\right)\right) \times Pr\left(mb_{k+1,t_4}|\Lambda_{t_3},\tau\right)\right] > p_k - \beta_{t_2}\nu \\
&(p_{k+1} - \beta_{t_2}\nu)\left[Pr\left(mb_{k+1,t_4}|\Lambda_{t_3},\tau\right) + \left(1 - Pr\left(mb_{k+1,t_4}|\Lambda_{t_3},\tau\right)\right) \times Pr\left(mb_{k+1,t_4}|\Lambda_{t_3},\tau\right)\right] > \\
&(\beta_{t_2}\nu - p_k)Pr\left(ms_{k,t_3}|\Lambda_{t_2},\tau\right) \times Pr\left(ms_{k,t_4}|\Lambda_{t_3},\tau\right), \\
&(p_{k+1} - \beta_{t_2}\nu)\left[Pr\left(mb_{k+1,t_4}|\Lambda_{t_3},\tau\right) + \left(1 - Pr\left(mb_{k+1,t_4}|\Lambda_{t_3},\tau\right)\right) \times Pr\left(mb_{k+1,t_4}|\Lambda_{t_3},\tau\right)\right] > \\
&(\beta_{t_2}\nu - p_{k+1})\left[Pr\left(ms_{k+1,t_3}|\Lambda_{t_2},\tau\right) + \left(Pr\left(nt_{k+1,t_3}|\Lambda_{t_2},\tau\right) + Pr\left(ls_{k+2,t_3}|\Lambda_{t_2},\tau\right)\right) \times Pr\left(ms_{k+1,t_4}|\Lambda_{t_3},\tau\right)\right]\Big)
\end{aligned}
\tag{133}
$$

Equation (133) can be simplified as follows:

$$Pr\left(ls_{k+1,t_2}|\Lambda_{t_1},\tau\right) = \begin{cases} Pr\left(\frac{p_k}{\nu} - \frac{\tau}{\nu}\frac{f}{1-f} < \beta_{t_2} < \frac{p_k}{\nu} + \frac{\tau}{\nu}\frac{f}{f+l}\right) & if\ Pr\left(ms_{k,t_2}|\Lambda_{t_1},\tau\right) > 0 \\ Pr\big((1-b) < \beta_{t_2} < \frac{p_k}{\nu} + \frac{\tau}{\nu}\frac{f}{f+l}\big) & otherwise \end{cases} \tag{134}$$

The probability that the 2^{nd} player submits a limit buy at p_k is:

$$\begin{aligned}
&Pr\left(lb_{k,t_2}|\Lambda_{t_1},\tau\right) = \\
&Pr\big((\beta_{t_2}\nu - p_k)Pr\left(ms_{k,t_3}|\Lambda_{t_2},\tau\right) \times Pr\left(ms_{k,t_4}|\Lambda_{t_3},\tau\right) > 0, \\
&(\beta_{t_2}\nu - p_k)Pr\left(ms_{k,t_3}|\Lambda_{t_2},\tau\right) \times Pr\left(ms_{k,t_4}|\Lambda_{t_3},\tau\right) > p_k - \beta_{t_2}\nu, \\
&(\beta_{t_2}\nu - p_k)Pr\left(ms_{k,t_3}|\Lambda_{t_2},\tau\right) \times Pr\left(ms_{k,t_4}|\Lambda_{t_3},\tau\right) > \\
&(p_{k+1} - \beta_{t_2}\nu)\left[Pr\left(mb_{k+1,t_4}|\Lambda_{t_3},\tau\right) + (1 - Pr\left(mb_{k+1,t_4}|\Lambda_{t_3},\tau\right)) \times Pr\left(mb_{k+1,t_4}|\Lambda_{t_3},\tau\right)\right], \\
&(\beta_{t_2}\nu - p_k)Pr\left(ms_{k,t_3}|\Lambda_{t_2},\tau\right) \times Pr\left(ms_{k,t_4}|\Lambda_{t_3},\tau\right) > \\
&(\beta_{t_2}\nu - p_{k+1})\left[Pr\left(ms_{k+1,t_3}|\Lambda_{t_2},\tau\right) + (Pr\left(nt_{k+1,t_3}|\Lambda_{t_2},\tau\right) + Pr\left(ls_{k+2,t_3}|\Lambda_{t_2},\tau\right)) \times Pr\left(ms_{k+1,t_4}|\Lambda_{t_3},\tau\right)\right]\big)
\end{aligned} \tag{135}$$

Equation (135) can be simplified in the following way:

$$Pr\left(lb_{k,t_2}|\Lambda_{t_1},\tau\right) = \begin{cases} Pr\big(\frac{p_k}{\nu} + \frac{\tau}{\nu}\frac{f}{f+l} < \beta_{t_2} < \frac{p_k}{\nu} - \frac{\tau}{\nu}\frac{g}{g-l}\big) & if\ \frac{p_k}{\nu} + \frac{\tau}{\nu}\frac{g}{g-l} < 1+b \\ Pr\big(\frac{p_k}{\nu} + \frac{\tau}{\nu}\frac{f}{f+l} < \beta_{t_2} < 1+b\big) & otherwise \end{cases} \tag{136}$$

Therefore the probability that the 2^{nd} player posts a limit buy at p_{k+1}, thus undercutting, is given by:

$$Pr\left(lb_{k+1,t_2}|\Lambda_{t_1},\tau\right) = \begin{cases} Pr\big(\frac{p_k}{\nu} - \frac{\tau}{\nu}\frac{g}{g-l} < \beta_{t_2} < 1+b\big) & if\ \frac{p_k}{\nu} + \frac{\tau}{\nu}\frac{g}{g-l} < 1+b \\ 0 & otherwise \end{cases} \tag{137}$$

Without loss of generality and using Lemma (1.2), if the 1^{st} player at t_1 is a buyer ($\beta_{t_1} > 1$), he posts a limit buy. The execution probability of a limit buy at price $p_k < p_{n^f}$ is:

$$
\begin{aligned}
&Pr\left(\Psi_{k,t_1}|\Lambda_{t_0},\tau\right) = Pr\left(ms_{k,t_2}|\Lambda_{t_1},\tau\right) + \\
&Pr\left(ls_{k+1,t_2}|\Lambda_{t_1},\tau\right)\left[Pr\left(ms_{k,t_3}|\Lambda_{t_2},\tau\right) + (1 - Pr\left(ms_{k,t_3}|\Lambda_{t_2},\tau\right))Pr\left(ms_{k,t_4}|\Lambda_{t_3},\tau\right)\right] + \\
&Pr\left(lb_{k,t_2}|\Lambda_{t_1},\tau\right)\left[Pr\left(ms_{k,t_3}|\Lambda_{t_2},\tau\right) + Pr\left(ls_{k+1,t_3}|\Lambda_{t_2},\tau\right)Pr\left(ms_{k,t_4}|\Lambda_{t_3},\tau\right)\right] + \\
&Pr\left(lb_{k+1,t_2}|\Lambda_{t_1},\tau\right)Pr\left(ms_{k+1,t_3}|\Lambda_{t_2},\tau\right)Pr\left(ms_{k,t_4}|\Lambda_{t_3},\tau\right)
\end{aligned}
\tag{138}
$$

and the execution probability of a limit buy at p_{n^f} is:

$$
\begin{aligned}
&Pr\left(\Psi_{n^f,t_1}|\Lambda_{t_0},\tau\right) = Pr\left(ms_{n^f,t_2}|\Lambda_{t_1},\tau\right) + \\
&Pr\left(lb_{n^f,t_2}|\Lambda_{t_1},\tau\right)\left[Pr\left(ms_{n^f,t_3}|\Lambda_{t_2},\tau\right) + Pr\left(nt_{n^f,t_3}|\Lambda_{t_2},\tau\right)Pr\left(ms_{n^f,t_4}|\Lambda_{t_3},\tau\right)\right]
\end{aligned}
\tag{139}
$$

Using equations (138) and (139), the equilibrium submission strategies of the 1^{st} player at price $p_k < p_{n^f}$ is:

$$
\begin{aligned}
&Pr\left(lb_{k,t_1}|\Lambda_{t_0},\tau\right) = \\
&Pr\Big((\beta_{t_1}\nu - p_k)Pr\left(\Psi_{k,t_1}|\Lambda_{t_0},\tau\right) > 0, \\
&(\beta_{t_1}\nu - p_k)Pr\left(\Psi_{k,t_1}|\Lambda_{t_0},\tau\right) > (\beta_{t_1}\nu - p_{\tilde{k}})Pr\left(\Psi_{\tilde{k},t_1}|\Lambda_{t_0},\tau\right), \\
&(\beta_{t_1}\nu - p_k)Pr\left(\Psi_{k,t_1}|\Lambda_{t_0},\tau\right) > (\beta_{t_1}\nu - p_{n^f})Pr\left(\Psi_{n^f,t_1}|\Lambda_{t_0},\tau\right)\Big)
\end{aligned}
\tag{140}
$$

In equilibrium, the 1^{st} player submits a limit order at p_{n^f} with probability:

$$
\begin{aligned}
&Pr\left(lb_{n^f,t_1}|\Lambda_{t_0},\tau\right) = \\
&Pr\Big((\beta_{t_1}\nu - p_{n^f})Pr\left(\Psi_{n^f,t_1}|\Lambda_{t_0},\tau\right) > 0, \\
&(\beta_{t_1}\nu - p_{n^f})Pr\left(\Psi_{n^f,t_1}|\Lambda_{t_0},\tau\right) > (\beta_{t_1}\nu - p_k)Pr\left(\Psi_{k,t_1}|\Lambda_{t_0},\tau\right)\Big)
\end{aligned}
\tag{141}
$$

We show that for any $\tau \in \left\{(0,\tau^{max})\,|n^f \geq 2\right\}$, at t_1 there exists at least one limit buy order, e.g., at p_{+1}, that dominates a limit buy at p_{+n^f}. As $n^f \geq 2$, only if by equation (21) $2b\nu \geq 3\tau$ the payoff of the 1^{st} player submitting a limit buy at p_{+1} (equation (140)) is strictly greater than the payoff from posting a limit buy at p_{n^f} (equation (141)). Therefore, the 1^{st} player never posts a limit buy at p_{+n^f}, which is a dominated strategy.

In the remaining part of the proof, we consider a generic τ such that $\tau \in (0, \tau^{max}) \, |n^f = 1$ and we show if there is only price level on each side of the book, the 1^{st} will lock the market with a positive probability. If the investor arriving at t_1 limit buys at p_{+1}, the submission probabilities of the 2^{nd} player are the following:

- $Pr\left(ms_{+1,t_2}|\Lambda_{t_1},\tau\right) = Pr\left(p_{+1} - \beta\nu > 0\right)$
- $Pr\left(nt_{+1,t_2}|\Lambda_{t_1},\tau\right) = 1 - Pr\left(ms_{+1,t_2}|\Lambda_{t_1},\tau\right)$

Having defined the payoff from both a limit buy at p_{-1} and a limit buy at p_{+1}, we can determine the associated probability of submission by equating the payoff from the two strategies:

$$\left(\beta_{t_1}\nu - p_{-1}\right) \times Pr\left(\Psi_{-1,t_1}|\Lambda_{t_0},\tau\right) = \left(\beta_{t_1}\nu - p_{+1}\right) \times Pr\left(\Psi_{+1,t_1}|\Lambda_{t_0},\tau\right) \tag{142}$$

Solving (142) by β_{t_1} we show that in the τ region ensuring $n^f = 1$ which is defined by $\tau \in (0, \tau^{max}) \, |2b\nu < 3\tau$, equation (142) admits an internal solution $\beta^\star_{t_1}$ and the 1^{st} player order submission probabilities are:

- $Pr\left(lb_{-1,t_1}|\Lambda_{t_0},\tau\right) = \frac{1}{\Gamma}\left[\beta^\star_{t_1} - 1\right]$
- $Pr\left(lb_{+1,t_1}|\Lambda_{t_0},\tau\right) = \frac{1}{\Gamma}\left[1 + b - \beta^\star_{t_1}\right]$

E.2 Proof of Proposition (6)

As for the proof of Proposition 4 (Appendix D.5), we now show how the equilibrium order submission probabilities and the associated welfare of the strategic game described in Appendix (E.1) change for $\tau \to 0^+$. As τ decreases, the number of feasible prices within the investor's support, $2b\nu$, increases. Approaching a continuum of prices, we indicate a generic feasible price as p. The order submission probabilities associated with the trading strategies of the 4^{rd} player are:

- $Pr\left(ms_{t_4}|\Lambda_{t_3}\right) = \frac{1}{\Gamma}\left(\frac{p}{v} - (1-b)\right)$

- $Pr\left(mb_{t_4}|\Lambda_{t_3}\right) = \frac{1}{\Gamma}\left((1+b) - \frac{p}{v}\right)$

The order submission probabilities for the 3^{rd} depend on the state of the book:

- If the book is empty ($\Lambda_{t_2} = \{lb_{k,t_1}, ms_{k,t_2}\}$), the 3^{rd} player submits a limit buy order at $p^{\star}_{t_3} = \frac{\nu}{2}\left(\beta_{t_3} + 1 - b\right)$ (equation (67))

- If the book opens with both a limit buy and a limit sell order ($\Lambda_{t_2} = \{lb_{k,t_1}, ls_{k+1,t_2}\}$), the 3^{rd} player submits market orders with the following probabilities:

  - $Pr\left(ms_{t_3}|\Lambda_{t_2}\right) = \frac{1}{\Gamma}\left(\frac{p}{v} - (1-b)\right)$
  - $Pr\left(mb_{t_3}|\Lambda_{t_2}\right) = \frac{1}{\Gamma}\left((1+b) - \frac{p}{v}\right)$.

- If the book opens with limit buy orders only ($\Lambda_{t_2} = \{lb_{k,t_1}, lb_{k+1,t_2}\}$ or $\Lambda_{t_2} = \{lb_{k,t_1}, lb_{k,t_2}\}$), the 3^{rd} player submits orders according to Appendix (D.5) equations (111) - (112) - (113).

The order submission probabilities of the 2^{nd} player for $\tau \to 0^+$ can be obtained by considering equations (132) - (134) - (136) - (137):

$$lim_{\tau \to 0^+} Pr\left(ms_{k,t_2}|\Lambda_{t_1}\right) = Pr\left(1 - b < \beta_{t_2} < \frac{p}{\nu}\right) = \left(\frac{p}{\Gamma\nu} - \frac{1-b}{\Gamma}\right) \tag{143}$$

$$lim_{\tau \to 0^+} Pr\left(ls_{k+1,t_2}|\Lambda_{t_1}\right) = 0 \tag{144}$$

$$lim_{\tau \to 0^+} Pr\left(lb_{k,t_2}|\Lambda_{t_1}\right) = 0 \tag{145}$$

$$lim_{\tau \to 0^+} Pr\left(lb_{k+1,t_2}|\Lambda_{t_1}\right) \approx Pr\left(\frac{p}{\nu} < \beta_{t_2} < (1+b)\right) \tag{146}$$

Considering the case of $\tau \to 0^+$, if the 2^{nd} player undercuts the 1^{st} player to the next adjacent price, he undercuts at a price $p_{t_2} = p + o(\epsilon)$ by an almost negligible quantity to gain price priority, hence $p_{t_2} \sim p$. As in the 3-period game, when τ approaches 0, in equilibrium the 2^{nd} player mainly focuses on aggressive strategies: he either market sells or undercuts the existing limit buy order; he neither limit sells nor queues behind the standing limit order.

For $\tau \to 0^+$, the probability of execution of a limit buy posted at p at t_1 is:

$$Pr\left(\Psi_{t_1}|\Lambda_{t_0}\right) = \big(Pr\left(ms_{k,t_2}|\Lambda_{t_1},\right) + Pr\left(lb_{k+1,t_2}|\Lambda_{t_1}\right) \times Pr\left(ms_{k,t_3}|\Lambda_{t_2}\right) \times Pr\left(ms_{k,t_4}|\Lambda_{t_3}\right)\big) =$$
$$\frac{1}{\Gamma}\left(\frac{p}{v} - (1-b)\right) + \left(\frac{1}{\Gamma}\left(\frac{p}{v} - (1-b)\right)\right)^2 - \left(\frac{1}{\Gamma}\left(\frac{p}{v} - (1-b)\right)\right)^3 \tag{147}$$

The generic payoff of the 1^{st} player for a limit order buy is:

$$\left(\beta_{t_1}\nu - p\right) Pr\left(\Psi_{t_1}|\Lambda_{t_0}\right) \tag{148}$$

Substituting (147) in (148) and taking the first order conditions $w.r.t.$ p, for any $\beta_{t_1} \in (1,\ 1+b)$, the 1^{st} player submits a limit buy order with probability 1 at the following price:

$$p^{\star}_{t_1} \approx \frac{\nu}{4}\left(\beta_{t_1} + 3 - b\right) \tag{149}$$

We can now compute the ex ante welfare of the players. Substituting (149) in (148), we obtain the 1^{st} player's welfare as:

$$\omega_{t_1}(lb_{t_1}) = \int_1^{1+b} \left(\beta_{t_1}\nu - p^{\star}_{t_1}\right) \times Pr\left(\Psi_{t_1}|\Lambda_{t_0}\right) \frac{1}{\Gamma} d\beta_{t_1} = \frac{10771}{61440} b\nu \tag{150}$$

The 2^{nd} player's welfare in case of market sell is:

$$\begin{aligned}\omega_{t_2}(ms_{t_2}) &= \int_1^{1+b}\left(\int_{1-b}^{\frac{p^{\star}_{t_1}}{\nu}} \left(p^{\star}_{t_1} - \beta_{t_2}\nu\right)\frac{1}{\Gamma} Pr\left(lb_{p^{\star}_{t_1},t_1}|\Lambda_{t_0}\right) d\beta_{t_2}|\beta_{t_1}\right)\frac{1}{\Gamma}d\beta_{t_1} \\ &= \int_1^{1+b}\left(\int_{1-b}^{\frac{p^{\star}_{t_1}}{\nu}} \left(p^{\star}_{t_1} - \beta_{t_2}\nu\right)\frac{1}{\Gamma} d\beta_{t_2}|\beta_{t_1}\right)\frac{1}{\Gamma}d\beta_{t_1} = \frac{37}{384} b\nu\end{aligned} \tag{151}$$

The 2^{nd} player's welfare in case of undercutting the limit buy posted by the 1^{st} player is:

$$\begin{aligned}\omega_{t_2}(lb_{t_2}) &= \int_1^{1+b}\left(\int_{\frac{p^\star_{t_1}}{\nu}}^{1+b}(\beta_{t_2}\nu - p^\star_{t_1})Pr\left(lb_{p^\star_{t_1},t_1}|\Lambda_{t_0}\right)Pr\left(ms_{p^\star_{t_1},t_3}|\Lambda_{t_2}\right)\frac{1}{\Gamma}d\beta_{t_2}|\beta_{t_1}\right)\frac{1}{\Gamma}d\beta_{t_1} \\ &= \int_1^{1+b}\left(\int_{\frac{p^\star_{t_1}}{\nu}}^{1+b}(\beta_{t_2}\nu - p^\star_{t_1})\frac{1}{\Gamma}\left(\frac{p^\star_{t_1}}{v}-(1-b)\right)\frac{1}{\Gamma}d\beta_{t_2}|\beta_{t_1}\right)\frac{1}{\Gamma}d\beta_{t_1} = \frac{845}{12288}b\nu\end{aligned} \tag{152}$$

The welfare of the 3^{rd} player at t_3 depends on his strategic action given the state of the book. If at t_3 the book opens empty - which happens with probability:

$$\int_1^{1+b}Pr\left(lb_{p,t_1}|\Lambda_{t_0}\right)Pr\left(ms_{p,t_2}|\Lambda_{t_2}\right)\frac{1}{\Gamma}d\beta_{t_1} = \int_1^{1+b}\frac{1}{\Gamma}\left(\frac{p^\star_{t_1}}{v}-(1-b)\right)\frac{1}{\Gamma}d\beta_{t_1} = \frac{7}{32}, \tag{153}$$

following Lemma (1), the 3^{rd} player submits a limit buy order and the associated welfare is given by:

$$\omega_{t_3}(lb_{t_3}) = \frac{7}{32}\int_1^{1+b}\left(\beta_{t_3}\nu - p^\star_{t_3}\right)\times Pr\left(ms_{p^\star_{t_3},t_4}|\Lambda_{t_3}\right)\frac{1}{\Gamma}d\beta_{t_3} = \frac{7}{32}\frac{7b\nu}{48} = \frac{49}{1536}b\nu \tag{154}$$

If instead the book at t_3 opens with two limit buy orders as the 2^{nd} player undercuts the 1^{st} player's limit buy order, the 3^{rd} player either market sells or further undercuts the standing limit order.

The welfare associated with a market sell order is:

$$\begin{aligned}\omega_{t_3}(ms_{t_3}) &= \int_1^{1+b}\left(\int_{1-b}^{\frac{p^\star_{t_1}}{\nu}}\left(p^\star_{t_1}-\beta_{t_3}\nu\right)\frac{1}{\Gamma}Pr\left(lb_{p^\star_{t_1},t_1}|\Lambda_{t_0}\right)Pr\left(lb_{p^\star_{t_1},t_2}|\Lambda_{t_1}\right)d\beta_{t_3}|\beta_{t_1}\right)\frac{1}{\Gamma}d\beta_{t_1} \\ &= \int_1^{1+b}\left(\int_{1-b}^{\frac{p^\star_{t_1}}{\nu}}\left(p^\star_{t_1}-\beta_{t_3}\nu\right)\left(\frac{1+b}{\Gamma}-\frac{p^\star_{t_1}}{\Gamma\nu}\right)\frac{1}{\Gamma}d\beta_{t_3}|\beta_{t_1}\right)\frac{1}{\Gamma}d\beta_{t_1} = \frac{659}{12288}b\nu\end{aligned} \tag{155}$$

Whereas the welfare associated to a limit buy order that undercuts the existing limit buy orders

is:

$$\omega_{t_3}(lb_{t_3}) = \int_1^{1+b} \left(\int_{1-b}^{\frac{p^\star_{t_1}}{\nu}} \left(\beta_{t_3}\nu - p^\star_{t_1}\right) \frac{1}{\Gamma} Pr\left(lb_{p^\star_{t_1},t_1}|\Lambda_{t_0}\right) Pr\left(lb_{p^\star_{t_1},t_2}|\Lambda_{t_1}\right) Pr\left(ms_{p^\star_{t_1},t_4}|\Lambda_{t_3}\right) d\beta_{t_3}|\beta_{t_1} \right) \frac{1}{\Gamma} d\beta_{t_1}$$

$$= \int_1^{1+b} \left(\int_{1-b}^{\frac{p^\star_{t_1}}{\nu}} \left(\beta_{t_3}\nu - p^\star_{t_1}\right) \left(\frac{1+b}{\Gamma} - \frac{p^\star_{t_1}}{\Gamma\nu}\right) \left(\frac{p^\star_{t_1}}{\Gamma\nu} - \frac{1-b}{\Gamma}\right) \frac{1}{\Gamma} d\beta_{t_3}|\beta_{t_1} \right) \frac{1}{\Gamma} d\beta_{t_1} = \frac{1589}{40960} b\nu \tag{156}$$

The welfare of the 4^{th} player is given by market selling either at $p^\star_{t_1}$ or at $p^\star_{t_3}$, conditional to the status of the book. In case the book opens empty at t_3 the 3^{rd} player submits $lb(p^\star_{t_3})$ with probability 1 and the welfare of the 4^{th} player is:

$$\begin{aligned} \omega_{t_4}(ms_{t_4}) &= \frac{7}{32} \int_1^{1+b} \left(\int_{1-b}^{\frac{p^\star_{t_3}}{\nu}} \left(p^\star_{t_3} - \beta_{t_4}\nu\right) \frac{1}{\Gamma} Pr\left(lb_{p^\star_{t_3},t_3}|\Lambda_{t_2}\right) d\beta_{t_4}|\beta_{t_3} \right) \frac{1}{\Gamma} d\beta_{t_3} \\ &= \frac{7}{32} \int_1^{1+b} \left(\int_{1-b}^{\frac{p^\star_{t_3}}{\nu}} \left(p^\star_{t_1} - \beta_{t_4}\nu\right) \frac{1}{\Gamma} d\beta_{t_4}|\beta_{t_3} \right) \frac{1}{\Gamma} d\beta_{t_3} = \frac{7}{32}\frac{7}{96} b\nu = \frac{49}{3072} b\nu \end{aligned} \tag{157}$$

If instead the book at t_3 opens with two limit buy order as the 2^{nd} player undercuts the 1^{st} player's limit buy orders, the 4^{th} player can always market sell at $p^\star_{t_1}$. The welfare of the 4^{th} player is:

$$\begin{aligned} \omega_{t_4}(ms_{t_4}) &= \int_1^{1+b} \left(\int_{1-b}^{\frac{p^\star_{t_1}}{\nu}} \left(p^\star_{t_1} - \beta_{t_4}\nu\right) \frac{1}{\Gamma} Pr\left(lb_{p^\star_{t_1},t_1}|\Lambda_{t_0}\right) Pr\left(lb_{p^\star_{t_1},t_2}|\Lambda_{t_1}\right) d\beta_{t_4}|\beta_{t_1} \right) \frac{1}{\Gamma} d\beta_{t_1} \\ &= \int_1^{1+b} \left(\int_{1-b}^{\frac{p^\star_{t_1}}{\nu}} \left(p^\star_{t_1} - \beta_{t_4}\nu\right) \left(\frac{1+b}{\Gamma} - \frac{p^\star_{t_1}}{\Gamma\nu}\right) \frac{1}{\Gamma} d\beta_{t_4}|\beta_{t_1} \right) \frac{1}{\Gamma} d\beta_{t_1} = \frac{659}{12288} b\nu \end{aligned} \tag{158}$$

The total welfare with a $\tau \to 0^+$ is hence given by:

$$\Omega(\tau \to 0^+) = \frac{65659}{122880} b\nu \approx 0.534\, b\nu \tag{159}$$

In order to show that $\tau \to 0^+$ is not the OTS, , we need to find a $\tau > 0$ with an associated welfare greater than $\frac{65659}{122880} b\nu$. For a generic combination of (b, ν), consider $\tau = \frac{b\nu}{2}$. The price grid and the associated submissions probabilities are:

Table 1.E: 4-period Game: Order Submission Probabilities
This table reports the order submission probabilities of the 4-period model for a generic combination of (b, ν), $\tau = \frac{b\nu}{2}$ and $p_k = \{p_{-2},\ p_{-1},\ p_{+1},\ p_{+2}\}$. Note that the equilibrium order submission strategies are those associated with $p_k = \{p_{-1}, p_{+1}\}$

Order Submission Probabilities	p_{+1}	p_{-1}
$Pr(lb_{k,t_1}\|\Lambda_{t_0},\tau)$	0.1185	0.3815
$Pr(ms_{k,t_2}\|\Lambda_{t_1},\tau)$	0.5485	0
$Pr(ls_{k+1,t_2}\|\Lambda_{t_1},\tau)$	0.1737	0.5946
$Pr(lb_{k,t_2}\|\Lambda_{t_1},\tau)$	0.2778	0.0613
$Pr(lb_{k+1,t_2}\|\Lambda_{t_1},\tau)$	0	0.3441
$Pr(ms_{k,t_3}\|\Lambda_{t_2}=\{lb,lb\},\tau)$	0.5893	0.225
$Pr(ls_{k+1,t_3}\|\Lambda_{t_2}=\{lb,lb\},\tau)$	0.2857	0.4
$Pr(lb_{k+1,t_3}\|\Lambda_{t_2}=\{lb,lb\},\tau)$	0.125	0.375
$Pr(lb_{k,t_3}\|\Lambda_{t_2}=\{lb,ms\},\tau)$	0	0.5
$Pr(ms_{k,t_3}\|\Lambda_{t_2}=\{lb,ls\},\tau)$	0.625	0.375
$Pr(mb_{k+1,t_3}\|\Lambda_{t_2}=\{lb,ls\},\tau)$	0.125	0.375
$Pr(ms_{k,t_4}\|\Lambda_{t_3},\tau)$	0.625	0.375
$Pr(mb_{k+1,t_4}\|\Lambda_{t_3},\tau)$	0.125	0.375

The 1^{st} player submits limit buy orders at p_{-1} and p_{+1} and his expected welfare is:

$$\begin{gathered}\omega_{t_1}(lb_{t_1}\ |\tau) = \\ \sum_{k=-1}^{+1}\big(Pr\left(ms_{k,t_2}|\Lambda_{t_1},\tau\right) + \\ Pr\left(ls_{k+1,t_2}|\Lambda_{t_1},\tau\right)\left[Pr\left(ms_{k,t_3}|\Lambda_{t_2},\tau\right) + (1 - Pr\left(ms_{k,t_3}|\Lambda_{t_2},\tau\right))Pr\left(ms_{k,t_4}|\Lambda_{t_3},\tau\right)\right] + \\ Pr\left(lb_{k,t_2}|\Lambda_{t_1},\tau\right)\left[Pr\left(ms_{k,t_3}|\Lambda_{t_2},\tau\right) + Pr\left(ls_{k+1,t_3}|\Lambda_{t_2},\tau\right)Pr\left(ms_{k,t_4}|\Lambda_{t_3},\tau\right)\right] + \\ Pr\left(lb_{k+1,t_2}|\Lambda_{t_1},\tau\right)Pr\left(ms_{k+1,t_3}|\Lambda_{t_2},\tau\right)Pr\left(ms_{k,t_4}|\Lambda_{t_3},\tau\right)\big)\int_{\beta_{t_1}\in B(\tau)}\frac{\beta_{t_1}\nu - p_k}{\Gamma}d\beta_{t_1} = 0.1793\,b\nu\end{gathered} \tag{160}$$

The welfare of the 2^{nd} player is :

$$
\begin{aligned}
&\omega_{t_2}(ms_{t_2} \ \vee ls_{t_2} \ \vee lb_{t_2}|\tau) = \\
&\sum_{k=-1}^{+1} Pr\left(lb_{k,t_1}|\Lambda_{t_0},\tau\right) \int_{\beta_{t_2}\in B(\tau)} \frac{p_k - \beta_{t_2}\nu}{\Gamma} d\beta_{t_2} + \\
&\sum_{k=-1}^{1} Pr\left(lb_{k,t_1}|\Lambda_{t_0},\tau\right) \Big[Pr\left(mb_{k+1,t_3}|\Lambda_{t_2},\tau\right) + \Big(1 - Pr\left(mb_{k+1,t_3}|\Lambda_{t_2},\tau\right)\Big) Pr\left(mb_{k+1,t_4}|\Lambda_{t_3},\tau\right)\Big] \\
&\int_{\beta_{t_2}\in B(\tau)} \frac{p_{k+1} - \beta_{t_2}\nu}{\Gamma} d\beta_{t_2} + \\
&\sum_{k=-1}^{+1} Pr\left(lb_{k,t_1}|\Lambda_{t_0},\tau\right) Pr\left(ms_{k,t_3}|\Lambda_{t_2},\tau\right) Pr\left(ms_{k,t_4}|\Lambda_{t_3},\tau\right) \int_{\beta_{t_2}\in B(\tau)} \frac{\beta_{t_2}\nu - p_k}{\Gamma} d\beta_{t_2} + \\
&\sum_{k=-1}^{1} Pr\left(lb_{k,t_1}|\Lambda_{t_0},\tau\right) \left[Pr\left(ms_{k+1,t_3}|\Lambda_{t_2},\tau\right) + Pr\left(ls_{k+2,t_3}|\Lambda_{t_2},\tau\right) Pr\left(ms_{k+1,t_4}|\Lambda_{t_3},\tau\right)\right] \\
&\int_{\beta_{t_2}\in B(\tau)} \frac{\beta_{t_2}\nu - p_{k+1}}{\Gamma} d\beta_{t_2} = 0.1861b\nu
\end{aligned}
\tag{161}
$$

The welfare of the 3^{rd} and 4^{th} player are conditional on the different path the trading game assumes. The welfare of the 3^{rd} player in case the 2^{nd} player immediately market sells the order posted at t_1 is:

$$
\omega_{t_3}(lb_{t_3} \ |\tau) = \sum_{k=-1}^{+1} Pr\left(lb_{k,t_1}|\Lambda_{t_0},\tau\right) Pr\left(ms_{k,t_2}|\Lambda_{t_1},\tau\right) Pr\left(ms_{-1,t_4}|\Lambda_{t_3},\tau\right) \int_{\beta_{t_3}\in B(\tau)} \frac{\beta_{t_3}\nu - p_{-1}}{\Gamma} d\beta_{t_3}
\tag{162}
$$

The welfare of the 3^{rd} player in case the 2^{nd} player posts a limit sell at the adjacent price level is:

$$
\begin{aligned}
&\omega_{t_3}(mb_{t_3} \ \vee ms_{t_3}|\tau) = \sum_{k=-1}^{1} Pr\left(lb_{k,t_1}|\Lambda_{t_0},\tau\right) Pr\left(ls_{k+1,t_2}|\Lambda_{t_1},\tau\right) \\
&\Big(\int_{(1-b)}^{\frac{p_k}{v}} \frac{p_k - \beta_{t_3}v}{\Gamma} d\beta_{t_3} + \int_{\frac{p_{k+1}}{v}}^{(1+b)} \frac{\beta_{t_3}v - p_{k+1}}{\Gamma} d\beta_{t_3}\Big)
\end{aligned}
\tag{163}
$$

Whereas the welfare of the 3^{rd} player in case the 2^{nd} player opts for either queuing or undercutting

is:

$$\begin{aligned}
&\omega_{t_3}(ms_{t_3} \ \vee ls_{t_3} \ \vee lb_{t_3}|\tau) = \sum_{k=-1}^{+1} Pr\left(lb_{k,t_1}|\Lambda_{t_0},\tau\right) Pr\left(lb_{k,t_2}|\Lambda_{t_1},\tau\right) \\
&(\int_{\beta_{t_3}\in B(\tau)} \frac{p_k - \beta_{t_3}\nu}{\Gamma} d\beta_{t_3} + Pr\left(mb_{k+1,t_4}|\Lambda_{t_3},\tau\right) \int_{\beta_{t_3}\in B(\tau)} \frac{p_{k+1} - \beta_{t_3}\nu}{\Gamma} d\beta_{t_3} + \\
&Pr\left(ms_{k+1,t_4}|\Lambda_{t_3},\tau\right) \int_{\beta_{t_3}\in B(\tau)} \frac{\beta_{t_3}\nu - p_{k+1}}{\Gamma} d\beta_{t_3}) + \\
&\sum_{k=-1}^{1} Pr\left(lb_{k,t_1}|\Lambda_{t_0},\tau\right) Pr\left(lb_{k+1,t_2}|\Lambda_{t_1},\tau\right) \\
&(\int_{\beta_{t_3}\in B(\tau)} \frac{p_{k+1} - \beta_{t_3}\nu}{\Gamma} d\beta_{t_3} + Pr\left(mb_{k+2,t_4}|\Lambda_{t_3},\tau\right) \int_{\beta_{t_3}\in B(\tau)} \frac{p_{k+2} - \beta_{t_3}\nu}{\Gamma} d\beta_{t_3} + \\
&Pr\left(ms_{k+2,t_4}|\Lambda_{t_3},\tau\right) \int_{\beta_3\in B(\tau)} \frac{\beta_{t_3}\nu - p_{k+2}}{\Gamma} d\beta_{t_3})
\end{aligned} \tag{164}$$

Therefore, the overall welfare of the 3^{rd} player is: $\omega_{t_3}(\cdot) = 0.1554b\nu$

The welfare of the 4^{th} player in case the book opens empty at t_3 and hence the 3^{rd} player posts a limit buy order (by Proposition (1)) is:

$$\omega_{t_4}(mb_{t_4}) = \sum_{k=-1}^{+1} Pr\left(lb_{k,t_1}|\Lambda_{t_0},\tau\right) Pr\left(ms_{k,t_2}|\Lambda_{t_1},\tau\right) Pr\left(lb_{-1,t_3}|\Lambda_{t_2},\tau\right) \int_{(1-b)}^{\frac{p_{-1}}{\nu}} \frac{p_{-1} - \beta_{t_4}\nu}{\Gamma} d\beta_{t_4} \tag{165}$$

The welfare of the 4^{th} player in case the 2^{nd} player posts a limit sell at the adjacent price level is:

$$\begin{aligned}
&\omega_{t_4}(ms_{t_4} \vee mb_{t_4}) = \sum_{k=-1}^{+1} Pr\left(lb_{k,t_1}|\Lambda_{t_0},\tau\right) Pr\left(ls_{k+1,t_2}|\Lambda_{t_1},\tau\right) \left(Pr\left(mb_{k+1,t_3}|\Lambda_{t_2},\tau\right) \int_{(1-b)}^{\frac{p_k}{v}} \frac{p_k - \beta_{t_4}\nu}{\Gamma} d\beta_{t_4} + \right. \\
&Pr\left(ms_{k,t_3}|\Lambda_{t_2},\tau\right) \int_{\frac{p_{k+1}}{\nu}}^{(1+b)} \frac{\beta_{t_4}\nu - p_{k+1}}{\Gamma} d\beta_{t_4} + Pr\left(nt_{k,t_3}|\Lambda_{t_2},\tau\right) (\int_{(1-b)}^{\frac{p_k}{\nu}} \frac{p_k - \beta_{t_4}\nu}{\Gamma} d\beta_{t_4} + \int_{\frac{p_{k+1}}{\nu}}^{(1+b)} \frac{\beta_{t_4}\nu - p_{k+1}}{\Gamma} d\beta_{t_4} \left.)\right)
\end{aligned} \tag{166}$$

Finally, the welfare of the 4^{th} player when the 2^{nd} player opts for either queuing or undercutting

is:

$$\omega_{t_4}(ms_{t_4} \vee mb_{t_4}) = (\sum_{k=-1}^{+1} Pr\left(lb_{k,t_1}|\Lambda_{t_0},\tau\right) Pr\left(lb_{k,t_2}|\Lambda_{t_1},\tau\right) \big(Pr\left(ms_{k,t_3}|\Lambda_{t_2},\tau\right) \int_{(1-b)}^{\frac{p_k}{\nu}} \frac{p_k - \beta_{t_4}\nu}{\Gamma} d\beta_{t_4} +$$
$$Pr\left(ls_{k+1,t_3}|\Lambda_{t_2},\tau\right) \big(\int_{(1-b)}^{\frac{p_k}{\nu}} \frac{p_k - \beta_{t_4}\nu}{\Gamma} d\beta_{t_4} + \int_{\frac{p_{k+1}}{\nu}}^{(1+b)} \frac{\beta_{t_4} v - p_{k+1}}{\Gamma} d\beta_{t_4}\big) + Pr\left(lb_{k+1,t_3}|\Lambda_{t_2},\tau\right) \int_{(1-b)}^{\frac{p_{k+1}}{\nu}} \frac{p_{k+1} - \beta_{t_4}\nu}{\Gamma} d\beta_{t_4}\big) +$$
$$\sum_{k=-1}^{+1} Pr\left(lb_{k,t_1}|\Lambda_{t_0},\tau\right) Pr\left(lb_{k+1,t_2}|\Lambda_{t_1},\tau\right) \big(Pr\left(ms_{k+1,t_3}|\Lambda_{t_2},\tau\right) \int_{(1-b)}^{\frac{p_k}{\nu}} \frac{p_k - \beta_{t_4}\nu}{\Gamma} d\beta_{t_4} +$$
$$Pr\left(ls_{k+2,t_3}|\Lambda_{t_2},\tau\right) \big(\int_{(1-b)}^{\frac{p_{k+1}}{\nu}} \frac{p_{k+1} - \beta_{t_4}\nu}{\Gamma} d\beta_{t_4} + \int_{\frac{p_{k+2}}{\nu}}^{(1+b)} \frac{\beta_{t_4} v - p_{k+2}}{\Gamma} d\beta_{t_4}\big) + Pr\left(lb_{k+2,t_3}|\Lambda_{t_2},\tau\right) \int_{(1-b)}^{\frac{p_{k+2}}{\nu}} \frac{p_{k+2} - \beta_{t_4}\nu}{\Gamma} d\beta_{t_4}\big) \tag{167}$$

The overall welfare of the 4^{th} player is: $\omega_{t_4}(\cdot) = 0.0670b\nu$ and the total welfare associated with a game with $\tau = \frac{b\nu}{2}$ is

$$\Omega(\frac{b\nu}{2}) = 0.5878b\nu \tag{168}$$

We can therefore conclude that $\tau \to 0^+$ is not the OTS is a 4-period model. ■

E.3 Model Solution of the Four Period Model

We solve the 4-period trading game by backward induction.

E.3.1 Period t_4

As for the previous trading games, the optimal order submission probabilities of investors arriving at t_4 are defined by Lemma (1) (point 3).

E.3.2 Period t_3

We now derive the optimal order submission strategies at t_3. At t_3, there are four possible state of the book:

- The book is empty ($\Lambda_{t_2} = \{lb_{k,t_1}, ms_{k,t_2}\}$), hence the 3^{rd} player will submit a limit buy order following Proposition (1).

- The book has a limit buy and a limit sell ($\Lambda_{t_2} = \{lb_{k,t_1}, ls_{k+1,t_2}\}$), hence the 3^{rd} player is a liquidity taker only and his order submission probabilities are defined by Lemma (1) (point 3).

- The book has a limit buy and a limit sell ($\Lambda_{t_2} = \{lb_{k,t_1}, ls_{k+d,t_2}\}$), with $d \geq 2$, hence the 3^{rd} player can either opt for market orders (sell at p_k and buy at p_{k+d}) or opt for limit orders inside the best bid ask spread. The probability of market sell at p_k is:

$$
\begin{aligned}
& Pr\left(ms_{k,t_3}|\Lambda_{t_2}, \tau\right) = \\
& Pr\big(p_k - \beta_{t_3}\nu > 0, \\
& p_k - \beta_{t_3}\nu > \left(p_l - \beta_{t_3}\nu\right) Pr\left(mb_{l,t_4}|\Lambda_{t_3}, \tau\right), \\
& p_k - \beta_{t_3}\nu > (\beta_{t_3}\nu - p_l) Pr\left(ms_{l,t_4}|\Lambda_{t_3}, \tau\right), \\
& p_k - \beta_{t_3}\nu > \beta_{t_3}\nu - p_{k+d}\big)
\end{aligned}
\tag{169}
$$

The conditions imposed in equation (169) ensure that a market sell at p_k is more profitable than no trade (nt), a limit sell and a limit buy at a generic price $p_l \in \{p_k,\ p_{k+d}\}$ and finally a market buy at p_{k+d}. The submission probability of a limit sell at a generic $p_l \in \{p_k,\ p_{k+d}\}$ is:

$$
\begin{aligned}
& Pr\left(ls_{l,t_3}|\Lambda_{t_2},\tau\right) = \\
& Pr\Big(\left(p_l - \beta_{t_3}\nu\right) Pr\left(mb_{l,t_4}|\Lambda_{t_3},\tau\right) > 0, \\
& \left(p_l - \beta_{t_3}\nu\right) Pr\left(mb_{l,t_4}|\Lambda_{t_3},\tau\right) > p_k - \beta_{t_3}\nu, \\
& \left(p_l - \beta_{t_3}\nu\right) Pr\left(mb_{l,t_4}|\Lambda_{t_3},\tau\right) > \left(p_{\tilde{l}} - \beta_{t_3}\nu\right) Pr\left(mb_{\tilde{l},t_4}|\Lambda_{t_3},\tau\right), \\
& \left(p_l - \beta_{t_3}\nu\right) Pr\left(mb_{l,t_4}|\Lambda_{t_3},\tau\right) > (\beta_{t_3}\nu - p_l) Pr\left(ms_{l,t_4}|\Lambda_{t_3},\tau\right), \\
& \left(p_l - \beta_{t_3}\nu\right) Pr\left(mb_{l,t_4}|\Lambda_{t_3},\tau\right) > \beta_{t_3}\nu - p_{k+d}\Big)
\end{aligned}
\tag{170}
$$

where $p_{\tilde{l}}$ is a generic price $\in \{p_k,\ p_{k+d}\}$ different from p_l. The submission probability of a limit buy at a generic $p_l \in \{p_k,\ p_{k+d}\}$ is:

$$
\begin{aligned}
& Pr\left(lb_{l,t_3}|\Lambda_{t_2},\tau\right) = \\
& Pr\Big((\beta_{t_3}\nu - p_l) Pr\left(ms_{l,t_4}|\Lambda_{t_3},\tau\right) > 0, \\
& (\beta_{t_3}\nu - p_l) Pr\left(ms_{l,t_4}|\Lambda_{t_3},\tau\right) > p_k - \beta_{t_3}\nu, \\
& (\beta_{t_3}\nu - p_l) Pr\left(ms_{l,t_4}|\Lambda_{t_3},\tau\right) > \left(p_l - \beta_{t_3}\nu\right) Pr\left(mb_{l,t_4}|\Lambda_{t_3},\tau\right), \\
& (\beta_{t_3}\nu - p_l) Pr\left(ms_{l,t_4}|\Lambda_{t_3},\tau\right) > (\beta_{t_3}\nu - p_{\tilde{l}}) Pr\left(ms_{\tilde{l},t_4}|\Lambda_{t_3},\tau\right), \\
& (\beta_{t_3}\nu - p_l) Pr\left(ms_{l,t_4}|\Lambda_{t_3},\tau\right) > \beta_{t_3}\nu - p_{k+d}\Big)
\end{aligned}
\tag{171}
$$

Finally the submission probability of a market buy at p_{k+d} is:

$$
\begin{aligned}
& Pr\left(mb_{k+d,t_3}|\Lambda_{t_2},\tau\right) = \\
& Pr\Big(\beta_{t_3}\nu - p_{k+d} > 0, \\
& \beta_{t_3}\nu - p_{k+d} > p_k - \beta_{t_3}\nu, \\
& \beta_{t_3}\nu - p_{k+d} > \left(p_l - \beta_{t_3}\nu\right) Pr\left(mb_{l,t_4}|\Lambda_{t_3},\tau\right), \\
& \beta_{t_3}\nu - p_{k+d} > (\beta_{t_3}\nu - p_l) Pr\left(ms_{l,t_4}|\Lambda_{t_3},\tau\right)\Big)
\end{aligned}
\tag{172}
$$

- The book is composed by limit buy orders only ($\Lambda_{t_2} = \{lb_{k,t_1}, lb_{k\pm d,t_2}\}$ with $d \in \left\{-n^f, n^f\right\}$), the 3^{rd} player's submission probability strategies are defined in Appendix D.1.2

E.3.3 Period $\mathbf{t_2}$

At t_1 the book opens empty. In addition, by Lemma (1) we know that at t_1 the incoming investor posts either a limit buy (if his $\beta > 1$) or a limit sell (if his $\beta < 1$) at p_k. Therefore, given that at t_2 the book symmetrically opens either with a limit buy or with a limit sell, without loss of generality we can consider a buyer arriving at t_1 so that the book opens with a limit buy at t_2. Hence, the incoming 2^{nd} player can either hit the previously posted limit buy by market selling at p_k, or limit sell at $p_d > p_k$, or he can limit buy still at $p_u > p_k$, or queue behind the 1^{st} player posting a limit buy at $p_q \leq p_k$ or decide not to trade (nt).

For a generic limit buy posted by the first player at p_k, the probability that the 2^{nd} player selects a market sell is given by:

$$
\begin{aligned}
&Pr\left(ms_{k,t_2}|\Lambda_{t_1},\tau\right) = \\
&Pr\big(p_k - \beta_{t_2}\nu > 0, \\
&p_k - \beta_{t_2}\nu > (p_{k+1} - \beta_{t_2}\nu)\left[Pr\left(mb_{k+1,t_3}|\Lambda_{t_2},\tau\right) + (1 - Pr\left(mb_{k+1,t_3}|\Lambda_{t_2},\tau\right))Pr\left(mb_{k+1,t_4}|\Lambda_{t_3},\tau\right)\right], \\
&p_k - \beta_{t_2}\nu > (p_l - \beta_{t_2}\nu)(Pr\left(mb_{l,t_3}|\Lambda_{t_2},\tau\right) + \left[\sum_{j=k+1}^{l-1} Pr\left(lb_{j,t_3}|\Lambda_{t_2},\tau\right) + Pr\left(ms_{k,t_3}|\Lambda_{t_2},\tau\right)\right] Pr\left(mb_{l,t_4}|\Lambda_{t_3},\tau\right)), \\
&p_k - \beta_{t_2}\nu > (\beta_{t_2}\nu - p_q)Pr\left(ms_{k,t_3}|\Lambda_{t_2},\tau\right)\ Pr\left(ms_{q,t_4}|\Lambda_{t_3},\tau\right), \\
&p_k - \beta_{t_2}\nu > (\beta_{t_2}\nu - p_u)(Pr\left(ms_{u,t_3}|\Lambda_{t_2},\tau\right) + (\sum_{j=u+1}^{+n^f} Pr\left(ls_{j,t_3}|\Lambda_{t_2},\tau\right) + Pr\left(nt_{u,t_3}|\Lambda_{t_2},\tau\right))Pr\left(ms_{u,t_4}|\Lambda_{t_3},\tau\right)\big)
\end{aligned}
\tag{173}
$$

Equation (173) guarantees that market selling is more profitable than any other possible action the 2^{nd} player can take. If the first player submits a limit buy at the most aggressive price level p_{n^f}, he locks the book in such a way that the 2^{nd} player can either market sell or queue behind his order In this special case, equation (173) reduces to:

$$
\begin{aligned}
&Pr\left(ms_{n^f,t_2}|\Lambda_{t_1},\tau\right) = Pr\Big(p_{n^f} - \beta_{t_2}\nu > 0,\\
&p_{n^f} - \beta_{t_2}\nu > (\beta_{t_2}\nu - p_q) Pr\left(ms_{k,t_3}|\Lambda_{t_2},\tau\right)\ \ Pr\left(ms_{q,t_4}|\Lambda_{t_3},\tau\right)\Big)
\end{aligned}
\tag{174}
$$

The probability that the 2^{nd} player selects a limit sell order at a price p_{k+1} is:

$$
\begin{aligned}
&Pr\left(ls_{k+1,t_2}|\Lambda_{t_1},\tau\right) =\\
&Pr\Big((p_{k+1} - \beta_{t_2}\nu)\left[Pr\left(mb_{k+1,t_3}|\Lambda_{t_2},\tau\right) + \Big(1 - Pr\left(mb_{k+1,t_3}|\Lambda_{t_2},\tau\right)\Big) Pr\left(mb_{k+1,t_4}|\Lambda_{t_3},\tau\right)\right] > 0,\\
&(p_{k+1} - \beta_{t_2}\nu)\left[Pr\left(mb_{k+1,t_3}|\Lambda_{t_2},\tau\right) + \Big(1 - Pr\left(mb_{k+1,t_3}|\Lambda_{t_2},\tau\right)\Big) Pr\left(mb_{k+1,t_4}|\Lambda_{t_3},\tau\right)\right] > (p_k - \beta_{t_2}\nu),\\
&(p_{k+1} - \beta_{t_2}\nu)\left[Pr\left(mb_{k+1,t_3}|\Lambda_{t_2},\tau\right) + \Big(1 - Pr\left(mb_{k+1,t_3}|\Lambda_{t_2},\tau\right)\Big) Pr\left(mb_{k+1,t_4}|\Lambda_{t_3},\tau\right)\right] >\\
&(p_l - \beta_{t_2}\nu)\left(Pr\left(mb_{l,t_3}|\Lambda_{t_2},\tau\right) + \left[\sum_{j=k+1}^{l-1} Pr\left(lb_{j,t_3}|\Lambda_{t_2},\tau\right) + Pr\left(ms_{k,t_3}|\Lambda_{t_2},\tau\right)\right] Pr\left(mb_{l,t_4}|\Lambda_{t_3},\tau\right)\right),\\
&(p_{k+1} - \beta_{t_2}\nu)\left[Pr\left(mb_{k+1,t_3}|\Lambda_{t_2},\tau\right) + \Big(1 - Pr\left(mb_{k+1,t_3}|\Lambda_{t_2},\tau\right)\Big) Pr\left(mb_{k+1,t_4}|\Lambda_{t_3},\tau\right)\right] >\\
&(\beta_{t_2}\nu - p_q) Pr\left(ms_{k,t_3}|\Lambda_{t_2},\tau\right)\ \ Pr\left(ms_{q,t_4}|\Lambda_{t_3},\tau\right),\\
&(p_{k+1} - \beta_{t_2}\nu)\left[Pr\left(mb_{k+1,t_3}|\Lambda_{t_2},\tau\right) + \Big(1 - Pr\left(mb_{k+1,t_3}|\Lambda_{t_2},\tau\right)\Big) Pr\left(mb_{k+1,t_4}|\Lambda_{t_3},\tau\right)\right] >\\
&(\beta_{t_2}\nu - p_u)(Pr\left(ms_{u,t_3}|\Lambda_{t_2},\tau\right) + (\sum_{j=u+1}^{+n^f} Pr\left(ls_{j,t_3}|\Lambda_{t_2},\tau\right) + Pr\left(nt_{u,t_3}|\Lambda_{t_2},\tau\right)) Pr\left(ms_{u,t_4}|\Lambda_{t_3},\tau\right)\Big)
\end{aligned}
\tag{175}
$$

The probability that the 2^{nd} player selects a limit sell order at a price p_l is:

$$
\begin{aligned}
&Pr\left(ls_{l,t_2}|\Lambda_{t_1},\tau\right) = \\
&Pr\Big((p_l-\beta_{t_2}\nu)\left(Pr\left(mb_{l,t_3}|\Lambda_{t_2},\tau\right)+\left[\sum_{j=k+1}^{l-1} Pr\left(lb_{j,t_3}|\Lambda_{t_2},\tau\right)+Pr\left(ms_{k,t_3}|\Lambda_{t_2},\tau\right)\right] Pr\left(mb_{l,t_4}|\Lambda_{t_3},\tau\right)\right)>0, \\
&(p_l-\beta_{t_2}\nu)\left(Pr\left(mb_{l,t_3}|\Lambda_{t_2},\tau\right)+\left[\sum_{j=k+1}^{l-1} Pr\left(lb_{j,t_3}|\Lambda_{t_2},\tau\right)+Pr\left(ms_{k,t_3}|\Lambda_{t_2},\tau\right)\right] Pr\left(mb_{l,t_4}|\Lambda_{t_3},\tau\right)\right)> \\
&(p_k-\beta_{t_2}\nu), \\
&(p_l-\beta_{t_2}\nu)\left(Pr\left(mb_{l,t_3}|\Lambda_{t_2},\tau\right)+\left[\sum_{j=k+1}^{l-1} Pr\left(lb_{j,t_3}|\Lambda_{t_2},\tau\right)+Pr\left(ms_{k,t_3}|\Lambda_{t_2},\tau\right)\right] Pr\left(mb_{l,t_4}|\Lambda_{t_3},\tau\right)\right)> \\
&(p_{k+1}-\beta_{t_2}\nu)\left[Pr\left(mb_{k+1,t_3}|\Lambda_{t_2},\tau\right)+\Big(1-Pr\left(mb_{k+1,t_3}|\Lambda_{t_2},\tau\right)\Big)Pr\left(mb_{k+1,t_4}|\Lambda_{t_3},\tau\right)\right], \\
&(p_l-\beta_{t_2}\nu)\left(Pr\left(mb_{l,t_3}|\Lambda_{t_2},\tau\right)+\left[\sum_{j=k+1}^{l-1} Pr\left(lb_{j,t_3}|\Lambda_{t_2},\tau\right)+Pr\left(ms_{k,t_3}|\Lambda_{t_2},\tau\right)\right] Pr\left(mb_{l,t_4}|\Lambda_{t_3},\tau\right)\right)> \\
&(p_{\tilde{l}}-\beta_{t_2}\nu)\left(Pr\left(mb_{\tilde{l},t_3}|\Lambda_{t_2},\tau\right)+\left[\sum_{j=k+1}^{\tilde{l}-1} Pr\left(lb_{j,t_3}|\Lambda_{t_2},\tau\right)+Pr\left(ms_{k,t_3}|\Lambda_{t_2},\tau\right)\right] Pr\left(mb_{\tilde{l},t_4}|\Lambda_{t_3},\tau\right)\right), \\
&(p_l-\beta_{t_2}\nu)\left(Pr\left(mb_{l,t_3}|\Lambda_{t_2},\tau\right)+\left[\sum_{j=k+1}^{l-1} Pr\left(lb_{j,t_3}|\Lambda_{t_2},\tau\right)+Pr\left(ms_{k,t_3}|\Lambda_{t_2},\tau\right)\right] Pr\left(mb_{l,t_4}|\Lambda_{t_3},\tau\right)\right)> \\
&(\beta_{t_2}\nu-p_q)Pr\left(ms_{k,t_3}|\Lambda_{t_2},\tau\right)\ Pr\left(ms_{q,t_4}|\Lambda_{t_3},\tau\right), \\
&(p_l-\beta_{t_2}\nu)\left(Pr\left(mb_{l,t_3}|\Lambda_{t_2},\tau\right)+\left[\sum_{j=k+1}^{l-1} Pr\left(lb_{j,t_3}|\Lambda_{t_2},\tau\right)+Pr\left(ms_{k,t_3}|\Lambda_{t_2},\tau\right)\right] Pr\left(mb_{l,t_4}|\Lambda_{t_3},\tau\right)\right)> \\
&(\beta_{t_2}\nu-p_u)(Pr\left(ms_{u,t_3}|\Lambda_{t_2},\tau\right)+(\sum_{j=u+1}^{+n^f} Pr\left(ls_{j,t_3}|\Lambda_{t_2},\tau\right)+Pr\left(nt_{u,t_3}|\Lambda_{t_2},\tau\right))Pr\left(ms_{u,t_4}|\Lambda_{t_3},\tau\right)\Big)
\end{aligned}
\tag{176}
$$

In the special case in which the 1^{st} player submits a limit buy at the most aggressive price level p_{n^f}, the probability of a limit sell is zero.

The probability that the 2^{nd} player selects a limit buy order at $p_q \leq p_k$ thus queuing the limit buy order posted at t_1 is:

$$
\begin{aligned}
&Pr\left(lb_{q,t_2}|\Lambda_{t_1},\tau\right) = \\
&Pr\Big((\beta_{t_2}\nu - p_q)Pr\left(ms_{k,t_3}|\Lambda_{t_2},\tau\right)\ \ Pr\left(ms_{q,t_4}|\Lambda_{t_3},\tau\right) > 0, \\
&(\beta_{t_2}\nu - p_q)Pr\left(ms_{k,t_3}|\Lambda_{t_2},\tau\right)\ \ Pr\left(ms_{q,t_4}|\Lambda_{t_3},\tau\right) > (p_k - \beta_{t_2}\nu), \\
&(\beta_{t_2}\nu - p_q)Pr\left(ms_{k,t_3}|\Lambda_{t_2},\tau\right)\ \ Pr\left(ms_{q,t_4}|\Lambda_{t_3},\tau\right) > \\
&(p_{k+1} - \beta_{t_2}\nu)\Big[Pr\left(mb_{k+1,t_3}|\Lambda_{t_2},\tau\right) + \Big(1 - Pr\left(mb_{k+1,t_3}|\Lambda_{t_2},\tau\right)\Big)Pr\left(mb_{k+1,t_4}|\Lambda_{t_3},\tau\right)\Big], \\
&(\beta_{t_2}\nu - p_q)Pr\left(ms_{k,t_3}|\Lambda_{t_2},\tau\right)\ \ Pr\left(ms_{q,t_4}|\Lambda_{t_3},\tau\right) > \\
&(p_l - \beta_{t_2}\nu)\left(Pr\left(mb_{l,t_3}|\Lambda_{t_2},\tau\right) + \left[\sum_{j=k+1}^{l-1} Pr\left(lb_{j,t_3}|\Lambda_{t_2},\tau\right) + Pr\left(ms_{k,t_3}|\Lambda_{t_2},\tau\right)\right] Pr\left(mb_{l,t_4}|\Lambda_{t_3},\tau\right)\right), \\
&(\beta_{t_2}\nu - p_q)Pr\left(ms_{k,t_3}|\Lambda_{t_2},\tau\right)\ \ Pr\left(ms_{q,t_4}|\Lambda_{t_3},\tau\right) > \\
&(\beta_{t_2}\nu - p_{\tilde{q}})Pr\left(ms_{k,t_3}|\Lambda_{t_2},\tau\right)\ \ Pr\left(ms_{\tilde{q},t_4}|\Lambda_{t_3},\tau\right), \\
&(\beta_{t_2}\nu - p_q)Pr\left(ms_{k,t_3}|\Lambda_{t_2},\tau\right)\ \ Pr\left(ms_{q,t_4}|\Lambda_{t_3},\tau\right) > \\
&(\beta_{t_2}\nu - p_u)(Pr\left(ms_{u,t_3}|\Lambda_{t_2},\tau\right) + (\sum_{j=u+1}^{+n^f} Pr\left(ls_{j,t_3}|\Lambda_{t_2},\tau\right) + Pr\left(nt_{u,t_3}|\Lambda_{t_2},\tau\right))Pr\left(ms_{u,t_4}|\Lambda_{t_3},\tau\right)\Big)
\end{aligned}
\tag{177}
$$

In the special case in which the 1^{st} player submits a limit buy at the most aggressive price level p_{n^f}, the probability of queuing is:

$$
\begin{aligned}
&Pr\left(lb_{q,t_2}|\Lambda_{t_1},\tau\right) = \\
&Pr\Big((\beta_{t_2}\nu - p_q)Pr\left(ms_{k,t_3}|\Lambda_{t_2},\tau\right)\ \ Pr\left(ms_{q,t_4}|\Lambda_{t_3},\tau\right) > 0, \\
&(\beta_{t_2}\nu - p_q)Pr\left(ms_{k,t_3}|\Lambda_{t_2},\tau\right)\ \ Pr\left(ms_{q,t_4}|\Lambda_{t_3},\tau\right) > (p_{n^f} - \beta_{t_2}\nu), \\
&(\beta_{t_2}\nu - p_q)Pr\left(ms_{k,t_3}|\Lambda_{t_2},\tau\right)\ \ Pr\left(ms_{q,t_4}|\Lambda_{t_3},\tau\right) > \\
&(\beta_{t_2}\nu - p_{\tilde{q}})Pr\left(ms_{k,t_3}|\Lambda_{t_2},\tau\right)\ \ Pr\left(ms_{\tilde{q},t_4}|\Lambda_{t_3},\tau\right)\Big)
\end{aligned}
\tag{178}
$$

The probability that the 2^{nd} player selects a limit buy order at $p_u > p_k$ thus undercutting the limit buy order posted at t_1 is:

$$
\begin{aligned}
&Pr\left(lb_{u,t_2}|\Lambda_{t_1},\tau\right) = \\
&Pr\Big((\beta_{t_2}\nu - p_u)(Pr\left(ms_{u,t_3}|\Lambda_{t_2},\tau\right) + (\sum_{j=u+1}^{+n^f} Pr\left(ls_{j,t_3}|\Lambda_{t_2},\tau\right) + Pr\left(nt_{u,t_3}|\Lambda_{t_2},\tau\right))Pr\left(ms_{u,t_4}|\Lambda_{t_3},\tau\right) > 0, \\
&(\beta_{t_2}\nu - p_u)(Pr\left(ms_{u,t_3}|\Lambda_{t_2},\tau\right) + (\sum_{j=u+1}^{+n^f} Pr\left(ls_{j,t_3}|\Lambda_{t_2},\tau\right) + Pr\left(nt_{u,t_3}|\Lambda_{t_2},\tau\right))Pr\left(ms_{u,t_4}|\Lambda_{t_3},\tau\right) > (p_k - \beta_{t_2}\nu), \\
&(\beta_{t_2}\nu - p_u)(Pr\left(ms_{u,t_3}|\Lambda_{t_2},\tau\right) + (\sum_{j=u+1}^{+n^f} Pr\left(ls_{j,t_3}|\Lambda_{t_2},\tau\right) + Pr\left(nt_{u,t_3}|\Lambda_{t_2},\tau\right))Pr\left(ms_{u,t_4}|\Lambda_{t_3},\tau\right) > \\
&(p_{k+1} - \beta_{t_2}\nu)\left[Pr\left(mb_{k+1,t_3}|\Lambda_{t_2},\tau\right) + \Big(1 - Pr\left(mb_{k+1,t_3}|\Lambda_{t_2},\tau\right)\Big)Pr\left(mb_{k+1,t_4}|\Lambda_{t_3},\tau\right)\right], \\
&(\beta_{t_2}\nu - p_u)(Pr\left(ms_{u,t_3}|\Lambda_{t_2},\tau\right) + (\sum_{j=u+1}^{+n^f} Pr\left(ls_{j,t_3}|\Lambda_{t_2},\tau\right) + Pr\left(nt_{u,t_3}|\Lambda_{t_2},\tau\right))Pr\left(ms_{u,t_4}|\Lambda_{t_3},\tau\right) > \\
&(p_l - \beta_{t_2}\nu)\left(Pr\left(mb_{l,t_3}|\Lambda_{t_2},\tau\right) + \left[\sum_{j=k+1}^{l-1} Pr\left(lb_{j,t_3}|\Lambda_{t_2},\tau\right) + Pr\left(ms_{k,t_3}|\Lambda_{t_2},\tau\right)\right] Pr\left(mb_{l,t_4}|\Lambda_{t_3},\tau\right)\right), \\
&(\beta_{t_2}\nu - p_u)(Pr\left(ms_{u,t_3}|\Lambda_{t_2},\tau\right) + (\sum_{j=u+1}^{+n^f} Pr\left(ls_{j,t_3}|\Lambda_{t_2},\tau\right) + Pr\left(nt_{u,t_3}|\Lambda_{t_2},\tau\right))Pr\left(ms_{u,t_4}|\Lambda_{t_3},\tau\right) > \\
&(\beta_{t_2}\nu - p_q)Pr\left(ms_{k,t_3}|\Lambda_{t_2},\tau\right)\ \ Pr\left(ms_{q,t_4}|\Lambda_{t_3},\tau\right), \\
&(\beta_{t_2}\nu - p_u)(Pr\left(ms_{u,t_3}|\Lambda_{t_2},\tau\right) + (\sum_{j=u+1}^{+n^f} Pr\left(ls_{j,t_3}|\Lambda_{t_2},\tau\right) + Pr\left(nt_{u,t_3}|\Lambda_{t_2},\tau\right))Pr\left(ms_{u,t_4}|\Lambda_{t_3},\tau\right) > \\
&(\beta_{t_2}\nu - p_{\tilde{u}})(Pr\left(ms_{\tilde{u},t_3}|\Lambda_{t_2},\tau\right) + (\sum_{j=\tilde{u}+1}^{+n^f} Pr\left(ls_{j,t_3}|\Lambda_{t_2},\tau\right) + Pr\left(nt_{\tilde{u},t_3}|\Lambda_{t_2},\tau\right))Pr\left(ms_{\tilde{u},t_4}|\Lambda_{t_3},\tau\right)\Big)
\end{aligned}
\tag{179}
$$

In the special case in which the 1^{st} player locks the market and submits a limit buy at the most aggressive price level p_{n^f}, the probability of undercutting at t_2 is zero. Finally, for any price submitted by the 1^{st} player, the probability that the 2^{nd} player chooses nt is zero. Indeed even by considering a set of actions always available to the 2^{nd} player - both market sell and queuing at p_k- no trade is a dominated strategy:

$$\begin{aligned}
&Pr\left(nt_{k,t_2}|\Lambda_{t_1},\tau\right) = \\
&Pr\big(0 > p_k - \beta_{t_2}\nu, \\
&0 > (\beta_{t_2}\nu - p_k)(\beta\nu - p_k)Pr\left(ms_{k,t_3}|\Lambda_{t_2},\tau\right)\ Pr\left(ms_{k,t_4}|\Lambda_{t_3},\tau\right)\big)
\end{aligned} \tag{180}$$

Given that $Pr\left(ms_{k,t_3}|\Lambda_{t_2},\tau\right)\ Pr\left(ms_{k,t_4}|\Lambda_{t_3},\tau\right)$ is positive, the conditions in (180) reduce to:

$$p_k > \beta_{t_2}v > p_k \tag{181}$$

which is impossible.

E.3.4 Period $\mathbf{t_1}$

Without loss of generality, using Lemma (1), if the 1^{st} player at t_1 is a buyer ($\beta_{t_1} > 1$), he can either limit buy at $p_k < p_{n^f}$, or limit buy at the most aggressive price p_{n^f}. The execution probability of a limit buy submitted at a generic price p_k is:

$$\begin{aligned}
&Pr\left(\Psi_{k,t_1}|\Lambda_{t_0},\tau\right) = Pr\left(ms_{k,t_2}|\Lambda_{t_1},\tau\right) + \\
&Pr\left(ls_{k+1,t_2}|\Lambda_{t_1},\tau\right)\left[Pr\left(ms_{k,t_3}|\Lambda_{t_2},\tau\right) + \left(1 - Pr\left(ms_{k,t_3}|\Lambda_{t_2},\tau\right)\right)Pr\left(ms_{k,t_4}|\Lambda_{t_3},\tau\right)\right] + \\
&\sum_{d>k+1} Pr\left(ls_{d,t_2}|\Lambda_{t_1},\tau\right)\left[Pr\left(ms_{k,t_3}|\Lambda_{t_2},\tau\right) + \left(Pr\left(mb_{d,t_3}|\Lambda_{t_2},\tau\right) + \sum_{k<l<d} Pr\left(ls_{l,t_3}|\Lambda_{t_2},\tau\right)\right)Pr\left(ms_{k,t_4}|\Lambda_{t_3},\tau\right)\right] + \\
&\sum_{q\leq k} Pr\left(lb_{q,t_2}|\Lambda_{t_1},\tau\right)\left[Pr\left(ms_{k,t_3}|\Lambda_{t_2},\tau\right) + \sum_{d>k} Pr\left(ls_{d,t_3}|\Lambda_{t_2},\tau\right)Pr\left(ms_{k,t_4}|\Lambda_{t_3},\tau\right)\right] + \\
&Pr\left(lb_{k+1,t_2}|\Lambda_{t_1},\tau\right)Pr\left(ms_{k+1,t_3}|\Lambda_{t_2},\tau\right)Pr\left(ms_{k,t_4}|\Lambda_{t_3},\tau\right) + \\
&\sum_{d>k} Pr\left(lb_{d,t_2}|\Lambda_{t_1},\tau\right)Pr\left(ms_{d,t_3}|\Lambda_{t_2},\tau\right)Pr\left(ms_{k,t_4}|\Lambda_{t_3},\tau\right)
\end{aligned} \tag{182}$$

The execution probability of a limit buy submitted at a generic price p_{+n^f} is:

$$Pr\left(\Psi_{+n^f,t_1}|\Lambda_{t_0},\tau\right) = Pr\left(ms_{+n^f,t_2}|\Lambda_{t_1},\tau\right) + \sum_{q\leq k} Pr\left(lb_{q,t_2}|\Lambda_{t_1},\tau\right)\left[Pr\left(ms_{k,t_3}|\Lambda_{t_2},\tau\right) + Pr\left(nt_{+n^f,t_3}|\Lambda_{t_2},\tau\right) Pr\left(ms_{+n^f,t_4}|\Lambda_{t_3},\tau\right)\right] \tag{183}$$

Therefore using equations (182) and (183), the submission probability of a limit buy at price p_k is:

$$\begin{aligned} &Pr\left(lb_{k,t_1}|\Lambda_{t_0},\tau\right) = \\ &Pr\Big((\beta_{t_1}\nu - p_k)Pr\left(\Psi_{k,t_1}|\Lambda_{t_0},\tau\right) > 0, \\ &(\beta_{t_1}\nu - p_k)\,\mathrm{Pr}\left(\Psi_{k,t_1}|\Lambda_{t_0},\tau\right) > (\beta_{t_1}\nu - p_{\tilde{k}})Pr\left(\Psi_{\tilde{k},t_1}|\Lambda_{t_0},\tau\right), \\ &(\beta_{t_1}\nu - p_k)Pr\left(\Psi_{k,t_1}|\Lambda_{t_0},\tau\right) > (\beta_{t_1}\nu - p_{+n^f})Pr\left(\Psi_{+n^f,t_1}|\Lambda_{t_0},\tau\right)\Big) \end{aligned} \tag{184}$$

where $p_{\tilde{k}} < p_{+n^f}$ different from p_k. In the extreme case of a limit buy at p_{+n^f}, the probability of submission is:

$$\begin{aligned} &Pr\left(lb_{+n^f,t_1}|\Lambda_{t_0},\tau\right) = \\ &Pr\Big((\beta_{t_1}\nu - p_{+n^f})Pr\left(\Psi_{+n^f,t_1}|\Lambda_{t_0},\tau\right) > 0, \\ &(\beta_{t_1}\nu - p_{+n^f})Pr\left(\Psi_{+n^f,t_1}|\Lambda_{t_0},\tau\right) > (\beta_{t_1}\nu - p_k)\,\mathrm{Pr}\left(\Psi_{k,t_1}|\Lambda_{t_0},\tau\right)\Big) \end{aligned} \tag{185}$$

E.4 Welfare Equations

The welfare of the 1^{st} player in the 4-period game is:

$$
\begin{aligned}
&\omega_{t_1}(lb_{t_1}\ |\tau) = \sum_{k=-n^f}^{+n^f} \big(Pr\,(ms_{k,t_2}|\Lambda_{t_1},\tau) + \\
&Pr\,(ls_{k+1,t_2}|\Lambda_{t_1},\tau)\,[Pr\,(ms_{k,t_3}|\Lambda_{t_2},\tau) + (1 - Pr\,(ms_{k,t_3}|\Lambda_{t_2},\tau))Pr\,(ms_{k,t_4}|\Lambda_{t_3},\tau)] + \\
&Pr\,(lb_{k,t_2}|\Lambda_{t_1},\tau)\,[Pr\,(ms_{k,t_3}|\Lambda_{t_2},\tau) + Pr\,(ls_{k+1,t_3}|\Lambda_{t_2},\tau)\,Pr\,(ms_{k,t_4}|\Lambda_{t_3},\tau)] + \\
&Pr\,(lb_{k+1,t_2}|\Lambda_{t_1},\tau)\,Pr\,(ms_{k+1,t_3}|\Lambda_{t_2},\tau)\,Pr\,(ms_{k,t_4}|\Lambda_{t_3},\tau)\,\big) \int_{\beta_{t_1}\in B(\tau)} \frac{\beta_{t_1}\nu - p_k}{\Gamma} d\beta_{t_1} + \\
&\mathbb{1}_G\Bigg\{\Bigg(\sum_{d>k+1} Pr\,(ls_{d,t_2}|\Lambda_{t_1},\tau)\left[Pr\,(ms_{k,t_3}|\Lambda_{t_2},\tau) + \left(Pr\,(mb_{d,t_3}|\Lambda_{t_2},\tau) + \sum_{k<l<d} Pr\,(ls_{l,t_3}|\Lambda_{t_2},\tau)\right) Pr\,(ms_{k,t_4}|\Lambda_{t_3},\tau)\right] + \\
&\quad \sum_{h<k} Pr\,(lb_{h,t_2}|\Lambda_{t_1},\tau)\left[Pr\,(ms_{k,t_3}|\Lambda_{t_2},\tau) + \sum_{d>k} Pr\,(ls_{d,t_3}|\Lambda_{t_2},\tau)\,Pr\,(ms_{k,t_4}|\Lambda_{t_3},\tau)\right] + \\
&\quad Pr\,(lb_{k,t_2}|\Lambda_{t_1},\tau) \sum_{d>k+1} Pr\,(ls_{d,t_3}|\Lambda_{t_2},\tau)\,Pr\,(ms_{k,t_4}|\Lambda_{t_3},\tau) + \\
&\quad \sum_{d>k+1} Pr\,(lb_{d,t_2}|\Lambda_{t_1},\tau)\,Pr\,(ms_{d,t_3}|\Lambda_{t_2},\tau)\,Pr\,(ms_{k,t_4}|\Lambda_{t_3},\tau)\Bigg) \int_{\beta_{t_1}\in B(\tau)} \frac{\beta_{t_1}\nu - p_k}{\Gamma} d\beta_{t_1}\Bigg\}
\end{aligned}
\tag{186}
$$

The welfare of the 1^{st} player is given by the product of the gain of a limit order multiplied by its probability of execution, defined in equation (186). The welfare of the 2^{nd} player is:

$$
\begin{aligned}
&\omega_{t_2}(ms_{t_2}\ \vee ls_{t_2}\ \vee lb_{t_2}\ |\tau) = \sum_{k=-n^f}^{+n^f} Pr\,(lb_{k,t_1}|\Lambda_{t_0},\tau)\Bigg(\int_{\beta_{t_2}\in B(\tau)} \frac{p_k - \beta_{t_2}\nu}{\Gamma} d\beta_{t_2} + \\
&\Big[Pr\,(mb_{k+1,t_3}|\Lambda_{t_2},\tau) + \Big(1 - Pr\,(mb_{k+1,t_3}|\Lambda_{t_2},\tau)\Big)Pr\,(mb_{k+1,t_4}|\Lambda_{t_3},\tau)\Big] \int_{\beta_{t_2}\in B(\tau)} \frac{p_{(k+1)} - \beta_{t_2}\nu}{\Gamma} d\beta_{t_2} + \\
&Pr\,(ms_{k,t_3}|\Lambda_{t_2},\tau) \quad Pr\,(ms_{k,t_4}|\Lambda_{t_3},\tau) \int_{\beta_{t_2}\in B(\tau)} \frac{\beta_{t_2}\nu - p_k}{\Gamma} d\beta_{t_2} \quad + \\
&[Pr\,(ms_{k+1,t_3}|\Lambda_{t_2},\tau) + Pr\,(ls_{k+2,t_3}|\Lambda_{t_2},\tau)\,Pr\,(ms_{k+1,t_4}|\Lambda_{t_3},\tau)] \int_{\beta_{t_2}\in B(\tau)} \frac{\beta_{t_2}\nu - p_{k+1}}{\Gamma} d\beta_{t_2} + \\
&\mathbb{1}_G\Bigg\{\Bigg(\sum_{d=k+2}^{+n^f} Pr\,(mb_{d,t_3}|\Lambda_{t_2},\tau) + \left[\sum_{j=k+1}^{k+d-1} Pr\,(lb_{j,t_3}|\Lambda_{t_2},\tau) + Pr\,(ms_{k,t_3}|\Lambda_{t_2},\tau)\right] Pr\,(mb_{d,t_4}|\Lambda_{t_3},\tau)\Bigg) \int_{\beta_{t_2}\in B(\tau)} \frac{p_d - \beta_{t_2}\nu}{\Gamma} d\beta_{t_2} + \\
&\quad \sum_{d=-n^f}^{k-1} Pr\,(ms_{k,t_3}|\Lambda_{t_2},\tau) \quad Pr\,(ms_{d,t_4}|\Lambda_{t_3},\tau) \int_{\beta_{t_2}\in B(\tau)} \frac{\beta_{t_2}\nu - p_d}{\Gamma} d\beta_{t_2} + \\
&\quad \sum_{d=k+2}^{+n^f} \left[Pr\,(ms_{d,t_3}|\Lambda_{t_2},\tau) + \sum_{j=d+1}^{+n^f} Pr\,(ls_{j,t_3}|\Lambda_{t_2},\tau)\,Pr\,(ms_{d,t_4}|\Lambda_{t_3},\tau)\right] \int_{\beta_{t_2}\in B(\tau)} \frac{\beta_{t_2}\nu - p_d}{\Gamma} d\beta_{t_2} + \\
&\quad \sum_{j=k+3}^{+n^f} Pr\,(ls_{j,t_3}|\Lambda_{t_2},\tau)\,Pr\,(ms_{k+1,t_4}|\Lambda_{t_3},\tau) \int_{\beta_{t_2}\in B(\tau)} \frac{\beta_{t_2}\nu - p_{k+1}}{\Gamma} d\beta_{t_2}\Bigg\}\Bigg)
\end{aligned}
\tag{187}
$$

The welfare of the 3^{rd} player is defined by the trading strategy implemented by the 2^{nd} player. The welfare of the 3^{rd} player in case the 2^{nd} player immediately markets sell:

$$\omega_{t_3}(ms_{t_3}\ |\tau) = \sum_{k=-n^f}^{+n^f} Pr\left(lb_{k,t_1}|\Lambda_{t_0},\tau\right) Pr\left(ms_{k,t_2}|\Lambda_{t_1},\tau\right) \sum_{k=-n^f}^{+n^f} Pr\left(ms_{k,t_4}|\Lambda_{t_3},\tau\right) \int_{\beta_{t_3}\in B(\tau)} \frac{\beta_{t_3}\nu - p_k}{\Gamma} d\beta_{t_3} \tag{188}$$

The welfare of the 3^{rd} player in case the 2^{nd} player posts a limit sell at p_{k+1}:

$$\omega_{t_3}(ms_{t_3}\ \vee mb_{t_3}\ |\tau) = \sum_{k=-n^f}^{+n^f-1} Pr\left(lb_{k,t_1}|\Lambda_{t_0},\tau\right) Pr\left(ls_{k+1,t_2}|\Lambda_{t_1},\tau\right) \left(\int_{(1-b)}^{\frac{p_k}{v}} \frac{p_k - \beta_{t_3} v}{\Gamma} d\beta_{t_3} + \int_{\frac{p_{k+1}}{v}}^{(1+b)} \frac{\beta_{t_3} v - p_{k+1}}{\Gamma} d\beta_{t_3} \right) \tag{189}$$

The welfare of the 3^{rd} player in case the 2^{nd} player posts a limit sell at p_{k+d} with $d \geq 2$:

$$\begin{aligned} \omega_{t_3}(lb_{t_1}\ |\tau) =& \mathbb{1}_G \Bigg\{ \sum_{k=-n^f}^{+n^f-2} Pr\left(lb_{k,t_1}|\Lambda_{t_0},\tau\right) \sum_{d=k+2}^{+n^f} Pr\left(ls_{d,t_2}|\Lambda_{t_1},\tau\right) \\ & \Bigg[\int_{\beta_{t_3}\in B(\tau)} \frac{p_k - \beta_{t_3}\nu}{\Gamma} d\beta_{t_3} + \sum_{j=k+1}^{d-1} Pr\left(mb_{j,t_4}|\Lambda_{t_3},\tau\right) \int_{\beta_{t_3}\in B(\tau)} \frac{p_j - \beta_{t_3}\nu}{\Gamma} d\beta_{t_3} + \\ & \sum_{j=k+1}^{d-1} Pr\left(ms_{j,t_4}|\Lambda_{t_3},\tau\right) \int_{\beta_{t_3}\in B(\tau)} \frac{\beta_{t_3}\nu - p_j}{\Gamma} d\beta_{t_3} + \int_{\beta_{t_3}\in B(\tau)} \frac{\beta_{t_3}\nu - p_d}{\Gamma} d\beta_{t_3} \Bigg] \Bigg\} \end{aligned} \tag{190}$$

The welfare of the 3^{rd} player in case the 2^{nd} player posts a limit buy behind the 1^{st} player:

$$\begin{aligned} &\omega_{t_3}(ms_{t_3}\ \vee ls_{t_3}\ \vee lb_{t_3}\ |\tau) = \sum_{k=-n^f}^{+n^f} Pr\left(lb_{k,t_1}|\Lambda_{t_0},\tau\right) Pr\left(lb_{k,t_2}|\Lambda_{t_1},\tau\right) \Bigg(\int_{\beta_{t_3}\in B(\tau)} \frac{p_k - \beta_{t_3}\nu}{\Gamma} d\beta_{t_3} + \\ &Pr\left(mb_{k+1,t_4}|\Lambda_{t_3},\tau\right) \int_{\beta_{t_3}\in B(\tau)} \frac{p_{k+1} - \beta_{t_3}\nu}{\Gamma} d\beta_{t_3} + Pr\left(ms_{k+1,t_4}|\Lambda_{t_3},\tau\right) \int_{\beta_{t_3}\in B(\tau)} \frac{\beta_{t_3}\nu - p_{k+1}}{\Gamma} d\beta_{t_3} \Bigg) + \\ &\mathbb{1}_G \Bigg\{ \sum_{k=-n^f}^{+n^f} Pr\left(lb_{k,t_1}|\Lambda_{t_0},\tau\right) \sum_{d=-n^f}^{k-1} Pr\left(lb_{d,t_2}|\Lambda_{t_1},\tau\right) \\ &\Bigg(\int_{\beta_{t_3}\in B(\tau)} \frac{p_k - \beta_{t_3}\nu}{\Gamma} d\beta_{t_3} + \sum_{j=k+1}^{+n^f} Pr\left(mb_{j,t_4}|\Lambda_{t_3},\tau\right) \int_{\beta_{t_3}\in B(\tau)} \frac{p_j - \beta_{t_3}\nu}{\Gamma} d\beta_{t_3} + \\ &\sum_{j=k+1}^{+n^f} Pr\left(ms_{j,t_4}|\Lambda_{t_3},\tau\right) \int_{\beta_{t_3}\in B(\tau)} \frac{\beta_{t_3}\nu - p_j}{\Gamma} d\beta_{t_3} \Bigg) + \sum_{k=-n^f}^{+n^f} Pr\left(lb_{k,t_1}|\Lambda_{t_0},\tau\right) Pr\left(lb_{k,t_2}|\Lambda_{t_1},\tau\right) \\ &\left(\sum_{j=k+2}^{+n^f} Pr\left(mb_{j,t_4}|\Lambda_{t_3},\tau\right) \int_{\beta_{t_3}\in B(\tau)} \frac{p_j - \beta_{t_3}\nu}{\Gamma} d\beta_{t_3} + \sum_{j=k+2}^{+n^f} Pr\left(ms_{j,t_4}|\Lambda_{t_3},\tau\right) \int_{\beta_{t_3}\in B(\tau)} \frac{\beta_{t_3}\nu - p_j}{\Gamma} d\beta_{t_3} \right) \Bigg\} \end{aligned} \tag{191}$$

The welfare of the 3^{rd} player in case the 2^{nd} player undercuts the limit buy posted by the 1^{st} player:

$$\begin{aligned}
&\omega_{t_3}(ms_{t_3} \ \vee ls_{t_3} \ \vee lb_{t_3} \ |\tau) = \sum_{k=-n^f}^{+n^f-1} Pr\left(lb_{k,t_1}|\Lambda_{t_0},\tau\right) Pr\left(lb_{k+1,t_2}|\Lambda_{t_1},\tau\right) \Bigg(\int_{\beta_{t_3}\in B(\tau)} \frac{p_{k+1}-\beta_{t_3}\nu}{\Gamma} d\beta_{t_3}+ \\
&Pr\left(mb_{k+2,t_4}|\Lambda_{t_3},\tau\right) \int_{\beta_{t_3}\in B(\tau)} \frac{p_{k+2}-\beta_{t_3}\nu}{\Gamma} d\beta_{t_3} + Pr\left(ms_{k+2,t_4}|\Lambda_{t_3},\tau\right) \int_{\beta_{t_3}\in B(\tau)} \frac{\beta_{t_3}\nu - p_{k+2}}{\Gamma} d\beta_{t_3} \Bigg) + \\
&\mathbb{1}_G \Bigg\{ \sum_{k=-n^f}^{+n^f-1} Pr\left(lb_{k,t_1}|\Lambda_{t_0},\tau\right) \sum_{d=k+2}^{n^f} Pr\left(lb_{d,t_2}|\Lambda_{t_1},\tau\right) \Bigg(\int_{\beta_{t_3}\in B(\tau)} \frac{p_d-\beta_{t_3}\nu}{\Gamma} d\beta_{t_3}+ \\
&\quad \sum_{j=d+1}^{+n^f} Pr\left(mb_{j,t_4}|\Lambda_{t_3},\tau\right) \int_{\beta_{t_3}\in B(\tau)} \frac{p_j-\beta_{t_3}\nu}{\Gamma} d\beta_{t_3} + \sum_{j=d+1}^{+n^f} Pr\left(ms_{j,t_4}|\Lambda_{t_3},\tau\right) \int_{\beta_{t_3}\in B(\tau)} \frac{\beta_{t_3}\nu - p_j}{\Gamma} d\beta_{t_3} \Bigg) + \\
&\quad \sum_{k=-n^f}^{+n^f-1} Pr\left(lb_{k,t_1}|\Lambda_{t_0},\tau\right) Pr\left(lb_{k+1,t_2}|\Lambda_{t_1},\tau\right) \Bigg(\\
&\quad \sum_{j=k+3}^{+n^f} Pr\left(mb_{j,t_4}|\Lambda_{t_3},\tau\right) \int_{\beta_{t_3}\in B(\tau)} \frac{p_j-\beta_{t_3}\nu}{\Gamma} d\beta_{t_3} + \sum_{j=k+3}^{+n^f} Pr\left(ms_{j,t_4}|\Lambda_{t_3},\tau\right) \int_{\beta_{t_3}\in B(\tau)} \frac{\beta_{t_3}\nu - p_j}{\Gamma} d\beta_{t_3} \Bigg) \Bigg\}
\end{aligned} \tag{192}$$

The overall welfare of the 3^{rd} player is hence given by the sum of equations (187)- (189)-(190)-(191)-(192).

The welfare of the 4^{th} player closely follows. The welfare of the 4^{th} player in case the 2^{nd} player immediately markets sell:

$$\omega_{t_4}(ms_{t_1} \ |\tau) = \sum_{k=-n^f}^{+n^f} Pr\left(lb_{k,t_1}|\Lambda_{t_0},\tau\right) Pr\left(ms_{k,t_2}|\Lambda_{t_1},\tau\right) \sum_{k=-n^f}^{+n^f} Pr\left(lb_{k,t_3}|\Lambda_{t_2},\tau\right) \int_{(1-b)}^{\frac{p_k}{v}} \frac{p_k-\beta_{t_4}v}{\Gamma} d\beta_{t_4} \tag{193}$$

The welfare of the 4^{th} player in case the 2^{nd} player posts a limit sell at p_{k+1}:

$$\begin{aligned}
&\omega_{t_4}(ms_{t_4} \ \vee mb_{t_4} \ |\tau) = \sum_{k=-n^f}^{+n^f-1} Pr\left(lb_{k,t_1}|\Lambda_{t_0},\tau\right) Pr\left(ls_{k+1,t_2}|\Lambda_{t_1},\tau\right) \Bigg(Pr\left(mb_{k+1,t_3}|\Lambda_{t_2},\tau\right) \int_{(1-b)}^{\frac{p_k}{v}} \frac{p_k-\beta_{t_4}v}{\Gamma} d\beta_{t_4}+ \\
&Pr\left(ms_{k,t_3}|\Lambda_{t_2},\tau\right) \int_{\frac{p_{k+1}}{v}}^{(1+b)} \frac{\beta_{t_4}v-p_{k+1}}{\Gamma} d\beta_{t_4} + Pr\left(nt_{k,t_3}|\Lambda_{t_2},\tau\right) \left(\int_{(1-b)}^{\frac{p_k}{v}} \frac{p_k-\beta_{t_4}v}{\Gamma} d\beta_{t_4} + \int_{\frac{p_{k+1}}{v}}^{(1+b)} \frac{\beta_{t_4}v-p_{k+1}}{\Gamma} d\beta_{t_4} \right) \Bigg)
\end{aligned} \tag{194}$$

The welfare of the 4^{th} player in case the 2^{nd} player posts a limit sell at p_{k+d} with $d \geq 2$:

$$\mathbb{1}_G \Bigg\{ \sum_{k=-n^f}^{+n^f-2} Pr\left(lb_{k,t_1}|\Lambda_{t_0},\tau\right) \sum_{d=k+2}^{+n^f} Pr\left(ls_{d,t_2}|\Lambda_{t_1},\tau\right)$$
$$\Bigg[Pr\left(ms_{k,t_3}|\Lambda_{t_2},\tau\right) \int_{\frac{p_d}{v}}^{(1+b)} \frac{\beta_{t_4}v - p_d}{\Gamma} d\beta_{t_4} + \sum_{j=k+1}^{d-1} Pr\left(ls_{j,t_3}|\Lambda_{t_2},\tau\right) \left(\int_{(1-b)}^{\frac{p_k}{v}} \frac{p_k - \beta_{t_4}v}{\Gamma} d\beta_{t_4} + \int_{\frac{p_j}{v}}^{(1+b)} \frac{\beta_{t_4}v - p_j}{\Gamma} d\beta_{t_4} \right) +$$
$$Pr\left(mb_{d,t_3}|\Lambda_{t_2},\tau\right) \int_{(1-b)}^{\frac{p_k}{v}} \frac{p_k - \beta_{t_4}v}{\Gamma} d\beta_{t_4} + \sum_{j=k+1}^{d-1} Pr\left(lb_{j,t_3}|\Lambda_{t_2},\tau\right) \left(\int_{(1-b)}^{\frac{p_j}{v}} \frac{p_j - \beta_{t_4}v}{\Gamma} d\beta_{t_4} + \int_{\frac{p_d}{v}}^{(1+b)} \frac{\beta_{t_4}v - p_d}{\Gamma} d\beta_{t_4} \right) \Bigg] \Bigg\} \tag{195}$$

The welfare of the 4^{th} player in case the 2^{nd} player posts a limit buy behind the 1^{st} player:

$$\omega_{t_4}(ms_{t_4} \vee mb_{t_4} |\tau) = \sum_{k=-n^f}^{+n^f} Pr\left(lb_{k,t_1}|\Lambda_{t_0},\tau\right) Pr\left(lb_{k,t_2}|\Lambda_{t_1},\tau\right) \Bigg(Pr\left(ms_{k,t_3}|\Lambda_{t_2},\tau\right) \int_{(1-b)}^{\frac{p_k}{v}} \frac{p_k - \beta_{t_4}v}{\Gamma} d\beta_{t_4} +$$
$$Pr\left(ls_{k+1,t_3}|\Lambda_{t_2},\tau\right) \left(\int_{(1-b)}^{\frac{p_k}{v}} \frac{p_k - \beta_{t_4}v}{\Gamma} d\beta_{t_4} + \int_{\frac{p_{k+1}}{v}}^{(1+b)} \frac{\beta_{t_4}v - p_{k+1}}{\Gamma} d\beta_{t_4} \right) + Pr\left(lb_{k+1,t_3}|\Lambda_{t_2},\tau\right) \int_{(1-b)}^{\frac{p_{k+1}}{v}} \frac{p_{k+1} - \beta_{t_4}v}{\Gamma} d\beta_{t_4} \Bigg) +$$
$$\mathbb{1}_G \Bigg\{ \sum_{k=-n^f}^{+n^f} Pr\left(lb_{k,t_1}|\Lambda_{t_0},\tau\right) \sum_{d=-n^f}^{k-1} Pr\left(lb_{d,t_2}|\Lambda_{t_1},\tau\right) \Bigg(Pr\left(ms_{k,t_3}|\Lambda_{t_2},\tau\right) \int_{(1-b)}^{\frac{p_d}{v}} \frac{p_d - \beta_{t_4}v}{\Gamma} d\beta_{t_4} +$$
$$\sum_{j=k+1}^{+n^f} Pr\left(ls_{j,t_3}|\Lambda_{t_2},\tau\right) \left(\int_{(1-b)}^{\frac{p_k}{v}} \frac{p_k - \beta_{t_4}v}{\Gamma} d\beta_{t_4} + \int_{\frac{p_j}{v}}^{(1+b)} \frac{\beta_{t_4}v - p_j}{\Gamma} d\beta_{t_4} \right) + \sum_{j=k+1}^{+n^f} Pr\left(lb_{j,t_3}|\Lambda_{t_2},\tau\right) \int_{(1-b)}^{\frac{p_j}{v}} \frac{p_j - \beta_{t_4}v}{\Gamma} d\beta_{t_4} \Bigg) +$$
$$\sum_{k=-n^f}^{+n^f} Pr\left(lb_{k,t_1}|\Lambda_{t_0},\tau\right) Pr\left(lb_{k,t_2}|\Lambda_{t_1},\tau\right) \Bigg($$
$$\sum_{j=k+2}^{+n^f} Pr\left(ls_{j,t_3}|\Lambda_{t_2},\tau\right) \left(\int_{(1-b)}^{\frac{p_k}{v}} \frac{p_k - \beta_{t_4}v}{\Gamma} d\beta_{t_4} + \int_{\frac{p_j}{v}}^{(1+b)} \frac{\beta_{t_4}v - p_j}{\Gamma} d\beta_{t_4} \right) + \sum_{j=k+2}^{+n^f} Pr\left(lb_{j,t_3}|\Lambda_{t_2},\tau\right) \int_{(1-b)}^{\frac{p_j}{v}} \frac{p_j - \beta_{t_4}v}{\Gamma} d\beta_{t_4} \Bigg) \Bigg\} \tag{196}$$

The welfare of the 4^{th} player in case the 2^{nd} player undercuts the limit buy posted by the 1^{st} player:

$$\omega_{t_4}(ms_{t_4} \vee mb_{t_4} \,|\tau) = \sum_{k=-n^f}^{+n^f} Pr\left(lb_{k,t_1}|\Lambda_{t_0},\tau\right) Pr\left(lb_{k+1,t_2}|\Lambda_{t_1},\tau\right)\Bigg(Pr\left(ms_{k+1,t_3}|\Lambda_{t_2},\tau\right) \int_{(1-b)}^{\frac{p_k}{v}} \frac{p_k-\beta_{t_4}v}{\Gamma} d\beta_{t_4} +$$
$$Pr\left(ls_{k+2,t_3}|\Lambda_{t_2},\tau\right)\left(\int_{(1-b)}^{\frac{p_{k+1}}{v}} \frac{p_{k+1}-\beta_{t_4}v}{\Gamma} d\beta_{t_4} + \int_{\frac{p_{k+2}}{v}}^{(1+b)} \frac{\beta_{t_4}v-p_{k+2}}{\Gamma} d\beta_{t_4}\right) + Pr\left(lb_{k+2,t_3}|\Lambda_{t_2},\tau\right)\int_{(1-b)}^{\frac{p_{k+2}}{v}} \frac{p_{k+2}-\beta_{t_4}v}{\Gamma} d\beta_{t_4}\Bigg) +$$
$$\mathbb{1}_G \Bigg\{ \sum_{k=-n^f}^{+n^f-1} Pr\left(lb_{k,t_1}|\Lambda_{t_0},\tau\right) \sum_{d=k+2}^{+n^f} Pr\left(lb_{d,t_2}|\Lambda_{t_1},\tau\right)\Bigg(Pr\left(ms_{d,t_3}|\Lambda_{t_2},\tau\right) \int_{(1-b)}^{\frac{p_k}{v}} \frac{p_k-\beta_{t_4}v}{\Gamma} d\beta_{t_4} +$$
$$\sum_{j=d+1}^{+n^f} Pr\left(ls_{j,t_3}|\Lambda_{t_2},\tau\right)\left(\int_{(1-b)}^{\frac{p_d}{v}} \frac{p_d-\beta_{t_4}v}{\Gamma} d\beta_{t_4} + \int_{\frac{p_j}{v}}^{(1+b)} \frac{\beta_{t_4}v-p_j}{\Gamma} d\beta_{t_4}\right) + \sum_{j=d+1}^{+n^f} Pr\left(lb_{j,t_3}|\Lambda_{t_2},\tau\right)\int_{(1-b)}^{\frac{p_j}{v}} \frac{p_j-\beta_{t_4}v}{\Gamma} d\beta_{t_4}\Bigg) +$$
$$\sum_{k=-n^f}^{+n^f} Pr\left(lb_{k,t_1}|\Lambda_{t_0},\tau\right) Pr\left(lb_{k+1,t_2}|\Lambda_{t_1},\tau\right)\Bigg($$
$$\sum_{j=k+3}^{+n^f} Pr\left(ls_{j,t_3}|\Lambda_{t_2},\tau\right)\left(\int_{(1-b)}^{\frac{p_k}{v}} \frac{p_k-\beta_{t_4}v}{\Gamma} d\beta_{t_4} + \int_{\frac{p_j}{v}}^{(1+b)} \frac{\beta_{t_4}v-p_j}{\Gamma} d\beta_{t_4}\right) + \sum_{j=k+3}^{+n^f} Pr\left(lb_{j,t_3}|\Lambda_{t_2},\tau\right)\int_{(1-b)}^{\frac{p_j}{v}} \frac{p_j-\beta_{t_4}v}{\Gamma} d\beta_{t_4}\Bigg)\Bigg\} \tag{197}$$

The overall welfare of the 4^{rd} player is hence given by the sum of equations (193)- (194)-(195)-(196)-(197). We are now in the position to define the total welfare of market participants, $\Omega(\tau)$, as the sum of the welfare of the four investors arriving respectively at time t_1, t_2, t_3 and t_4 of the 4-period trading game. The SP will choose the tick size that maximizes $\Omega(\tau)$:

$$\max_{\tau\in(0,\tau^{max})} \Omega(\tau) = \sum_{i=i}^{4} \omega_{t_i}(\cdot\,|\tau) \tag{198}$$

Given the optimization problems solved by traders and the SP, we can define the equilibrium of our trading game:

Definition 4. *A sub-game Perfect Nash Equilibrium of the trading game is the set of limit order submission probabilities and their respective execution probabilities (defined in Appendix E.3) that solve the optimization problem of investors at t_1, t_2,, t_3 and t_4 and that are consistent with the tick size, $\tau^\star \in (0, \quad \tau^{max})$, set by the SP to maximize total welfare $\Omega(\tau)$.*

E.5 Comparative Analysis Submission Strategies in the 5-period Game

Table 3.E: Comparative Analysis of the Player's Equilibrium Submission Probabilities
This table reports the equilibrium submission probabilities of the 5-period game solved for the OTS of both the 4-period and 5-period trading game. Panel A and B summarize the submission probabilities of the first two players. The first column reports the prices associated to the equilibrium order submission strategies of the 1^{st} player $Pr\left(lb_{k,t_1}|\Lambda_{t_0},\tau\right)$. The columns $3-6$ of Panel A and B report the probabilities of market sell at t_2 ($Pr\left(ms_{k,t_2}|\Lambda_{t_1},\tau\right)$), of limit sell ($Pr\left(ls_{>k,t_2}|\Lambda_{t_1},\tau\right)$), of queuing ($Pr\left(lb_{\leq k,t_2}|\Lambda_{t_1},\tau\right)$) and of undercutting ($Pr\left(lb_{>k,t_2}|\Lambda_{t_1},\tau\right)$). Panel C and D report the equilibrium unconditional order submission probabilities of the 3^{rd} and 4^{th} players. We report in Panel C the unconditional probability of market sell at t_3 ($Pr\left(ms_{t_3}\right)$) (column 2), of limit sell (undercutting) ($Pr\left(ls_{>k,t_3}\right)$) (column 3), of limit sell (queuing) (column 4) ($Pr\left(ls_{\leq k,t_3}\right)$), of no trade ($Pr\left(nt_{k,t_3}\right)$), of limit buy (queuing) ($Pr\left(lb_{\leq k,t_3}\right)$), of limit buy (undercutting) ($Pr\left(lb_{>k,t_3}\right)$) and of market buy ($Pr\left(mb_{k,t_3}\right)$). We report in Panel D the analogous t_4 unconditional order submission probabilities of the 4^{th} player. Results are reported for the baseline parameterization ($b = 0.06$ and $\nu = 10$).

Panel A: 5-period game 1^{st} and 2^{nd} player conditional order submission strategies with OTS $4P$ (0.214)

Price p_k	Limit Buy $t1$ $Pr\left(lb_{k,t_1}\|\Lambda_{t_0},\tau\right)$	Market Sell t_2 $Pr\left(ms_{k,t_2}\|\Lambda_{t_1},\tau\right)$	Limit Sell t_2 $Pr\left(ls_{>k,t_2}\|\Lambda_{t_1},\tau\right)$	Queuing t_2 $Pr\left(lb_{\leq k,t_2}\|\Lambda_{t_1},\tau\right)$	Undercutting t_2 $Pr\left(lb_{>k,t_2}\|\Lambda_{t_1},\tau\right)$
10.107	0.283	0.503	0.107	0.293	0.097
9.893	0.208	0.000	0.627	0.000	0.373
9.679	0.009	0.000	0.510	0.000	0.490

Panel B: 5-period game 1^{st} and 2^{nd} player conditional order submission strategies with OTS $5P$ (0.160)

Price p_k	Limit Buy $t1$ $Pr\left(lb_{k,t_1}\|\Lambda_{t_0},\tau\right)$	Market Sell t_2 $Pr\left(ms_{k,t_2}\|\Lambda_{t_1},\tau\right)$	Limit Sell t_2 $Pr\left(ls_{>k,t_2}\|\Lambda_{t_1},\tau\right)$	Queuing t_2 $Pr\left(lb_{\leq k,t_2}\|\Lambda_{t_1},\tau\right)$	Undercutting t_2 $Pr\left(lb_{>k,t_2}\|\Lambda_{t_1},\tau\right)$
10.080	0.272	0.450	0.157	0.190	0.203
9.920	0.186	0.097	0.508	0.000	0.395
9.760	0.042	0.000	0.525	0.000	0.475

Panel C: 5-period game - 3^{rd} player unconditional order submission strategies

	Market Sell $Pr\left(ms_{k,t_3}\right)$	Undercutting $Pr\left(ls_{>k,t_3}\right)$	Queuing $Pr\left(ls_{\leq k,t_3}\right)$	No Trade $Pr\left(nt_{k,t_3}\right)$	Queuing $Pr\left(lb_{\leq k,t_3}\right)$	Undercutting $Pr\left(lb_{>k,t_3}\right)$	Market Buy $Pr\left(mb_{k,t_3}\right)$
OTS 4P (0.214)	0.142	0.054	0.024	0.000	0.047	0.106	0.056
OTS 5P (0.160)	0.149	0.056	0.010	0.000	0.052	0.120	0.042

Panel D: 5-period game - 4^{th} player unconditional order submission strategies

	Market Sell $Pr\left(ms_{k,t_4}\right)$	Undercutting $Pr\left(ls_{>k,t_4}\right)$	Queuing $Pr\left(ls_{\leq k,t_4}\right)$	No Trade $Pr\left(nt_{k,t_4}\right)$	Queuing $Pr\left(lb_{\leq k,t_4}\right)$	Undercutting $Pr\left(lb_{>k,t_4}\right)$	Market Buy $Pr\left(mb_{k,t_4}\right)$
OTS 4P (0.214)	0.150	0.099	0.000	0.018	0.000	0.095	0.048
OTS 5P (0.160)	0.162	0.097	0.000	0.013	0.000	0.108	0.047

F Appendix: Infinite Number of Investors/Periods

In this session, we present a stylized LOB trading protocol with an infinite number of investors arriving at the market and show that a zero tick size does not maximize global welfare. The protocol is the following:

- In each period t_i - with i=1,..,∞ - a risk-neutral investor arrives at the market with certainty with a private valuation $\beta_{t_i}\nu$, where β_{t_i} is drawn from a uniform distribution, $\beta_{t_i} \overset{i.i.d.}{\sim} U[\underline{\beta}, \overline{\beta}]$, centered around the publicly known asset value ν, $\underline{\beta} = 1 - b$, $\overline{\beta} = 1 + b$ and $b \in (0, 1)$.

- For tractability, we assume that each arriving investor bids a price that reveals his private valuation of the asset, $\beta_{t_i}\nu$.

We first consider the case of a zero tick size. When $\tau = 0$, each player posts a limit order with a limit price equal to his $\beta_{t_i}\nu$ evaluation. As soon as two players with different evaluation have arrived (two limit orders sequentially posted), there is a trade, and total welfare is $|\beta_{t_i}\nu - \beta_{t_l}\nu|$.[39] As $\beta_{t_i}\nu$ are randomly drawn from a continuous distribution, the probability that two extracted evaluations are the same is zero. Therefore, every two periods the market clears. The expected total welfare from each trade is:

$$E[|\beta_{t_i}\nu - \beta_{t_l}\nu|] = \int_{1-b}^{1+b}\int_{1-b}^{1+b} \frac{|\beta_{t_i}\nu - \beta_{t_l}\nu|}{\Gamma^2} d\beta_{t_i} d\beta_{t_l} = \frac{2b\nu}{3} \tag{199}$$

We then consider the case with $\tau > 0$, for example $\tau = b\nu$. By equations (1) and (2), the price grid is formed by $p_{+1} = \nu(1 + \frac{1}{2}b)$ and $p_{-1} = \nu(1 - \frac{1}{2}b)$. Given truthful reporting, the equilibrium submission strategies are the following:

- Each trader with evaluation $\beta_{t_i}\nu < p_{-1}$, submits a limit sell at p_{-1}.

- Each trader with evaluation $p_{-1} < \beta_{t_i}\nu < \nu$, submits a limit sell at p_{+1}.

- Each trader with evaluation $\nu < \beta_{t_i}\nu < p_{+1}$, submits a limit buy at p_{-1}

[39]Given two different evaluations, assuming $\beta_{t_i} > \beta_{t_l}$ without loss of generality, any price $p \in (\beta_{t_l}\nu, \beta_{t_i}\nu)$ is a possible transaction price, therefore the total welfare generated by such trade is $(\beta_{t_i}\nu - p) + (p - \beta_{t_l}\nu) = \beta_{t_i}\nu - \beta_{t_l}\nu$.

- Each trader with evaluation $\beta_{t_i}\nu > p_{+1}$, submits a limit buy at p_{+1}.[40]

Therefore any transaction at p_{+1} involves a player with $\beta_{t_i}\nu \in (p_{-1},\ \nu)$ and one with $\beta_{t_l}\nu \in (p_{+1},\ (1+b)\nu]$. Symmetrically, any transaction at p_{-1} involves one player with $\beta_{t_i}\nu \in (\nu,\ p_{+1})$ and one with $\beta_{t_l}\nu \in [(1-b)\nu,\ p_{-1})$. The expected total welfare associated with each trade at p_{+1} is:

$$
\begin{aligned}
&E[\omega(p_{+1})] = E[\beta_{t_l}\nu - \beta_{t_i}\nu|\beta_{t_l}\nu > p_{+1},\ p_{-1} \leq \beta_{t_i}\nu \leq \nu] \times Pr(\beta_{t_l}\nu > p_{+1},\ p_{-1} \leq \beta_{t_i}\nu \leq \nu)\\
&= E[\beta_{t_l}\nu|\beta_{t_l}\nu > p_{+1}] - E[\beta_{t_i}\nu|p_{-1} \leq \beta_{t_i}\nu \leq \nu]\\
&= \int_{1+\frac{b}{2}}^{1+b} \frac{\beta_{t_l}\nu}{\frac{b}{2}} d\beta_{t_l} - \int_{1-\frac{b}{2}}^{1} \frac{\beta_{t_l}\nu}{\frac{b}{2}} d\beta_{t_i} = \nu(1+\frac{3}{4}b) - (\nu - \frac{b\nu}{4}) = b\nu
\end{aligned}
\tag{200}
$$

where $Pr(\beta_{t_l}\nu > p_{+1},\ p_{-1} \leq \beta_{t_i}\nu \leq \nu)$ is equal to 1 given that an infinite number of players are supposed to arrive at the market. Symmetric results apply for a trade at p_{-1}.

By comparing (199) and (200), we can therefore conclude that there exists at least one tick size value such that the expected total welfare is greater than the one associated with each trade when the tick size is zero. We can also conclude that $\tau = 0$ is not the OTS. In addition, it should be noticed that $\tau = 2b\nu$ would cause a market breakdown as all players would submit either a limit buy at p_{-1} or a limit sell at p_{+1}, thus confirming that the OTS is an interior point of the open interval $(0, 2b\nu)$.

[40] Players with an evaluation equal to p_{-1} or p_{+1} have zero expected payoff and do not trade, while players with an evaluation of ν are indifferent between a limit buy at p_{-1}, and a limit sell at p_{+1}. The probability of drawing such events is zero given that $\beta_{t_i}\nu$ is a continuous distribution.

G Appendix: Empirical Analysis

Table 1.G: Effects of MiFID II on each Book Level

This table reports the coefficients of a tick size increase ($\mathbb{I}_{inc} \times AFTER$) and decrease from the Difference in Difference (DD) regression analysis around the introduction of the MiFID II regime using the following specification:

$$MQ_{i,t,l} = \alpha + \gamma_i + \delta_t + \phi_1 \tau_{i,t} + \beta_1(\mathbb{I}_{inc} \times AFTER) + \beta_2(\mathbb{I}_{dec} \times AFTER) + \phi_2 Volat_{i,t} + \phi_3 EUVIX_t + \epsilon_{i,t}$$

where $MQ_{i,t,l}$ is a market quality metric - Spread, %-Spread (bps) and Depth - for stock i, day t and level l of the book with $1 \leq l \leq 10$; $\tau_{i,t}$ is the daily tick size; $AFTER$ is an indicator variable equal to 1 after January the 1^{st} 2018 and 0 otherwise; $\mathbb{I}_{inc}$ is an indicator variable equal to 1 if the tick associated to stock i increased after MiFID II and 0 otherwise; $\mathbb{I}_{dec}$ is an indicator variable is equal to 1 if the tick associated to stock i decreased after MiFID II and 0 otherwise; $Volat_{i,t}$ is the daily volatility at the stock level, while $EUVIX_t$ is the STOXX volatility index at daily level. We report t-statistics in parentheses obtained from robust standard errors clustered by stock. $\star\star\star$: 1% significance, $\star\star$: 5% significance, and $\star$: 10% significance, respectively (results statically significant at 10% level at least are grey shaded).

		Spread	%Spread(bps)	Depth
L1	$\mathbb{I}_{inc} \times AFTER$	-0.049	-0.654	-0.004
		(−0.220)	(−0.500)	(−0.249)
	$\mathbb{I}_{dec} \times AFTER$	-0.174	-1.130	-0.026
		(−1.503)	(−1.234)	(−1.216)
L2	$\mathbb{I}_{inc} \times AFTER$	-0.252	-2.318	-0.012
		(−0.903)	(−1.272)	(−0.398)
	$\mathbb{I}_{dec} \times AFTER$	−0.424*	−3.173*	-0.062
		(−1.932)	(−1.705)	(−1.477)
L3	$\mathbb{I}_{inc} \times AFTER$	-0.480	−4.855*	-0.020
		(−1.157)	(−1.770)	(−0.582)
	$\mathbb{I}_{dec} \times AFTER$	−0.744**	−5.651*	-0.076
		(−2.098)	(−1.949)	(−1.530)
L4	$\mathbb{I}_{inc} \times AFTER$	-0.709	−7.222**	-0.022
		(−1.250)	(−2.054)	(−0.698)
	$\mathbb{I}_{dec} \times AFTER$	−1.094**	−8.274**	−0.087*
		(−2.315)	(−2.194)	(−1.828)
L5	$\mathbb{I}_{inc} \times AFTER$	-0.812	−8.359*	-0.010
		(−1.135)	(−1.948)	(−0.433)
	$\mathbb{I}_{dec} \times AFTER$	−1.249**	−9.988**	−0.079**
		(−2.271)	(−2.195)	(−2.238)
L6	$\mathbb{I}_{inc} \times AFTER$	-1.074	−9.779*	0.009
		(−1.184)	(−1.928)	(0.575)
	$\mathbb{I}_{dec} \times AFTER$	−1.542**	−12.314**	−0.058**
		(−2.241)	(−2.218)	(−2.832)
L7	$\mathbb{I}_{inc} \times AFTER$	-1.368	−12.113**	0.011
		(−1.250)	(−1.993)	(1.039)
	$\mathbb{I}_{dec} \times AFTER$	−1.872**	−14.140**	−0.034**
		(−2.249)	(−2.189)	(−2.451)
L8	$\mathbb{I}_{inc} \times AFTER$	-1.665	−15.147**	0.011
		(−1.334)	(−2.300)	(1.233)
	$\mathbb{I}_{dec} \times AFTER$	−2.047**	−15.800**	-0.016
		(−2.242)	(−2.334)	(−1.291)
L9	$\mathbb{I}_{inc} \times AFTER$	-1.737	−16.509**	0.003
		(−1.250)	(−2.306)	(0.330)
	$\mathbb{I}_{dec} \times AFTER$	−2.340**	−17.698**	-0.005
		(−2.245)	(−2.311)	(−0.402)
L10	$\mathbb{I}_{inc} \times AFTER$	-1.936	−18.694**	-0.005
		(−1.242)	(−2.290)	(−0.764)
	$\mathbb{I}_{dec} \times AFTER$	−2.537**	−19.088**	-0.002
		(−2.191)	(−2.256)	(−0.155)