A Real Investment-based Model of Asset Pricing

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Abstract

We recover a stochastic discount factor (SDF) for asset returns from a firm's investment Euler equation. Given a parametric statistical specification of the SDF and profitability process, we solve for the firms' optimal investment decision with approximate analytical solutions and provide a dissection of the determinants of real investment. We estimate a specification of the model to discipline the free parameters of the SDF by matching moments of both aggregate real quantities and asset prices. We use the estimated parameters to recover the latent SDF from data on aggregate investment rates, risk-free rates, and profitability growth rates. Innovations in the recovered SDF are driven dominantly by innovations in investment rates and marginally by innovations in risk-free rates and profitability growth rates. The recovered SDF exhibits strong counter-cyclicality with large jumps in recessions and prices standard Fama-French portfolios out of sample reasonably well. Our model allows us to explicitly characterize the risk-free rate, the equity premium, the term structure of interest rates, and the term structure of equity risk premia. The framework we propose here is general and can be extended to accommodate several additional aggregate shocks and frictions that have been proposed in the literature.

Keyword: stochastic discount factor, investment, investment-based asset pricing

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1 Introduction

The stochastic discount factor (SDF) is the foundation of modern asset pricing theory[1]. The fundamental theorem of asset pricing, credited to Ross (1978) and Harrison and Kreps (1979), states that there exists a positive stochastic discount factor that prices all assets in absence of arbitrage opportunities. Investigating the behavior of the stochastic discount factor and identifying its empirical measure is the key to understanding asset prices. Barring a few exceptions (discussed below), stochastic discount factors are recovered from consumption-based models. In the consumption-based approach, pioneered by Rubinstein (1976), Lucas (1978), and Breeden (1979), a stochastic discount factor is recovered from equilibrium marginal rates of substitution (MRS) inferred from consumers' first-order condition of utility maximization.

This paper proposes a novel investment-based approach to recover the stochastic discount factor from the supply side of the economy. We consider a stylized partial-equilibrium neoclassical model of investment with a value-maximizing representative producer. Taking as given exogenous processes of the profitability growth and the stochastic discount factor, the producer optimally invests to maximize the market value of the firm. The optimal investment rate (investment-to-capital ratio) can be solved explicitly in a first-order approximation and expressed as a linear function of state variables that drive both the profitability growth and the stochastic discount factor. We estimate a specification of the model to match moments of investment rates, risk-free rates, equity premium, and profitability growth rates, and minimize mean squared pricing errors of in-sample test portfolios. The estimated model allows us to recover realizations of underlying state variables from data on investment rates, risk-free rates, and profitability growth rates, and recover shocks to state variables due to parametric statistical assumptions of exogenous processes. Finally, we recover the stochastic discount factor as a function of investment rates, risk-free rates, and profitability growth rates. We investigate properties of the recovered stochastic discount factor and examine its pricing performance on out-of-sample portfolios.

Our baseline specification of the model closely matches most moments of investment, profitability growth, and asset prices. Specifically, the estimated model generates a log investment rates with a mean of -3.29 and a standard deviation of 8.39%, identical to those in the data. The autocorrelation coefficient of the investment rate is 0.88 in the model and close to 0.97 in the data. The log risk-free rate implied by the model has a mean of 0.50% and a standard deviation of 0.57%, identical to 0.50% and 0.57% in the data. The autocorrelation coefficient of the model-implied risk-free rate is 0.69%, very close to 0.68% in the data. The model-implied excess return has a mean of 1.28% and a standard deviation of 8.62%, almost identical to 1.29% and 8.61% in the data. The investment rate has a positive correlation of 0.35 with the real risk-free rate, close to 0.34 in the data, and has a negative correlation of -0.0006 with the

1. See, for example, Cochrane (2005) and Campbell (2018) for a textbook treatment of the stochastic discount factor in asset pricing.

excess return, falling short of -0.18 in the data.

Central to our paper is the investment-implied SDF. In the simplest specification of the model, innovations in the recovered SDF (22) are driven by innovations in investment rates and innovations in profitability. Our model goes beyond the simple return predictability of investment and profitability and delivers the prediction that the expected return of an asset is determined by the covariance between asset returns and two factors, innovations in investment rates and innovations in profitability. In the more realistic model specification taken to estimation, innovations in risk-free rates are also included in the recovered SDF (50). While data on risk-free rates and profitability growth are used, in our estimation, variations in the recovered SDF are driven dominantly by innovations in investment rates and only marginally by innovations in risk-free rates and profitability.

Using the estimated parameters, we recover the SDF from data on investment rates, risk-free rates, and profitability growth rates. The time series is plotted in Figure 3. The recovered SDF peaked during all major recessions, most significantly during the recent COVID-19 pandemic in 2020, followed by the Great Recession in 2008. A variance decomposition of unexpected innovations in the log SDF reveals that shocks to expected profitability growth account for 106.7% of total variations and innovations in investment rates account for 98.9% of total variations. This result highlights the role of shocks to expected profitability growth or innovations in investment rates in driving the SDF.

The recovered SDF prices 10 size-sorted portfolios in the sample with a mean absolute pricing error of only 0.06%. The recovered SDF also prices out-of-sample portfolios reasonably well. The recovered SDF prices 6 size-book-to-market-sorted portfolios, and 6-size-profitability-sorted portfolios, 6-size-investment-sorted portfolios, with mean absolute pricing errors are 0.36%, 0.41%, 0.39%, respectively. Similarly, a covariance decomposition shows that the covariance between portfolio returns and shocks to expected profitability growth or innovations in investment rates matters the most among all components. Therefore, we should expect that a SDF with innovations in investment rates as the single factor prices the cross section with a very close performance to that of the complete recovered SDF.

The intuition behind our approach recovering the stochastic discount factor from investment is straightforward. Despite being not state-contingent, investment is forward-looking. More specifically, investment decisions are made based on the joint conditional distribution of the stochastic discount factor and the profitability growth. Therefore, investment data contain information about both the stochastic discount factor and the profitability growth. The information of the stochastic discount factor can be disentangled from that of the profitability growth once the structure between these two objects is specified. Agnostic about its form in the absence of household preference, we make parametric assumptions that both the stochastic discount factor and the profitability growth are driven by the same state variables, which are the expected profitability growth and the profitability growth uncertainty, and that the stochastic discount factor is subject to shocks to profitability growth, shocks to expected profitability growth, and shocks to profitability growth uncertainty with free parameters.[2]

Through a first-order approximation of the investment return, the investment Euler equation yields the optimal investment rate as a linear function of state variables, given the stochastic discount factor with the assumed functional form and the exogenously specified profitability growth. The coefficients of the optimal investment rate are elasticities of investment to the expected profitability growth and to the profitability growth uncertainty, as functions of primitive parameters of the SDF and state variables. The elasticity of investment to the expected profitability growth summarizes the net effect of a change in the expected profitability growth on investment through both a cash flow channel and a risk-free rate channel. The elasticity of investment to the profitability growth uncertainty summarizes the net effect of a change in conditional volatility of the profitability growth on investment through both a cash flow channel and a discount rate channel (both the risk-free rate and the risk premium).

A valid stochastic discount factor should jointly explain both real quantities and asset prices. To discipline free parameters of the stochastic discount factor (and state variables), we estimate the model to match moments of investment rates, profitability growth rates, risk-free rates, and equity premia. The estimation in turn yields estimated elasticities of investment rates and risk-free rates to state variables, with which we are able to recover the latent expected profitability growth and the profitability growth uncertainty from investment rates and risk-free rates. Finally, we recover the stochastic discount factor by assembling recovered state variables and shocks to state variable with estimated parameters of the stochastic discount factor.

We focus on a simple specification of the model in this paper to illustrate our new approach. Nevertheless, this framework is flexible and can incorporate additional aggregate shocks and frictions that have been proposed in the literature. For example, one may incorporate labor market frictions to jointly account for investment, employment, and asset prices and recover a SDF from investment and employment data. One may also include investment-specific shocks or similarly adjustment cost shocks to introduce an additional source of risk. It is also possible to extend the model to a multi-sector setting in which the SDF can be recovered from multiple sectoral investment rates.

Related Literature

Our paper contributes to the relatively small branch of the production-based asset pricing literature that attempts to recover a stochastic discount factor from production/investment decisions. Our paper is closest to Cochrane (1993) and Belo (2010).[3] They propose an approach to infer the stochastic discount factor from a producers' marginal rates of transformation (MRT) across states of nature, without any information about the consumer side of the economy and without having to parameterize stochastic processes that drive a firm's decision. Because the MRT across states of nature is not well defined for

2. In the simpler case, we assume that the stochastic discount factor is driven only by the expected profitability growth and subject to shocks to profitability growth and shocks to expected profitability growth. Adding time-varying uncertainty is crucial to generate the time-varying equity premium and other features of both investment and asset prices.

3. Cochrane (2020) provides a detailed exposition of this pure production-based approach.

standard representations of the technology (see Figure 1 in Belo (2010)), they propose a flexible production technology that allows a producer to transform productivity across states. While this approach is theoretically appealing and parallel to the consumption-based approach, the empirical identification of the latent productivity process that drives the SDF can be challenging. We recover a SDF for asset returns using a conventional representation of the production technology and directly observable data on investment and asset prices. We achieve this result by making, relative to the previous work, additional parametric and statistical assumptions about underlying stochastic processes that drive firm's decisions in the economy (the profitability growth and the SDF).

Another closely related paper is Cochrane (1996). It proposes a multi-factor representation of the SDF with two sectoral investment returns as factors[4]. While the proposed SDF prices 10 size-sorted portfolios remarkably well, it only motivates but does not theoretically establish the SDF. Subsequently, Li, Vassalou, and Xing (2006) experiments further with this approach and proposes using three sectoral investment growth rates as factors. The proposed SDF prices the 25 Fama-French size-sorted and book-to-market-sorted portfolios with a performance comparable to that of Fama-French 3-factor model. As Campbell (2018) points out, "A satisfying economic explanation should at a minimum derive the risk prices of multiple factors from deeper equilibrium considerations, such as the preferences of investors and the production possibilities of the economy". Our paper provides a complete theory for recovering the SDF from observable investment based on the firm's optimal investment decision, deriving and imposing theoretical prices of risk on factors in the SDF. A multi-sector version of our model can rationalize the specification of SDFs with multiple sectoral investment rates as factors.

In the same spirit, Cochrane (1988) and Jermann (2010) seeks to use investment returns to recover state prices in a discrete Markov setting. However, this approach requires that the production technology have as many types of capital inputs as the number of states of nature (in the baseline setting they focus on a two-state representation of nature). Our model does not require such "complete technology" and hence can be used with continuous random variables and infinite state space.

Following Cochrane (1991), Liu, Whited, and Zhang (2009) builds upon the equality between investment and stock returns and provides a characteristic-based explanation of the cross-sectional variation in average stock returns by modeling investment returns directly. As a result of focusing on the investment return itself and overlooking its connection with the SDF, their approach is unable to link expected returns to exposure to aggregate risks. In contrast, our approach goes beyond the simple return predictability of investment and profitability and delivers the prediction that the expected return of an asset is determined by the covariance between asset returns and two factors, innovations in investment rates and innovations in profitability (in the baseline case), which are linked to aggregate risks in the model. In addition, as pointed out by Campbell (2018), different parameter values are required to fit different

4. Investment returns are constructed as a function of investment rates and marginal product of capital.

test assets in their approach and in the vast majority of investment-based asset pricing models. Our approach obtains parameters structurally from matching moments of aggregate quantities and prices and then apply the SDF with a fixed set of parameters to price the cross section.

Departing from the characteristic-based approach, Hou, Xue, and Zhang (2015) proposes a multi-factor reduced-form SDF including an investment factor and a profitability factor in addition to a market factor and a size factor. The investment factor and the profitability factor are motivated by the return predictability of investment and profitability and constructed by building factor-mimicking portfolios sorted on investment and profitability in the same way as Fama and French (1993).[5] Despite its empirical success, the economic mechanism driving the results are still poorly understood (as in the Fama-French 3-factor model), in the sense that return predictability of firm characteristics does not directly imply return comovement among firms with similar characteristics. Our approach theoretically recovers a SDF with innovations in investment rates and in profitability as factors.

The rest of the paper is organized as follows. Section 2 sets up and explicitly solves two specifications of the model of increasing complexity: Case I considers an simple homoscedastic environment, and Case II considers an economy with time-varying conditional volatility. The solution method is illustrated and the intuition of our model is discussed. Section 3 estimates the model (Case II) by matching both quantities and asset prices to obtain model parameters. Section 4 recovers state variables and the SDF. Subsequently, the model-implied equity premium, term structure of interest rates, and term structure of dividend strips are also presented. Section 5 concludes.

2 Model

This section presents a standard neoclassical model of investment with a representative producer. Sector 2.1 sets up the model and derives the optimal condition of investment, i.e., the investment Euler equation, which is a joint restriction on the SDF and the investment return. Section 2.2 (Case I) illustrates the solution method and discusses the intuition in a simple homoscedastic exogenous environment. Section 2.5 (Case II) extends the specification to feature time-varying conditional volatility in order to capture missing salient features of investment and asset prices in Case I. We derive in both cases the recovered SDF as a function of investment rates (coupled with or without risk-free rates) and profitability growth rates in section 2.3 and 2.6. We provide in both cases analytical solutions of the risk-free rate, the equity premium, the term structure of interest rates, and the term structure of equity premia in section 2.4 and 2.7. All detailed derivations are provided in the Appendix.

5. Zhang (2017) provides a comprehensive review of the q-factor model and the literature of investment-based asset pricing.

2.1 Investment Euler Equation

Consider a representative producer producing a single good to be consumed or invested. The production function exhibits constant return-to-scale in capital and labor.

$$Y_t = A_t K_t^{\alpha} N_t^{1-\alpha} \tag{1}$$

where A_t is the exogenous total factor of productivity (TFP), K_t is the stock of physical capital, N_t is the number of total labor hours. Assuming the labor is costlessly adjustable, we obtain the profit function by profit maximization, $\Pi_t = \max_{N_t} A_t K_t^{\alpha} N_t^{1-\alpha} - W_t N_t$.

$$\Pi(\mathcal{E}_t, K_t) = \left[[\alpha(1-\alpha)^{\frac{1-\alpha}{\alpha}}] A_t^{\frac{1}{\alpha}} W_t^{\frac{\alpha-1}{\alpha}} \right] K_t \equiv \mathcal{E}_t K_t \tag{2}$$

where the profitability is given by $\mathcal{E}_t = [\alpha(1-\alpha)^{\frac{1-\alpha}{\alpha}}] A_t^{\frac{1}{\alpha}} W_t^{\frac{\alpha-1}{\alpha}}$ and the profit function is constant returns-to-scale in capital only. For convenience of exposition, we will use profitability for the rest of the paper.

The capital stock is subject to depreciation and accumulated by investment.

$$K_{t+1} = (1-\delta) K_t + I_t \tag{3}$$

where $\delta \in (0,1)$ denotes the depreciation rate and I_t denotes investment.

Investment is costly in the sense that it costs more than the purchase cost of investment goods to install new capital. One can interpret additional costs of investment as output lost during installation of new capital. Following Merz and Yashiv (2007), we assume that the total investment cost is convex and proportional to output.[6]

$$\Phi(I_t, K_t, \mathcal{E}_t) = \frac{\kappa}{\eta+1} \left(\frac{I_t}{K_t} \right)^{\eta+1} \Pi(\mathcal{E}_t, K_t) \tag{4}$$

where $\kappa > 0$ is the scalar of the total investment cost, and $\eta \geq 0$ controls the curvature of the total investment cost. For example, installing I_t units of capital costs in total $\kappa \frac{I_t}{K_t} \Pi(\mathcal{E}_t, K_t)$ units of goods when $\eta = 0$, and costs $\frac{\kappa}{2} \frac{I_t^2}{K_t} \Pi(\mathcal{E}_t, K_t)$ units of goods when $\eta = 1$. The convexity implies that installing new capital is increasingly costly in the investment rate. Higher curvature η implies lower elasticity of investment to marginal value of capital and more elastic supply of capital (Jermann, 1998).

6. In some papers, the total investment cost is specified as the sum of purchase cost of investment goods and investment adjustment costs $I_t + \Phi(I_t, K_t)$. We fold the purchase cost of investment goods I_t into the total cost of investment for algebraic convenience without loss of generality, as in Merz and Yashiv (2007) and Kogan and Papanikolaou (2012).

The producer pays out its residual profits after investment as dividends to its equity holders.

$$D_t = \mathit{\Pi}(\mathcal{E}_t, K_t) - \mathit{\Phi}(I_t, K_t, \mathcal{E}_t) \tag{5}$$

To simplify the analysis, the production is assumed to be fully equity-financed. A negative dividend payout refers to equity issuance to equity holders.

Taking as given exogenous processes of the SDF and the profitability, the producer optimally chooses investment to maximize its cum-dividend value. The producer's problem can be written conveniently in a recursive manner. Denote a vector of state variables $X_t = (\mathcal{E}_t, K_t)$, the Bellman equation of the producer is given by

$$V(X_t) = \max_{\{I_t\}} \left\{ D_t + \mathbb{E}_t[M_{t+1} V(X_{t+1})] \right\}$$

subject to the flow of fund constraint (5) and the capital's law of motion (3).

The first-order condition of the producer's value maximization, often called the investment Euler equation, is given by

$$1 = \mathbb{E}_t[M_{t+1} R^I_{t+1}] \tag{6}$$

$$R^I_{t+1} = \frac{\mathcal{E}_{t+1}[1 + \kappa \frac{\eta}{\eta+1} IK^{\eta+1}_{t+1} + (1-\delta)\kappa IK^{\eta}_{t+1}]}{\mathcal{E}_t[\kappa IK^{\eta}_t]} \tag{7}$$

where $IK_t \equiv I_t/K_t$ denotes the investment-capital ratio, or the investment rate, and R^I_{t+1} denotes the marginal investment return.

First derived by Cochrane (1991), the investment Euler equation states that, in equilibrium, the producer adjusts the investment until the marginal cost of investment, $\mathcal{E}_t[\kappa IK^{\eta}_t]$, equals the expected discounted marginal benefit of investment, $\mathbb{E}_t\big[M_{t+1}\mathcal{E}_{t+1}[1+\kappa\frac{\eta}{\eta+1} IK^{\eta+1}_{t+1} + (1-\delta)\kappa IK^{\eta}_{t+1}]\big]$. Alternatively put, the marginal investment return, given by the marginal benefit of investment divided by the marginal cost of investment, has a price of one in equilibrium.

More importantly, in absence of arbitrage and under constant returns-to-scale, the investment return equals ex post the stock return at all times and across all states of nature.[7]

$$\frac{\mathcal{E}_{t+1}[1 + \kappa \frac{\eta}{\eta+1} IK^{\eta+1}_{t+1} + (1-\delta)\kappa IK^{\eta}_{t+1}]}{\mathcal{E}_t[\kappa IK^{\eta}_t]} \equiv R^I_{t+1} = R^S_{t+1} \equiv \frac{P_{t+1} + D_{t+1}}{P_t} \equiv \frac{V(X_{t+1})}{V(X_t) - D_t} \tag{8}$$

where R^S_{t+1} denotes the stock return and P_t denotes the ex-dividend value.

The investment Euler equation is a restriction on the joint process of the SDF, profitability and investment and can be interpreted in two ways. From the perspective of intertemporal optimization,

7. Restoy and Rockinger (1994) provides general conditions for the equality of investment returns and stock returns. Critical conditions are absence of arbitrage and linearly homogenous production technology and adjustment cost function. The equality is robust to considering external funding constraints and taxation. Liu, Whited, and Zhang (2009) incorporates debt and taxes and derives that the investment return equals the weighted average cost of capital (WACC).

the investment Euler equation yields the optimal investment given an exogenous SDF and profitability, analogous to the consumption Euler equation solving the household's optimal consumption and portfolio choice problem given an exogenous SDF and asset returns. Investment is essentially driven by expectations of cash flows and discount rates and therefore contains information about both. From the perspective of asset pricing, the investment Euler equation relates asset returns to production variables, investment and profitability. All else equal, high investment predicts low returns, while high expected profitability predicts high returns. These predictability patterns are consistent with the intuition of the classic net present value (NPV) rule in capital budgeting. Lower discount rates imply higher NPV and investment. Higher expected profitability relative to current investment implies higher discount rates.

Our approach of recovering the SDF takes advantage of both interpretations of the investment Euler equation. We first solve explicitly the optimal investment rate given an exogenous SDF and profitability, driven by common underlying state variables. We then estimate model parameters by matching moments of quantities and prices, both of which are interrelated via the investment Euler equation. We finally use estimated parameters and observed investment data to recover underlying state variables and therefore recover the SDF. Using the recovered SDF, we can obtain the conditional equity premium and evaluate prices of both claims to risky dividends and claims to risk-less payoffs across different maturities.

The producer's problem cannot be explicitly solved under general conditions due to its dynamic and stochastic nature. Since our ultimate goal is to recover the SDF as a function of observed production variables, our approach requires tractability. Therefore, we log-linearize the investment return to obtain analytical solutions.

The log investment return is linearized with respect to the investment rate around its long-term unconditional mean.[8]

$$\begin{aligned} r^I_{t+1} \equiv \log R^I_{t+1} &= (a_1 + b_1 ik_{t+1}) - (a_2 + b_2 ik_{t+1}) + \mathit{\Delta}\varepsilon_{t+1} \\ &= (a_1 - a_2) + (b_1 ik_{t+1} - b_2 ik_t) + \mathit{\Delta}\varepsilon_{t+1} \end{aligned} \tag{9}$$

where a_1, b_1 are linearization constants, $a_2 = \log\kappa$, $b_2 = \eta$.

This log-linearized investment return is the production counterpart of the return on wealth log-linearized with respect to the price-consumption ratio in Campbell and Shiller (1988), $r_{t+1} = \kappa_0 + \kappa_1 pc_{t+1} - pc_t + \mathit{\Delta}c_{t+1}$. In our model, the investment rate takes the central role as the price-to-consumption ratio or the wealth-to-consumption ratio does in the consumption framework. From the perspective of asset pricing, this expression says that the investment return from time t to $t+1$ is determined by the investment rate at time t and time $t+1$ and the profitability growth from time t to $t+1$. Alternatively,

8. In principle, the investment rate can be negative and therefore its logarithm is not well defined. The aggregate investment, however, is never negative in the data. As shown in Engsted, Pedersen, and Tanggaard (2012), the upper bound on mean errors of a first-order approximation is minimized by setting the point of linearization to the unconditional mean of variables linearized.

from the perspective of the theory of investment, $b_2 ik_t = -r^I_{t+1} + b_1 ik_{t+1} + \Delta\varepsilon_{t+1}$, this expression yields that the optimal investment is driven by expectations of the return, the investment rate, and the profitability growth in the next period. Echoing the point made in Gomes (2001), cash flows contributes to predicting investment even in the absence of financial constraints. In fact, the expectation of cash flow growth is an crucial component of investment decisions, and its empirical importance is left to be examined in later sections.

2.2 Case I: investment under constant uncertainty

In Case I, we illustrate the solution method and discuss the intuition of our model in a simple homoscedastic exogenous system. The solution method proceed as follows.

We assume that in this economy there exists a latent state variable that captures time-varying business conditions and drives both the profitability growth and the SDF. This latent state variable is specified to be an AR(1) process with constant conditional volatility.

$$s_{t+1} = \mu_s + \rho_s s_t + \sigma_s e_{s,t+1} \tag{10}$$

where $\mu_s \equiv (1-\rho_s)\bar{s}$ is a constant, $\bar{s}$ denotes the unconditional mean, and $e_{s,t+1} \overset{i.i.d}{\sim} N(0,1)$ is the state variable shock.

The log profitability growth is driven by the state variable and subject to a transitory profitability growth shock.[9]

$$\Delta\varepsilon_{t+1} = \mu_\varepsilon + s_t + \sigma_\varepsilon e_{\varepsilon,t+1} \tag{11}$$

where $e_{\varepsilon,t+1} \overset{i.i.d}{\sim} N(0,1)$ is the profitability growth shock and uncorrelated with $e_{s,t+1}$. The state variable can be interpreted as the time-varying expected profitability growth, and the state variable shock is the shock to the expected profitability growth. Shocks to this state variable can have non-trivial long-run effects on the profitability growth, upon assuming high persistence of the state variable as in Max Croce (2014).

The log SDF is also driven by the state variable s_t and subject to the profitability growth shock and the expected profitability growth shock with the price of risk λ^m_ε and λ^m_s, respectively.

$$m_{t+1} = -\mu_m - \rho^m_s s_t - \lambda^m_\varepsilon \sigma_\varepsilon e_{\varepsilon,t+1} - \lambda^m_s \sigma_s e_{s,t+1} \tag{12}$$

We conjecture a linear functional form for the policy function of log investment. That is, the producer

9. Notations: uppercase letters denote the level of variables while lowercase letters denote their logarithm counterparts. For example, the level of profitability is denoted by $\mathcal{E}_t$, and its log counterpart is ε_t.

observes the current realization of the state variable s_t and invests log-linearly.[10]

$$ik_t = \alpha + \beta s_t \tag{13}$$

where α is a constant and β is the elasticity of the investment rate to expected profitability.

Substituting the conjectured investment rate into the investment return, we have

$$r^I_{t+1} = [(a_1 - a_2) + (b_1 - b_2)\alpha + \mu_\varepsilon + b_1\beta\mu_s] + [(b_1\rho_s - b_2)\beta + 1]s_t + \sigma_\varepsilon e_{\varepsilon,t+1} + (b_1\beta)\sigma_s e_{s,t+1} \tag{14}$$

The investment return is risky in that it bears 1 unit of profitability growth risk and $b_1\beta$ units of expected profitability growth risk.

Finally, using the property that the Investment Euler equation holds at all times,

$$0 = \mathbb{E}_t[m_{t+1}] + \mathbb{E}_t[r^I_{t+1}] + \frac{1}{2}\mathbb{V}_t[m_{t+1}] + \frac{1}{2}\mathbb{V}_t[r^I_{t+1}] + COV_t(m_{t+1}, r^I_{t+1}) \tag{15}$$

we solve analytically the coefficients of the policy function of investment (14) by method of undetermined coefficients.

$$\beta = \frac{\rho^m_s - 1}{b_1\rho_s - b_2} \tag{16}$$

$$\alpha = \frac{\mu_m - \frac{1}{2}(\lambda^m_\varepsilon - 1)^2\sigma^2_\varepsilon - \frac{1}{2}(\lambda^m_s - b_1\beta)^2\sigma^2_s - (a_1 - a_2) - \mu_\varepsilon - b_1\beta\mu_s}{b_1 - b_2} \tag{17}$$

The solution shows the elasticity of investment to the expected profitability growth.[11] For a one-percent increase in the expected profitability growth, the investment rate increases by β percent. Exogenous changes in the expected profitability growth incentivize the producer to invest or divest through both the cash flow channel and the discount rate channel. A one-percent increase in the expected profitability growth implies, according to (10), a one-percent increase in the expected investment return before changing any investment, leaving value-enhancing investment opportunities for the producer. This one-percent increase in the expected profitability growth is also correlated with a $-\rho^m_s$ percent change in the inverse of the risk-free rate as in (13), which is the present value of one unit of goods in the next period, while the risk premium remains unchanged due to the constant conditional covariance between the investment return and the SDF. Suppose the producer invests by x percent, the expected investment return, which is the expected marginal benefit of investment less the marginal cost of investment, now increases in total by $[(b_1\rho_s - b_2)x + 1]$ percent according to (15). In equilibrium, the producer invests until the expected discounted net marginal value of investment diminishes to zero, $(b_1\rho_s - b_2)x + 1 - \rho^m_s = 0$,

10. For example, the producer can observe the current realization of the state variable s_t by observing the current one-period risk-free rate, which is the inverse of the conditional expectation of the SDF.

11. Given $\log \mathbb{E}_t[\Delta\mathcal{E}_{t+1}] = \mu_\varepsilon + \frac{1}{2}\sigma^2_\varepsilon + s_t$, rewrite (14) as $\log IK_t = (\alpha - \beta\mu_\varepsilon - \frac{1}{2}\beta\sigma^2_\varepsilon) + \beta \log \mathbb{E}_t[\Delta\mathcal{E}_{t+1}]$.

yielding the optimal investment coefficient $\beta = x = \frac{\rho_s^m - 1}{b_1 \rho_s - b_2}$.

2.3 Case I: recovery of SDF

The assumed SDF (13) is composed of one state variable and two exogenous shocks. The expected profitability growth, shocks to the expected profitability growth, and shocks to profitability growth can be recovered from data of the investment rate and the profitability growth.

$$s_t = (ik_t - \alpha)/\beta \tag{18}$$

$$\sigma_s e_{s,t+1} = s_{t+1} - \rho_s s_t - \mu_s \tag{19}$$

$$\sigma_\varepsilon e_{\varepsilon,t+1} = \Delta\varepsilon_{t+1} - s_t - \mu_\varepsilon \tag{20}$$

As a result, the SDF is recovered as

$$m_{t+1} = -\mu_m' - \rho_{ik}^m ik_t - \lambda_\varepsilon^m \Delta\varepsilon_{t+1} - \lambda_{ik}^m \Delta ik_{t+1} \tag{21}$$

where coefficients are given by

$$\mu_m' = \mu_m - [\rho_s^m - \lambda_\varepsilon^m + \lambda_s^m(1-\rho_s)]\alpha/\beta - \lambda_\varepsilon^m \mu_\varepsilon - \lambda_s^m \mu_s \tag{22}$$

$$\rho_{ik}^m = [\rho_s^m - \lambda_\varepsilon^m - \lambda_s^m(1-\rho_s)]/\beta \tag{23}$$

$$\lambda_{ik}^m = \lambda_s^m/\beta \tag{24}$$

The recovered SDF is driven by the investment rate and subject to the profitability growth and the investment rate growth as shocks, with prices of risks $\lambda_\varepsilon^m, \lambda_{ik}^m$, respectively. Our model goes beyond the return predictability of investment and profitability and delivers the prediction that the expected return of an asset is determined by the covariance between asset returns and investment rate growth and profitability growth. Although our goal is to generate a structural investment-implied SDF, this expression conveniently suggests an empirical two-factor representation of the SDF for excess returns with investment rate growth and profitability growth as factors.

2.4 Case I: asset prices

We are able to characterize a wide range of asset pricing variables in this economy with the solution of the optimal investment. The risk-free rate is the inverse of the conditional mean of SDF, $R_t^f = \mathbb{E}_t[M_{t+1}]^{-1}$, so the log risk-free rate in this case is given by

$$r_t^f = -\mathbb{E}_t[m_{t+1}] - \frac{1}{2}\mathbb{V}_t[m_{t+1}] = \mu_m - \frac{1}{2}(\lambda_\varepsilon^m)^2\sigma_\varepsilon^2 - \frac{1}{2}(\lambda_s^m)^2\sigma_s^2 + \rho_\varepsilon^m s_t \tag{25}$$

where ρ_s^m is the elasticity of the risk-free rate to expected profitability growth.

Rearranging the investment Euler equation (16) and using the expression of the risk-free rate (26), we have the conditional log equity premium equal the covariance between the log SDF and the log investment return.[12]

$$\mathbb{E}_t[r_{t+1}^I] - r_t^f + \frac{1}{2}\mathbb{V}_t[r_{t+1}^I] = -COV_t(m_{t+1}, r_{t+1}^I) = \lambda_\varepsilon^m \sigma_\varepsilon^2 + \lambda_s^m (b_1\beta)\sigma_s^2 \tag{26}$$

Holding the investment return is compensated by $\lambda_\varepsilon^m \sigma_\varepsilon^2$ for bearing 1 unit of profitability growth risk and $\lambda_s^m(b_1\beta)\sigma_s^2$ for bearing $b_1\beta$ unit of expected profitability growth risk.

We can characterize prices and yields of risk-less bonds in our model in the same way as in Vasicek (1977) except that our specification designates the expected profitability growth as the latent factor. Denote B_t^n as the time-t price of a risk-less bond that pays the one unit of goods in n periods. The price of a n-period bond is given by $B_t^n = \mathbb{E}_t[M_{t+1}...M_{t+n}]$. The yield-to-maturity of a n-period bond is $Y_t^n = (1/B_t^n)^{1/n}$, or $y_t^n = -b_t^n/n$. Using the recursive relation of bond prices $B_t^n = \mathbb{E}_t[M_{t+1}B_{t+1}^{n-1}]$, we can compute prices of n-period bonds recursively as functions of the state variable. In our model, the log price of a n-period bond is affine in the expected log profitability growth.

$$b_t^n = A_n + B_n s_t \tag{27}$$

where B_n is the elasticity of the price of a n-period bond to the expected profitability growth and coefficients are recursively computed.

$$A_{n+1} = A_n + B_n\mu_s - \mu_m + \frac{1}{2}(\lambda_\varepsilon^m)^2\sigma_\varepsilon^2 + \frac{1}{2}(\lambda_s^m - B_n)^2\sigma_s^2 \tag{28}$$

$$B_{n+1} = B_n\rho_s - \rho_s^m = \frac{\rho_s^m}{1-\rho_s}(\rho_s^{n+1} - 1) \tag{29}$$

where $A_1 = -\mu_m + \frac{1}{2}(\lambda_\varepsilon^m)^2\sigma_\varepsilon^2 + \frac{1}{2}(\lambda_s^m)^2\sigma_s^2$, $B_1 = -\rho_s^m$.

The one-period holding return on a n-period bond is defined by $R_{t+1}^{bn} = B_{t+1}^{n-1}/B_t^n$, and the log expected excess return is given by

$$\mathbb{E}_t[r_{t+1}^{bn}] - r_t^f + \frac{1}{2}\mathbb{V}_t[r_{t+1}^{bn}] = -COV_t[m_{t+1}, b_{t+1}^{n-1}] = \lambda_s^m B_{n-1}\sigma_s^2 \tag{30}$$

Holding the n-period bond for one period bears B_{n-1} unit of expected profitability growth risk. When the price of expected profitability growth risk is positive, $\lambda_s^m > 0$, a positive shock to expected profitability growth drives down simultaneously the SDF and the bond price as $B_n < 0$ given $\rho_s^m > 0$. In this case, bonds are hedges because bond prices are high during bad states when the SDF is low. The opposite

12. The expected geometric excess return is given by $\mathbb{E}_t[R_{t+1}^e] \equiv \mathbb{E}_t[R_{t+1}^I]/R_t^f = \exp(\mathbb{E}_t[r_{t+1}^I] + \frac{1}{2}\mathbb{V}_t[r_{t+1}^I] - r_t^f)$

interpretation that holding bonds is risky holds when $\lambda_\varepsilon^m < 0$ and $\rho_s^m < 0$.

We can also characterize prices of dividend claims and the term structure of equity premia. Denote D_t^n as the time-t price of a claim to the dividend paid in n periods D_{t+n}. The price of a n-period dividend claim is given by $D_t^n = \mathbb{E}_t[M_{t+1}...M_{t+n}D_{t+n}]$ and follows the recursive relation $D_t^n = \mathbb{E}_t[M_{t+1}D_{t+1}^{n-1}]$. For example, when $n = 0$, the 0-period dividend claim is a claim to the current dividend, $D_t^0 = D_t$. When $n = 1$, the price of a 1-period dividend claim is the expected discounted value of the next-period dividend, $D_t^1 = \mathbb{E}_t[M_{t+1}D_{t+1}]$. It can be explicitly solved in our model as follows, $\frac{D_t^1}{P_t} = \mathbb{E}_t\left[M_{t+1}\frac{D_{t+1}}{P_{t+1}}\frac{P_{t+1}}{P_t}\right]$. We can then compute prices of n-period dividend claims recursively using the relation, $\frac{D_t^n}{P_t} = \mathbb{E}_t\left[M_{t+1}\frac{D_{t+1}^{n-1}}{P_{t+1}}\frac{P_{t+1}}{P_t}\right]$.

The log price of a n-period dividend claim scaled by the ex-dividend value is given by

$$d_t^n - p_t = A'_n + B'_n s_t \tag{31}$$

where B'_n is the elasticity of the price of a n-period dividend claim scaled by the ex-dividend value to the expected profitability growth and coefficient are recursively computed.

$$\begin{aligned} A'_{n+1} &= A'_n + a_4 + b_4\alpha + \mu_\varepsilon + [B'_n + (b_4+\eta)\beta]\mu_s - \mu_m \\ &\quad + \frac{1}{2}(\lambda_\varepsilon^m - 1)^2\sigma_\varepsilon^2 + \frac{1}{2}[\lambda_s^m - (B'_n + (b_4+\eta)\beta)]^2\sigma_s^2 \end{aligned} \tag{32}$$

$$B'_{n+1} = [B'_n + (b_4+\eta)\beta]\rho_s - \eta\beta + 1 - \rho_s^m \tag{33}$$

where $A'_1 = a_6 + b_6\alpha + \mu_\varepsilon + (b_6+\eta)\beta\rho_s - \mu_m + \frac{1}{2}(\lambda_\varepsilon^m - 1)^2\sigma_\varepsilon^2 + \frac{1}{2}[\lambda_s^m - (b_6+\eta)\beta]^2\sigma_s^2$ and $B'_1 = [(b_6+\eta)\rho_s - \eta]\beta + 1 - \rho_s^m$.

The one-period holding return on a n-period dividend claim is defined by $R_{t+1}^{dn} = D_{t+1}^{n-1}/D_t^n$, and the log expected excess return is given by

$$\mathbb{E}_t[r_{t+1}^{dn}] - r_t^f + \frac{1}{2}\mathbb{V}_t[r_{t+1}^{dn}] = -COV_t(m_{t+1}, d_{t+1}^{n-1} - p_{t+1}) = \lambda_\varepsilon^m\sigma_\varepsilon^2 + \lambda_s^m[B'_{n-1} + (b_4+\eta)\beta]\sigma_s^2 \tag{34}$$

Holding a n-period dividend claim for one period bears 1 unit of profitability growth risk and $[B'_{n-1} + (b_4+\eta)\beta]$ unit of expected profitability growth risk with price of risk λ_ε^m and λ_s^m.

The Case I fails to capture several salient features of investment and asset returns well documented in the literature. First, many papers document that investment is sensitive to volatility. Bloom (2009) shows that positive shocks to equity volatility predicts lower investment. The optimal investment in Case I, however, is only responding to the current state of the economy and not affected by uncertainty over future states. A related and undesirable consequence is that investment does not predict excess returns, at odds with data. Second, asset pricing models predict that the real risk-free rate is lower when the uncertainty of growth is higher due to precautionary motives. Hartzmark (2016) also empirically documents a negative relationship between the interest rate and macroeconomic uncertainty. Similar to the investment rate, the risk-free rate in Case I is neither affected by uncertainty. Third, a large strand of empirical asset pricing literature finds that the discount rate is varying substantially over time for both equity and treasuries. Cochrane (2011) surveys this literature and places the time-varying discount rate at the heart of modern asset pricing theory. Fourth, while there is controversy over the sign of the average slope of the term structure of equity premia, Binsbergen et al. (2013) and Bansal et al. (2021) both find empirically that the slope is time-varying and pro-cyclical.

2.5 Case II: investment under time-varying uncertainty

In order to capture important features overlooked in Case I, we introduce in Case II a time-varying uncertainty. For parsimony, the profitability growth and the expected profitability growth share the same stochastic conditional volatility process, which is also AR(1).

$$\Delta\varepsilon_{t+1} = \mu_\varepsilon + s_t + \sigma_t e_{\varepsilon,t+1} \tag{35}$$

$$s_{t+1} = \mu_s + \rho_s s_t + \varphi_s \sigma_t e_{s,t+1} \tag{36}$$

$$\sigma^2_{t+1} = \mu_\sigma + \rho_\sigma \sigma^2_t + \sigma_\sigma e_{\sigma,t+1} \tag{37}$$

where $\mu_s \equiv (1-\rho_s)\bar{s}$ is a constant, $\bar{s}$ denotes the unconditional mean, $\mu_\sigma \equiv (1-\rho_\sigma)\bar{\sigma}^2$, $\bar{\sigma}^2$ is the unconditional mean of the conditional variance, and $e_{\varepsilon,t+1}, e_{s,t+1}, e_{\sigma,t+1} \overset{i.i.d}{\sim} N(0,1)$ and are orthogonal to each other.

Accordingly, the SDF is assumed to be driven by two exogenous state variables, the expected profitability growth and the uncertainty of profitability growth, and subject to three sources of shocks, shocks to profitability growth, shocks to expected profitability growth and shocks to uncertainty, with prices of risks λ^m_ε, λ^m_s and λ^m_σ, respectively.

$$m_{t+1} = -\mu_m - \rho^m_s s_t - \rho^m_\sigma \sigma^2_t - \lambda^m_\varepsilon \sigma_t e_{\varepsilon,t+1} - \lambda^m_s \sigma_t e_{s,t+1} - \lambda^m_\sigma \sigma_\sigma e_{\sigma,t+1} \tag{38}$$

The optimal investment is conjectured to be linear in both the expected profitability growth and the profitability growth uncertainty. That is, the producer observes the expected profitability growth s_t and the conditional variance of profitability growth σ^2_t and invests log-linearly.

$$ik_t = \alpha + \beta s_t + \phi \sigma^2_t \tag{39}$$

where α is a constant, β is the elasticity of the investment rate to profitability, and ϕ is the elasticity of the investment rate to uncertainty (conditional volatility) of future profitability. Substituting the conjectured investment rate into the investment return, we have

$$r^I_{t+1} = [(a_1 - a_2) + (b_1 - b_2)\alpha + \mu_\varepsilon + b_1\beta\mu_s + b_1\phi\mu_\sigma] + [(b_1\rho_s - b_2)\beta + 1]s_t + [(b_1\rho_\sigma - b_2)\phi]\sigma_t^2$$
$$+ \sigma_t e_{\varepsilon,t+1} + (b_1\beta\phi_s)\sigma_t e_{s,t+1} + (b_1\phi)\sigma_\sigma e_{\sigma,t+1} \tag{40}$$

Again using (16) we solve for coefficients of the optimal investment rate.

$$\beta = \frac{\rho_s^m - 1}{b_1\rho_s - b_2} \tag{41}$$
$$\phi = \frac{\rho_\sigma^m - \frac{1}{2}(\lambda_\varepsilon^m - 1)^2 - \frac{1}{2}(\lambda_s^m - b_1\beta\varphi_s)^2}{b_1\rho_\sigma - b_2} \tag{42}$$
$$\alpha = \frac{\mu_m - \frac{1}{2}(\lambda_\sigma^m - b_1\phi)^2\sigma_\sigma^2 - (a_1 - a_2) - \mu_\varepsilon - b_1\beta\mu_s - b_1\phi\mu_\sigma}{b_1 - b_2} \tag{43}$$

The interpretation of β remains the same as in Case I. We here focus on the interpretation of ϕ. For a one-percent increase in the exponential uncertainty $\exp(\sigma_t^2)$, the investment rate increases by ϕ percent. Alternatively, holding the expected log profitability growth constant, for a one-percent increase in the expected profitability growth, the investment rate increases by 2ϕ percent.[13] Exogenous changes in the uncertainty of the profitability growth incentivize the producer to invest or divest through both the cash flow channel and the discount rate channel. A one-percent increase in the exponential uncertainty $\exp(\sigma_t^2)$ is associated with a $-\rho_\sigma^m + \frac{1}{2}(\lambda_\varepsilon^m)^2 + \frac{1}{2}(\lambda_s^m)^2$ percent change in the inverse of the risk-free rate and a $[\lambda_\varepsilon^m + \lambda_s^m(b_1\beta\varphi_s)]$ percent change in the risk premium before changing any investment. Suppose the producer invests by y percent to counterbalance effects of the change in the uncertainty. This y-percent change in investment results in $[(b_1\rho_\sigma - b_2)y + \frac{1}{2} + \frac{1}{2}(b_1\beta\varphi_s)^2]$ percent. In equilibrium, the expected discounted net marginal value of investment equals zero, $(b_1\rho_\sigma - b_2)y - \rho_\sigma^m + \frac{1}{2} + \frac{1}{2}(b_1\beta\varphi_s)^2 - \lambda_\varepsilon^m - \lambda_s^m(b_1\beta\varphi_s) + \frac{1}{2}(\lambda_\varepsilon^m)^2 + \frac{1}{2}(\lambda_s^m)^2 = 0$, yielding the solution of $\phi = y = \frac{\rho_\sigma^m - \frac{1}{2}(\lambda_\varepsilon^m - 1)^2 - \frac{1}{2}(\lambda_s^m - b_1\beta\varphi_s)^2}{b_1\rho_\sigma - b_2}$.

2.6 Recovery of SDF

The SDF is composed of two state variables and three exogenous shocks. The expected profitability growth and the uncertainty of the profitability growth and their shocks can be recovered from data of the risk-free rate and the investment rate.

$$\begin{bmatrix} r_t^f \\ ik_t \end{bmatrix} = \begin{bmatrix} a \\ \alpha \end{bmatrix} + \begin{bmatrix} b & c \\ \beta & \phi \end{bmatrix} \begin{bmatrix} s_t \\ \sigma_t^2 \end{bmatrix}$$

where, for convenience, $a \equiv \mu_m - \frac{1}{2}(\lambda_\sigma^m)^2\sigma_\sigma^2,\ b \equiv \rho_s^m,\ c \equiv \rho_\sigma^m - \frac{1}{2}(\lambda_\varepsilon^m)^2 - \frac{1}{2}(\lambda_s^m)^2$.

13. We can rewrite the log investment rate as $ik_t = (\alpha - \beta\mu_\varepsilon) + (\beta - 2\phi)\mathbb{E}_t[\log \Delta\mathcal{E}_{t+1}] + 2\phi \log \mathbb{E}_t[\Delta\mathcal{E}_{t+1}]$, where $\mathbb{E}_t[\log \Delta\mathcal{E}_{t+1}] = \mu_\varepsilon + s_t$ and $\log \mathbb{E}_t[\Delta\mathcal{E}_{t+1}] = \mu_\varepsilon + s_t + \frac{1}{2}\sigma_t^2$.

The solution is given by

$$\begin{bmatrix} s_t \\ \sigma_t^2 \end{bmatrix} = \begin{bmatrix} b & c \\ \beta & \phi \end{bmatrix}^{-1} \begin{bmatrix} r_t^f - a \\ ik_t - \alpha \end{bmatrix} = \frac{1}{b\phi - \beta c} \begin{bmatrix} \phi & -c \\ -\beta & b \end{bmatrix} \begin{bmatrix} r_t^f - a \\ ik_t - \alpha \end{bmatrix}$$

In short-hand notations, state variables are recovered as

$$s_t = \eta_{s,0} + \eta_{s,1} r_t^f + \eta_{s,2} ik_t \tag{44}$$

$$\sigma_t^2 = \eta_{\sigma,0} + \eta_{\sigma,1} r_t^f + \eta_{\sigma,2} ik_t \tag{45}$$

Their shocks can be recovered as

$$\varphi_s \sigma_t e_{s,t+1} = (1-\rho_s)(\eta_{s,0} - \bar{s}) + \eta_{s,1}(r_{t+1}^f - \rho_s r_t^f) + \eta_{s,2}(ik_{t+1} - \rho_s ik_t) \tag{46}$$

$$\sigma_\sigma e_{\sigma,t+1} = (1-\rho_\sigma)(\eta_{\sigma,0} - \bar{\sigma}^2) + \eta_{\sigma,1}(r_{t+1}^f - \rho_\sigma r_t^f) + \eta_{\sigma,2}(ik_{t+1} - \rho_\sigma ik_t) \tag{47}$$

The remaining shock to the profitability growth can be recovered from profitability growth.

$$\sigma_t e_{\varepsilon,t+1} = \Delta\varepsilon_{t+1} - s_t - \mu_\varepsilon \tag{48}$$

The SDF therefore can be recovered as

$$m_{t+1} = -\mu_m' - \rho_{r_f}^m r_t^f - \rho_{ik}^m ik_t - \lambda_{r_f}^m \Delta r_{t+1}^f - \lambda_{ik}^m \Delta ik_{t+1} - \lambda_\varepsilon^m \Delta\varepsilon_{t+1} \tag{49}$$

where coefficients are given by

$$\begin{aligned} \mu_m' = \mu_m - \lambda_\varepsilon^m \mu_\varepsilon + (\rho_s^m - \lambda_\varepsilon^m)\eta_{s,0} + \rho_\sigma^m \eta_{\sigma,0} \\ + \lambda_s^m \frac{1}{\varphi_s}(1-\rho_s)(\eta_{s,0} - \bar{s}) + \lambda_\sigma^m (1-\rho_\sigma)(\eta_{\sigma,0} - \bar{\sigma}^2) \end{aligned} \tag{50}$$

$$\rho_{r_f}^m = (\rho_s^m - \lambda_\varepsilon^m)\eta_{s,1} + \rho_\sigma^m \eta_{\sigma,1} + \lambda_s^m \frac{1}{\varphi_s}\eta_{s,1}(1-\rho_s) + \lambda_\sigma^m \eta_{\sigma,1}(1-\rho_\sigma) \tag{51}$$

$$\rho_{ik}^m = (\rho_s^m - \lambda_\varepsilon^m)\eta_{s,2} + \rho_\sigma^m \eta_{\sigma,2} + \lambda_s^m \frac{1}{\varphi_s}\eta_{s,2}(1-\rho_s) + \lambda_\sigma^m \eta_{\sigma,2}(1-\rho_\sigma) \tag{52}$$

$$\lambda_{r_f}^m = \lambda_s^m \frac{1}{\varphi_s}\eta_{s,1} + \lambda_\sigma^m \eta_{\sigma,1} \tag{53}$$

$$\lambda_{ik}^m = \lambda_s^m \frac{1}{\varphi_s}\eta_{s,2} + \lambda_\sigma^m \eta_{\sigma,2} \tag{54}$$

The recovered SDF is driven by the risk-free rate and the investment rate and subject to three sources of shocks, i.e., the risk-free rate growth, the investment rate growth, and the profitability growth, with prices of risks $\lambda_{r_f}^m, \lambda_{ik}^m, \lambda_\varepsilon^m$, respectively. This expression suggests an empirical three-factor representation of the SDF for excess returns with investment rate growth, profitability growth, and real rate growth as factors.

2.7 Case II: asset prices

We can compute asset pricing variables again in Case II. The risk-free rate is given by

$$r_t^f = [\mu_m - \frac{1}{2}(\lambda_\sigma^m)^2\sigma_\sigma^2] + \rho_s^m s_t + [\rho_\sigma^m - \frac{1}{2}(\lambda_\varepsilon^m)^2 - \frac{1}{2}(\lambda_s^m)^2]\sigma_t^2 \tag{55}$$

The risk premium is given by

$$\mathbb{E}_t[r_{t+1}^I] - r_t^f + \frac{1}{2}\mathbb{V}_t[r_{t+1}^I] = \lambda_\varepsilon^m\sigma_t^2 + \lambda_s^m(b_1\beta\varphi_s)\sigma_t^2 + \lambda_\sigma^m(b_1\phi)\sigma_\sigma^2 \tag{56}$$

Prices of n-period bonds are given by

$$b_t^n = A_n + B_n s_t + C_n\sigma_t^2 \tag{57}$$

where coefficients are given by

$$A_{n+1} = A_n + B_n\mu_s + C_n\mu_\sigma - \mu_m + \frac{1}{2}(\lambda_\sigma^m - C_n)^2\sigma_\sigma^2 \tag{58}$$

$$B_{n+1} = B_n\rho_s - \rho_s^m = \frac{\rho_s^m}{1-\rho_s}(\rho_s^{n+1} - 1) \tag{59}$$

$$C_{n+1} = C_n\rho_\sigma - \rho_\sigma^m + \frac{1}{2}(\lambda_\varepsilon^m)^2 + \frac{1}{2}(\lambda_s^m - B_n\varphi_s)^2 \tag{60}$$

where $A_1 = [-\mu_m + \frac{1}{2}(\lambda_\sigma^m)^2\sigma_\sigma^2]$, $B_1 = -\rho_s^m$, $C_1 = -\rho_\sigma^m + \frac{1}{2}(\lambda_\varepsilon^m)^2 + \frac{1}{2}(\lambda_s^m)^2$.

Risk premia on n-period bonds are given by

$$\mathbb{E}_t[r_{t+1}^{bn}] - r_t^f + \frac{1}{2}\mathbb{V}_t[r_{t+1}^{bn}] = \lambda_s^m B_{n-1}\varphi_s\sigma_t^2 + \lambda_\sigma^m C_{n-1}\sigma_\sigma^2 \tag{61}$$

Prices of n-period dividend claims are given by

$$d_t^n - p_t = A_n' + B_n' s_t + C_n'\sigma_t^2 \tag{62}$$

where coefficient are given by

$$A'_{n+1} = A'_n + a_4 + b_4\alpha + \mu_\varepsilon + [B'_n + (b_4+\eta)\beta]\mu_s + [C'_n + (b_4+\eta)\phi]\mu_\sigma$$
$$- \mu_m + \frac{1}{2}[\lambda^m_\sigma - (C'_n + (b_4+\eta)\phi)]^2\sigma^2_\sigma \tag{63}$$

$$B'_{n+1} = [B'_n + (b_4+\eta)\beta]\rho_s - \eta\beta + 1 - \rho^m_s \tag{64}$$

$$C'_{n+1} = [C'_n + (b_4+\eta)\phi]\rho_\sigma - \rho^m_\sigma + \frac{1}{2}(\lambda^m_\varepsilon - 1)^2 + \frac{1}{2}(\lambda^m_s - (B_n + (b_4+\eta)\beta)\varphi_s)^2 \tag{65}$$

where $A'_1 = a_6 + b_6\alpha + \mu_\varepsilon + (b_6+\eta)\beta\mu_s + (b_6+\eta)\phi\mu_\sigma - \mu_m + \frac{1}{2}[\lambda^m_\sigma - (b_6+\eta)\phi]^2\sigma^2_\sigma$, $B'_1 = (b_6+\eta)\beta\rho_s - \eta\beta + 1 - \rho^m_s$, $C'_1 = (b_6+\eta)\phi\rho_\sigma - \rho^m_\sigma + \frac{1}{2}(\lambda^m_\varepsilon - 1)^2 + \frac{1}{2}[\lambda^m_s - (b_6+\eta)\beta\varphi_s]^2$.

Risk premia on n-period dividend claims are given by

$$\mathbb{E}_t[r^{dn}_{t+1}] - r^f_t + \frac{1}{2}\mathbb{V}_t[r^{dn}_{t+1}] = \lambda^m_\varepsilon\sigma^2_t + \lambda^m_s[B'_{n-1} + (b_4+\eta)\beta]\varphi_s\sigma^2_t + \lambda^m_\sigma[C'_{n-1} + (b_4+\eta)\phi]\sigma^2_\sigma \tag{66}$$

3 Estimation

3.1 Data

We estimate parameters in Case II of the model using quarterly data. The estimation requires data of investment rates, investment returns, risk-free rates, profitability growth rates, and portfolio returns.

We retrieve investment data from Bureau of Economic Research (BEA) and measure the real aggregate physical capital investment (I_t) as private fixed nonresidential investment (PFNI) (NIPA Table 1.1.5, line 9), deflated by the corresponding price index (NIPA Table 1.1.4, line 9), available from 1947:Q1 to 2021:Q4. Following Cochrane (1991), investment rates (IK_t) are constructed using the perpetual inventory method, as quarterly data of the private fixed nonresidential capital stock are unavailable. Specifically, we compute investment rates recursively, using the following equation implied by the law of motion of physical capital (3).

$$IK_t = \frac{I_t}{I_{t-1}}\frac{IK_{t-1}}{(1-\delta) + IK_{t-1}} \tag{67}$$

We set the first investment rate (1947:Q1) to the "steady-state" value $IK^* \equiv \mathbb{E}(\Delta I) - (1-\delta)$, which is defined by the fixed point of (68) with the investment growth set to its mean value and is an accurate approximation to the mean investment rate. We then use (68) to recursively construct the full time series of investment rates. We set the quarterly depreciation rate to $\delta = 2.6\%$, equivalent to a 10% annual depreciation rate.

We assume that the CRSP value-weighted return (VWRET) retrieved from CRSP Stock Market Indexes files and multiplied by the target equity-to-asset ratio proxies the investment return (R^I_t) on the claim to private fixed nonresidential capital stock. Following Barro (2006), we set the target equity-to-asset ratio to 2/3.[14]

We retrieve Treasury returns from CRSP US Treasury and Inflation Indexes Files. We take returns of 3-month Treasury bills (T90RET) as nominal one-period risk-free rates. We retrieve the seasonally adjusted consumer price index (CPI) from Bureau of Labor Statistics (BLS). Quarterly inflation are calculated as the log growth of the CPI in the final month in the current quarter over the final month in the previous quarter. Ex-ante real risk-free rates are fitted values from regressing ex-post real risk-free rates on nominal risk-free rates and lagged inflation in past four quarters.

The log profitability growth is given by the weighted average of log TFP growth and log wage growth, $\Delta\varepsilon_{t+1} \equiv \frac{1}{\alpha}\Delta a_{t+1} + \frac{\alpha-1}{\alpha}\Delta w_{t+1}$, according to (2). We retrieve utilization-adjusted quarterly TFP growth (Δa_{t+1}) and capital share (α) from John Fernald's website.We retrieve seasonally adjusted average hourly earnings of production and nonsupervisory employees on private nonfarm payrolls, starting from 1964 and available at monthly frequency, from BLS (Employment Situation Table B-8).[15] Quarterly real wage growth (Δw_{t+1}) is calculated as the inflation-adjusted log growth of hourly earnings in the final month in the current quarter over the final month in the previous quarter. We use time-varying capital share to calculate the log profitability growth, although using the sample average capital share $\alpha = 1/3$ barely affects any results.

We retrieve portfolio returns from Kenneth French's website for both in-sample estimation and out-of-sample testing.

To align with the sample period of wage growth, our final sample starts from 1964:Q2 to 2021:Q4. In matching investment rates with stock returns, we follow the "beginning-of-period" convention in Campbell (2003). Specifically, the asset return in quarter t is contemporaneous to the investment growth as well as the TFP growth from quarter t to quarter $t+1$.

3.2 Estimation strategy

We choose the parameter vector Θ that minimizes weighted mean squared errors between the model-implied moments $X(\Theta)$ and actual moments of data X.

$$\hat{\Theta} = \arg\min_{\Theta} \big(X - X(\Theta)\big)' W \big(X - X(\Theta)\big)$$

We have 16 primitive parameters in total, listed in Table 1. We set quarterly depreciation rate to $\delta = 2.6\%$ as above. We calibrate the rest of 15 parameters. While we have 16 parameters, we have even more moments in order to over-identify the system. Target unconditional moments are listed in Table 2

14. This is equivalent to a debt-tot-equity ratio $D/E = 1/2$ and implies a mean quarterly asset excess return of 0.86% for a mean quarterly equity excess return of 1.29% .

15. BLS provides monthly data of average hourly earnings of all employees on private nonfarm payrolls (Employment Situation Table B-3) starting only from 2006. Neverthelss, monthly data of average hourly earnings of production and nonsupervisory employees on private nonfarm payrolls have growth rates almost identical to and highly correlated (0.94) with those of the former time series over 2006 to 2021. A Welch two-sample t-test of sample means yield p-value of 0.6, suggesting that the difference between two sample means is statistically indistingushiable.

and derived in the Appendix, including (1) the mean, variance, and autocorrelation of the risk-free rate and the investment rate, (2) the mean and variance of market excess returns, (3) correlations between the investment rate and the risk-free rate and the excess return, (4) the mean and variance of profitability growth, and (5) mean squared pricing errors of 10 size-sorted portfolios. For the last set of moments, we do not have analytical predictions for portfolio returns sorted on firm characteristics or other variables since we do not model the cross section of firms explicitly. We proceed as follows. For each set of parameter values during minimization, we recover the SDF using data as in (50) and calculate the mean squared pricing errors of in-sample test portfolios. We set its empirical counterpart to zero, corresponding to a perfect fit. We use 10 size-sorted portfolios as in-sample test assets because Cochrane (1996) has shown that these portfolios can well be priced by a SDF with residential and nonresidential sectoral investment returns as factors. Our choices of out-of-sample test assets are 6 size-book-to-market-sorted portfolios, 6-size-investment-sorted portfolios, and 6-size-profitability-sorted portfolios.

3.3 Estimation results

Estimated parameters are presented in Table 1. The expected profitability growth, has a quarterly autocorrelation coefficient of 0.88, equivalent to an annual autocorrelation coefficient of 0.60. Max Croce (2014) estimates the persistence of the long-run component in the profitability growth to be between 0.66 and 0.99 using annual data, and sets it to be 0.80 in his calibration. Our estimate of the persistence of the long-run component is close to the lower bound of those empirical estimates. This long-run component is also small given that the conditional correlation between the expected profitability growth and the profitability growth is $\varphi_s = 0.29$.

Our conditional variance is neither highly persistent. The autocorrelation coefficient of the conditional variance is estimated to be 0.63. Its mean and standard deviation are 0.0033 and 0.0001, implying a rather smooth process. This smoothness is required implicitly to have non-negative values in the recovered time series of conditional variance.

Prices of risk for profitability growth shock and expected profitability growth shock are both positive, and they are 1.53 and 15.85, respectively. Positive prices of risk imply that a positive profitability growth shock or a positive expected profitability growth shock drives down the SDF corresponding to good states of the economy. The uncertainty shock, on contrary, carries a negative price of risk of -2321, implying that a positive uncertainty shock drives up the SDF corresponding to bad states of the economy.

The total investment cost is estimated to have curvature of $\eta = 0.52$ and scalar $\kappa = 2.02$, implying that on average the investment cost is about 0.9% of profits.

The optimal investment rate is characterized by elasticities of investment to expected profitability growth and conditional variance of profitability growth. The elasticity of investment to expected profitability growth β is estimated to be 2.3 and the elasticity of investment to profitability growth uncertainty ϕ is -132. If the expected log profitability growth increases by one standard deviation, the log investment rate increases by 0.08, equivalent to a 8.3% increase in the level of the investment rate. If the profitability growth uncertainty increases by one standard deviation, the log investment rate decreases by -0.0095, equivalent to a 0.95% decrease in the level of the investment rate.

The model fit is presented in Panel A of Table 2. The calibrated model generates a risk-free rate with a mean of 0.5% and a standard deviation of 0.57%, identical to those in the data. The autocorrelation of the risk-free rate is 0.69 in the model against 0.68 in the data. The investment rate implied by the model has a mean of -3.29 and a standard deviation of 8.39%, identical to -3.29 and 8.39% in the data. The autocorrelation coefficient of the investment rate is 0.88 in the model and 0.97 that in the data. The model-implied asset excess return has a mean 0f 1.28% almost identical to 1.29% in the data, and has a standard deviation of 8.62% against 8.61% in the data. The investment rate has a positive correlation of 0.35 with the real risk-free rate, almost identical to 0.34 in the data, and has a slightly negative correlation of -0.0006 with the excess return, in contrast to -0.18 in the data. The Panel B of Table 2 presents the model-implied average returns of 10 size-sorted portfolios. Our predicted average returns closely match the decreasing pattern of realized average returns from small to big portfolios in the data with a mean absolute pricing errors of only 0.059%.

4 Recovery of state variables and SDF

The ultimate goal of our paper is to recover the SDF. To that end, we first recover underlying state variables from investment rates and real rates. Using (45) and (46), we obtain realizations of the expected profitability growth s_t and the conditional variance of profitability growth σ_t^2. Figure 1 shows the real risk-free rate and the real investment rate in the top panel and recovered state variables in the bottom panel. The recovered s_t exhibits an long-term downward trend, implying the expected profitability growth of the economy has been declining. The recovered σ_t^2 exhibits an downward trend since 1980s with the notable exception of a spike among the great recession. This result echoes findings of the reduction in volatility of aggregate variables documented in Fernandez-Villaverde and Rubio-Ramirez (2006) and Justiniano and Primiceri (2008).

Correspondingly, the model-implied equity premium in Figure 2 is also trending down. This result is consistent with the notion of declining equity premium in the late 20th century proposed by Blanchard (1993), Jagannathan, McGrattan, and Scherbina (2001), and Fama and French (2002). Lettau, Ludvigson, and Wachter (2008) attributes the decline in equity premium to the reduction in macroeconomic volatility, especially consumption growth volatility. Their estimated volatility also exhibits a significant downward trend since 1980s'. In our model, the decline in equity premium to the reduction in the conditional volatility of productivity growth.

Figure 3 plots the recovered SDF. The SDF peaks during all major recessions, most significantly during the recent COVID-19 pandemic in 2020, followed by the Great Recession in 2008. Figure 4 shows time series of unexpected innovations in SDF and three components in terms of both primitive shocks and observable shocks. We also provide a variance decomposition of unexpected innovations in SDF in Table 3. In terms of primitive shocks, shocks to expected profitability growth account for 106.7% of total variations, highlighting the role of the long-run component in driving the SDF. In terms of observable shocks, innovations in investment rates account for 98.9% of total variations. Other variances and covariances are trivial in both cases. This result suggests that a SDF with shocks to expected profitability growth or innovations in investment rates as the single factor should be able to proxy for the complete recovered SDF.

The recovered SDF prices 10 size-sorted portfolios in the sample with a mean absolute pricing error of only 0.06% as shown in Panel B of Table 2 and Figure 5. The recovered SDF also prices out-of-sample portfolios reasonably well. Table 4 and Figure 6 show the realized and predicted average returns of 6 size-book-to-market-sorted portfolios, 6-size-investment-sorted portfolios, 6-size-profitability-sorted portfolios, and all 18 of them. Mean absolute pricing errors are 0.36%, 0.41%, 0.39%, and 0.39%, respectively. As illustrated above, we should also expect that a SDF with shocks to expected profitability growth or innovations in investment rates as the single factor should be able to price the cross section with a very close performance to that of the complete recovered SDF. Figure 7 and 8 show that this is indeed the case. The covariance between unexpected innovations in the log SDF and log portfolio returns are decomposed into covariances between each component and log portfolios returns. As expected, shocks to expected profitability growth or innovations in investment rates accomplish almost all the work in pricing the cross section of portfolio returns. Other components are trivial in contrast.

The recovered SDF has implications for term structures of both interest rates and equity premia, although we do not attempt to match both term structures in this paper, nor do we include their moments in our estimation. Figure 9 plots the model-implied yield curve and time variations in yields. The recovered SDF implies an slightly downward sloping yield curve. Figure 10 plots the model-implied term structure of bond risk premia and time variations in term premia. Similarly term premia of risk-free bonds are downward sloping across maturities. Figure 11 plots the model-implied term structure of equity risk premia and time variations in risk premia. The term structure of equity premia is steeply upward sloping.

5 Conclusion

This paper proposes a new systematic approach to recovering the stochastic discount factor from firms' investment decisions. Our approach builds on the simple intuition that investment is forward looking.

More specifically, investment decisions are made on the joint conditional distribution of the profitability growth and the SDF. We approach the investment Euler equation from the perspective of intertemporal optimization to solve for the optimal investment and from the perspective of asset pricing to link asset prices to investment and profitability dynamics. We finally recover the the SDF using estimated parameters and data on investment rates, risk-free rates, and profitability growth rates. Innovations in the recovered SDF are driven dominantly by innovations in investment rates and marginally by innovations in risk-free rates and profitability growth rates. Our model goes beyond the return predictability of investment and profitability and delivers the prediction that the expected return of an asset is determined by the covariance between asset returns and two factors, innovations in investment rates and innovations in profitability. The recovered SDF exhibits strong counter-cyclicality with large jumps in recessions. While our model is estimated to match moments of aggregate quantities and prices, the recovered SDF is capable of pricing the cross section of asset returns out of sample reasonably well. Our model explicitly derives and have implications for the term structure of interest rates and the term structure of equity premia. Our approach is general and flexible to accommodate several additional aggregate shocks and frictions that have been proposed in the literature.

References

Bansal, R., S. Miller, D. Song, and A. Yaron. 2021. "The term structure of equity risk premia." *Journal of Financial Economics* 142 (3): 1209–1228.

Barro, R. J. 2006. "Rare Disasters and Asset Markets in the Twentieth Century*." *The Quarterly Journal of Economics* 121, no. 3 (August): 823–866.

Belo, F. 2010. "Production-based measures of risk for asset pricing." *Journal of Monetary Economics* 57 (2): 146–163.

Binsbergen, J. van, W. Hueskes, R. Koijen, and E. Vrugt. 2013. "Equity yields." *Journal of Financial Economics* 110 (3): 503–519.

Blanchard, O. 1993. "Movements in the Equity Premium." *Brookings Papers on Economic Activity* 24 (2): 75–138.

Bloom, N. 2009. "The Impact of Uncertainty Shocks." *Econometrica* 77 (3): 623–685.

Breeden, D. T. 1979. "An intertemporal asset pricing model with stochastic consumption and investment opportunities." *Journal of Financial Economics* 7 (3): 265–296.

Campbell, J. Y. 2003. "Chapter 13 Consumption-based asset pricing." In *Financial Markets and Asset Pricing,* 1:803–887. Handbook of the Economics of Finance. Elsevier.

———. 2018. *Financial decisions and markets: A course in asset pricing.* Princeton University Press.

Campbell, J. Y., and R. J. Shiller. 1988. "The Dividend-Price Ratio and Expectations of Future Dividends and Discount Factors." *The Review of Financial Studies* 1 (3): 195–228.

Cochrane, J. H. 1988. "Production-based asset pricing." *Working paper,* 2776.

———. 1991. "Production-based asset pricing and the link between stock returns and economic fluctuations." *The Journal of Finance,* no. 46, 209–237.

———. 1993. "Rethinking production under uncertainty." *Working paper.*

———. 1996. "A Cross-Sectional Test of an Investment-Based Asset Pricing Model." *Journal of Political Economy* 104 (3): 572–621.

———. 2005. *Asset Pricing (Revised Edition).* Princeton University Press.

———. 2011. "Presidential Address: Discount Rates." *The Journal of Finance* 66 (4): 1047–1108.

———. 2020. "Rethinking Production under Uncertainty." *The Review of Asset Pricing Studies* 11, no. 1 (June): 1–59. ISSN: 2045-9920.

Engsted, T., T. Q. Pedersen, and C. Tanggaard. 2012. "The Log-Linear Return Approximation, Bubbles, and Predictability." *The Journal of Financial and Quantitative Analysis* 47 (3): 643–665.

Fama, E., and K. French. 1993. "Common risk factors in the returns on stocks and bonds." *Journal of Financial Economics* 33 (1): 3–56.

———. 2002. "The Equity Premium." *The Journal of Finance* 57 (2): 637–659.

Fernandez-Villaverde, J., and J. F. Rubio-Ramirez. 2006. "Solving DSGE models with perturbation methods and a change of variables." *Journal of Economic Dynamics and Control* 30, no. 12 (December): 2509–2531.

Gomes, Joao F. 2001. "Financing Investment." *American Economic Review* 91, no. 5 (December): 1263–1285.

Harrison, J. M., and D. Kreps. 1979. "Martingales and arbitrage in multiperiod securities markets." *Journal of Economic Theory* 20 (3): 381–408.

Hartzmark, S. M. 2016. "Economic Uncertainty and Interest Rates." *The Review of Asset Pricing Studies* 6 (2): 179–220.

Hou, K., C. Xue, and L. Zhang. 2015. "Digesting Anomalies: An Investment Approach." *The Review of Financial Studies* 28, no. 3 (March): 650–705.

Jagannathan, R., E. McGrattan, and A. Scherbina. 2001. *The Declining U.S. Equity Premium.* NBER Working Papers 8172. National Bureau of Economic Research, Inc.

Jermann, U. 2010. "The equity premium implied by production." *Journal of Financial Economics* 98 (2): 279–296.

Justiniano, A., and G. E. Primiceri. 2008. "The Time-Varying Volatility of Macroeconomic Fluctuations." *American Economic Review* 98, no. 3 (June): 604–41.

Kogan, L., and D. Papanikolaou. 2012. "Economic Activity of Firms and Asset Prices." *Annual Review of Financial Economics* 4 (1): 361–384.

Lettau, M., S. Ludvigson, and J. Wachter. 2008. "The Declining Equity Premium: What Role Does Macroeconomic Risk Play?" *Review of Financial Studies* 21 (4): 1653–1687.

Li, Q., M. Vassalou, and Y. Xing. 2006. "Sector Investment Growth Rates and the Cross Section of Equity Returns." *The Journal of Business* 79 (3): 1637–1665.

Liu, L. X., T. Whited, and L. Zhang. 2009. "Investment-Based Expected Stock Returns." *Journal of Political Economy* 117 (6): 1105–1139.

Lucas, Jr, R. E. 1978. "Asset Prices in an Exchange Economy." *Econometrica* 46, no. 6 (November): 1429–1445.

Max Croce, M. 2014. "Long-run productivity risk: A new hope for production-based asset pricing?" *Journal of Monetary Economics* 66 (C): 13–31.

Merz, M., and E. Yashiv. 2007. "Labor and the Market Value of the Firm." *American Economic Review* 97, no. 4 (September): 1419–1431.

Restoy, F., and M. Rockinger. 1994. "On Stock Market Returns and Returns on Investment." *Journal of Finance* 49 (2): 543–56.

Ross, S. 1978. "A Simple Approach to the Valuation of Risky Streams." *The Journal of Business* 51 (3): 453–75.

Rubinstein, M. 1976. "The Valuation of Uncertain Income Streams and the Pricing of Options." *Bell Journal of Economics* 7 (2): 407–425.

Vasicek, O. 1977. "An equilibrium characterization of the term structure." *Journal of Financial Economics* 5 (2): 177–188.

Zhang, Lu. 2017. "The Investment CAPM." *European Financial Management* 23 (4): 545–603.

Table 1: Model parameters

Parameter description	Symbol	Value
Stochastic discount factor		
Constant	μ_m	-0.205
Loading on expected profitability growth	ρ_s^m	0.0774
Loading on conditional variance	ρ_σ^m	180.869
Price of profitability growth risk	λ_ε^m	1.53
Price of expected profitability growth risk	λ_s^m	15.845
Price of uncertainty risk	λ_σ^m	-2321.150
Profitability growth		
Constant	μ_ε	-0.580
Expected profitability growth		
Mean	$\bar{s}$	0.601
Persistence of expected profitability growth	ρ_s	0.882
Conditional correlation with profitability growth	φ_s	0.291
Uncertainty		
Mean of conditional variance of profitability	$\bar{\sigma}^2$	0.0033
Persistence of conditional variance	ρ_σ	0.627
Volatility of conditional variance	σ_σ	0.0001
Technology		
Investment adjustment cost scalar	κ	2.021
Investment adjustment cost curvature	η	0.518
Depreciation rate	δ	0.026
Investment coefficients (composite parameters)		
Constant	α	-4.268
Elasticity of investment to expected profitability growth	β	2.346
Elasticity of investment to profitability growth uncertainty	ϕ	-132.099

This table presents estimated values of parameters used in Case II of the model, using the method of weighted non-linear least squares. There are 6 parameters for the SDF, 1 for the profitability growth, 3 for expected profitability growth, 3 for profitability growth uncertainty, and 2 for investment cost. The first 15 parameters in the table are estimated, and the depreciation rate is calibrated. The last three investment coefficients are composite parameters.

Table 2: Model fit

Panel A			Panel B		
Moments	Data	Model	Portfolios	Realized	Predicted
$\mathbb{E}(r^f)$	0.0050	0.0050	Small	2.61	2.59
$\sigma(r^f)$	0.0057	0.0057	ME2	2.46	2.47
$Cor(r_t^f, r_{t-1}^f)$	0.6788	0.6859	ME3	2.57	2.52
$\mathbb{E}(ik)$	-3.2899	-3.2905	ME4	2.38	2.42
$\sigma(ik)$	0.0839	0.0839	ME5	2.46	2.37
$Cor(ik_t, ik_{t-1})$	0.9689	0.8769	ME6	2.21	2.21
$\mathbb{E}(r^e)$	0.0129	0.0128	ME7	2.25	2.33
$\sigma(r^e)$	0.0861	0.0862	ME8	2.17	2.06
$\mathbb{E}(\Delta\varepsilon_t)$	0.0207	0.0207	ME9	1.93	2.03
$\sigma(\Delta\varepsilon_t)$	0.0648	0.0672	Big	1.54	1.62
$Cor(ik_t, r_t^f)$	0.3435	0.3479	MAE	0.0587	
$Cor(ik_t, r_{t+1}^e)$	-0.1783	-0.0006			

This table presents actual and model-implied moments of risk-free rates, investment rates, equity premium, profitability growth in Panel A and realized and predicted average excess returns of 10 size-sorted portfolios in Panel B. Expressions of unconditional moments listed in the table are provided in the Appendix. Predicted average excess returns are calculated as the negative unconditional covariance between the recovered SDF and portfolios returns divided by the unconditional mean of the recovered SDF.

Table 3: Variance decomposition of unexpected innovations in the log SDF

Decomposition into primitive shocks					
$\frac{\mathbb{V}(-\lambda^m_\varepsilon e_{\varepsilon,t+1})}{\mathbb{V}(m_{+1}-\mathbb{E}_t(m_{t+1}))}$	$\frac{\mathbb{V}(-\lambda^m_s e_{s,t+1})}{\mathbb{V}(m_{+1}-\mathbb{E}_t(m_{t+1}))}$	$\frac{\mathbb{V}(-\lambda^m_\sigma e_{\sigma,t+1})}{\mathbb{V}(m_{+1}-\mathbb{E}_t(m_{t+1}))}$	$\frac{COV(-\lambda^m_\varepsilon e_{\varepsilon,t+1},-\lambda^m_s e_{s,t+1})}{\mathbb{V}(m_{+1}-\mathbb{E}_t(m_{t+1}))}$	$\frac{COV(-\lambda^m_\varepsilon e_{\varepsilon,t+1},-\lambda^m_\sigma e_{\sigma,t+1})}{\mathbb{V}(m_{+1}-\mathbb{E}_t(m_{t+1}))}$	$\frac{COV(-\lambda^m_s e_{s,t+1},-\lambda^m_\sigma e_{\sigma,t+1})}{\mathbb{V}(m_{+1}-\mathbb{E}_t(m_{t+1}))}$
4.2%	106.7%	10.5%	-8.2/%	0.6%	-13.7%
Decomposition into observable shocks					
$\frac{\mathbb{V}(-\lambda^m_{rf}\Delta r^f_{t+1})}{\mathbb{V}(m_{+1}-\mathbb{E}_t(m_{t+1}))}$	$\frac{\mathbb{V}(-\lambda^m_{ik}\Delta ik_{t+1})}{\mathbb{V}(m_{+1}-\mathbb{E}_t(m_{t+1}))}$	$\frac{\mathbb{V}(-\lambda^m_\varepsilon \Delta\varepsilon_{t+1})}{\mathbb{V}(m_{+1}-\mathbb{E}_t(m_{t+1}))}$	$\frac{COV(-\lambda^m_{rf}\Delta r^f_{t+1},-\lambda^m_{ik}\Delta ik_{t+1})}{\mathbb{V}(m_{+1}-\mathbb{E}_t(m_{t+1}))}$	$\frac{COV(-\lambda^m_{rf}\Delta r^f_{t+1},-\lambda^m_\varepsilon \Delta\varepsilon_{t+1})}{\mathbb{V}(m_{+1}-\mathbb{E}_t(m_{t+1}))}$	$\frac{COV(-\lambda^m_{ik}\Delta ik_{t+1},-\lambda^m_\varepsilon \Delta\varepsilon_{t+1})}{\mathbb{V}(m_{+1}-\mathbb{E}_t(m_{t+1}))}$
1.4%	98.9%	4.0%	-0.8%	-0.2%	-3.3%

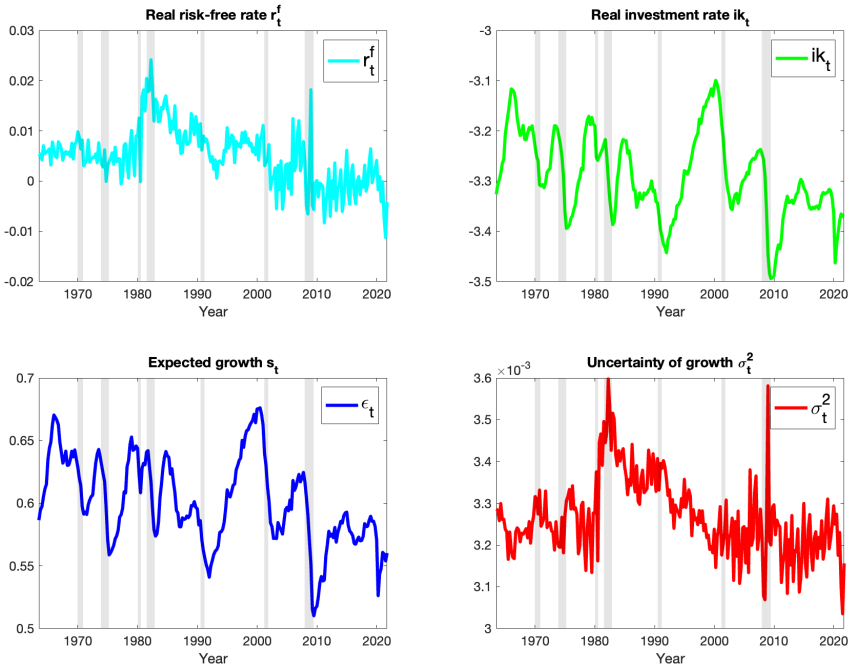
This table presents variance decomposition of unexpected innovations in the log SDF, $m_{t+1}-\mathbb{E}_t[m_{t+1}]$, into three primitive shocks, profitability growth shock, $-\lambda^m_\varepsilon e_{\varepsilon,t+1}$, expected profitability growth shock, $-\lambda^m_s e_{s,t+1}$, and profitability growth uncertainty shock $-\lambda^m_\sigma e_{\sigma,t+1}$ in the top panel, and into three observable shocks, investment rate growth $-\lambda^m_{ik}\Delta ik_{t+1}$, profitability growth rates $-\lambda^m_\varepsilon \Delta\varepsilon_{t+1}$, and real rates growth $-\lambda^m_{r_f}\Delta r^f_{t+1}$ in the bottom panel.

Table 4: Pricing of 6 SZ/BM & 6 SZ/OP & 6 SZ/INV portfolios

	6 SZ/BM		6 SZ/OP		6 SZ/INV	
	Realized	Predicted	Realized	Predicted	Realized	Predicted
Small-X1	1.96	2.14	2.10	2.42	3.06	2.62
Small-X2	2.75	2.35	2.67	2.32	2.87	2.35
Small-X3	3.16	2.90	3.01	2.79	1.96	2.45
Big-X1	1.74	1.43	1.33	2.11	2.11	1.46
Big-X2	1.66	2.07	1.58	1.92	1.68	1.80
Big-X3	2.19	2.79	1.96	1.52	1.69	1.83
MAE	0.36		0.41		0.39	

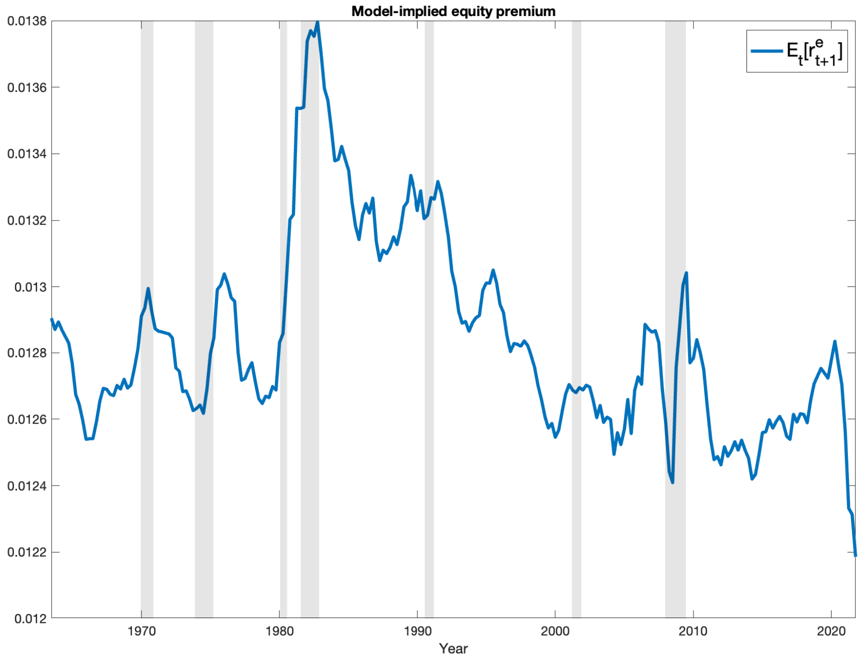
This table presents realized and predicted average excess returns of 6 size-book-to-market-sorted portfolios, 6-size-investment-sorted portfolios, 6-size-profitability-sorted portfolios. $X1, X2, X3$ represents Predicted average excess returns are calculated as the negative unconditional covariance between the recovered SDF and portfolios returns divided by the unconditional mean of the recovered SDF.

Figure 1: Recovery of underlying state variables



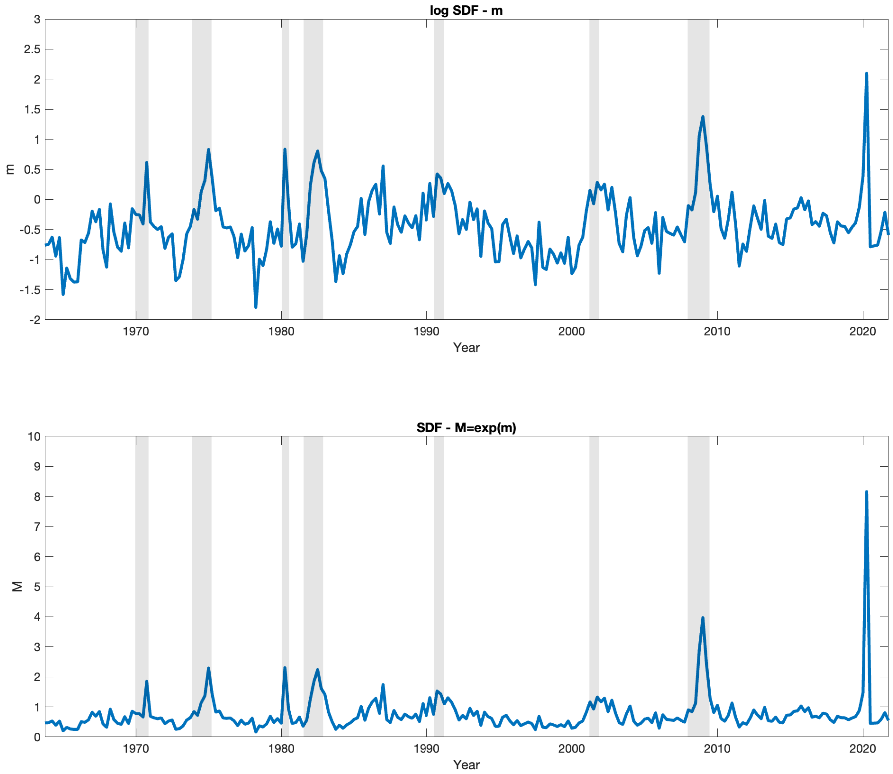
Notes: This figure shows the time series of ex ante real risk-free rates, real investment rates, recovered expected profitability growth rates, and recovered profitability growth uncertainty. Ex ante real risk-free rates are fitted values from regressing ex post real risk-free rates on nominal risk-free rates and inflation in the past four quarters. Real investment rates are constructed using the perpetual inventory method.

Figure 2: Model-implied equity premium



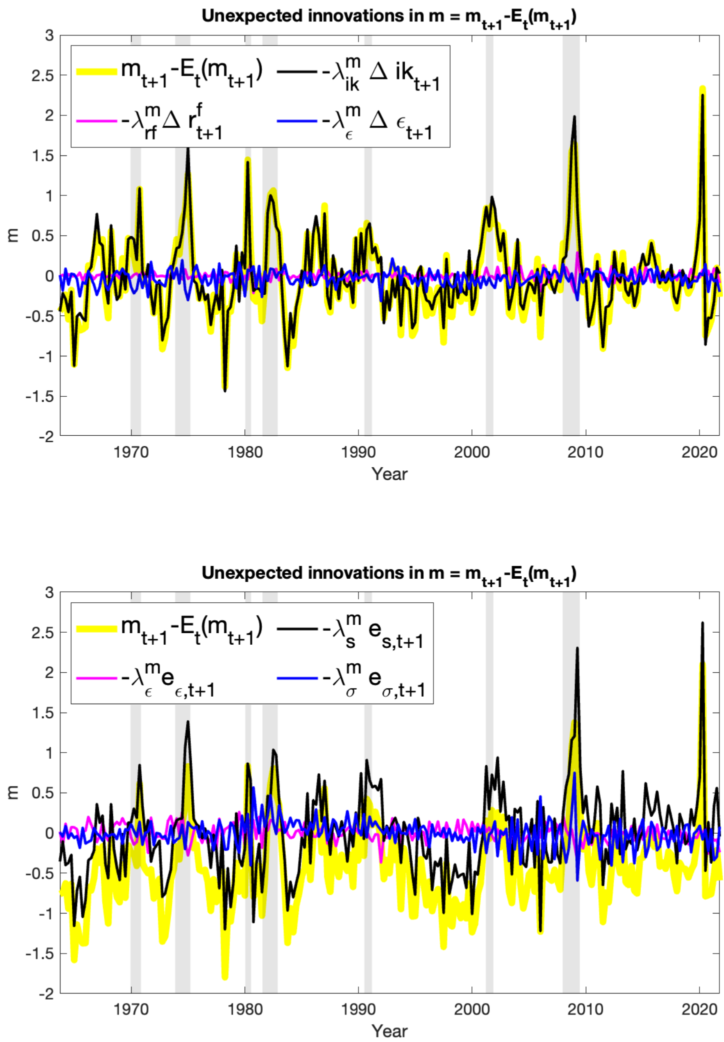
Notes: This figure presents the time series of model-implied equity premia. The equity premium is calculated as the model-implied expected investment excess return adjusted by the leverage ratio. Quarterly excess returns are then smoothed by taking four-quarter moving average.

Figure 3: Recovered SDF



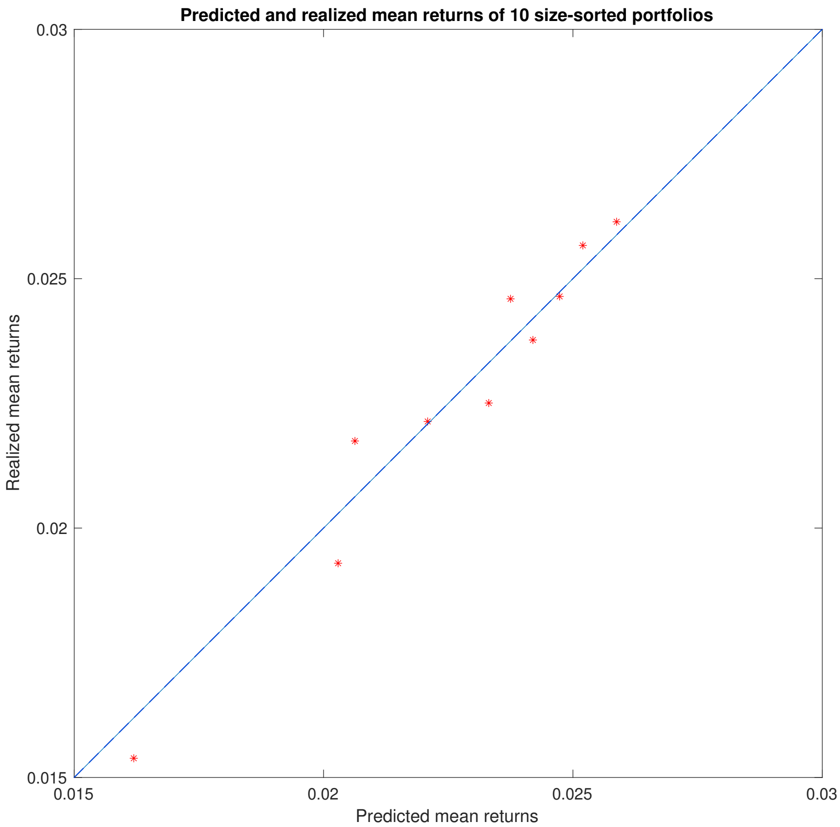
Notes: This figure presents the time series of the stochastic discount factor recovered from real investment rates, profitability growth rates, and real rates.

Figure 4: Unexpected innovations in the log SDF and decomposition into primitive shocks



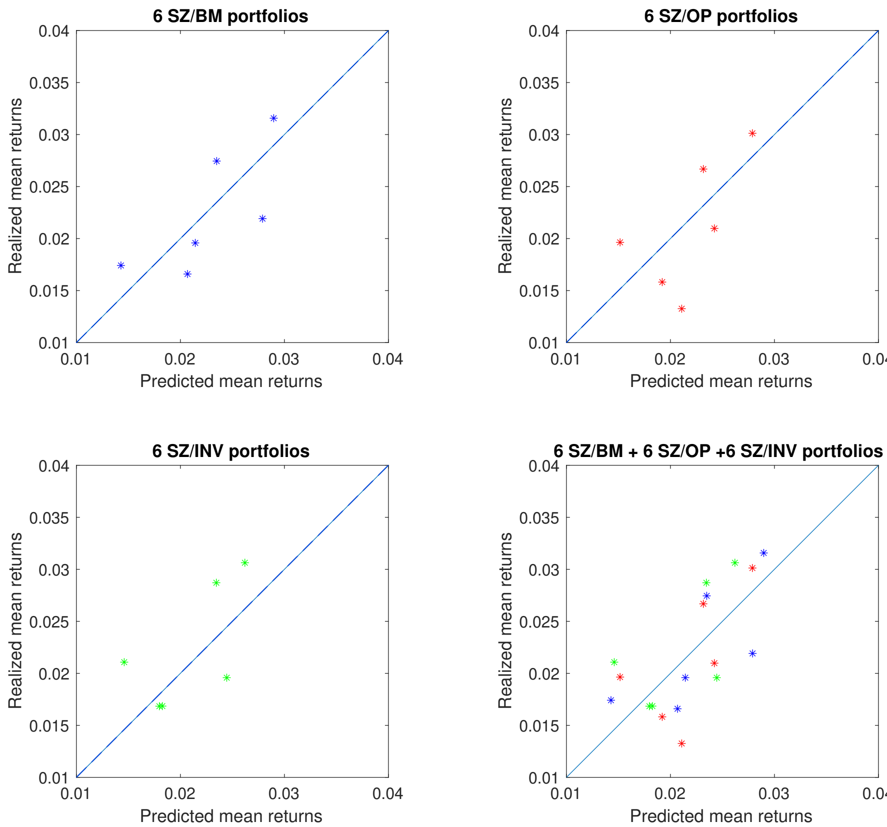
Notes: The top panel of this figure presents the time series of unexpected innovations in the log SDF, $m_{t+1} - \mathbb{E}_t[m_{t+1}] = -\lambda_\varepsilon^m e_{\varepsilon,t+1} - \lambda_s^m e_{s,t+1} - \lambda_\sigma^m e_{\sigma,t+1}$, and three components, profitability growth shock, $-\lambda_\varepsilon^m e_{\varepsilon,t+1}$, expected profitability growth shock, $-\lambda_s^m e_{s,t+1}$, and profitability growth uncertainty shock $-\lambda_\sigma^m e_{\sigma,t+1}$, respectively. The bottom panel of this figure presents the time series of unexpected innovations in the log SDF, $m_{t+1} - \mathbb{E}_t[m_{t+1}] = -\lambda_{ik}^m \Delta ik_{t+1} - \lambda_\varepsilon^m \Delta\varepsilon_{t+1} - \lambda_{r_f}^m \Delta r_{t+1}^f$, and three components, investment rate growth $-\lambda_{ik}^m \Delta ik_{t+1}$, profitability growth rates $-\lambda_\varepsilon^m \Delta\varepsilon_{t+1}$, and real rates $-\lambda_{r_f}^m \Delta r_{t+1}^f$, respectively.

Figure 5: Pricing of 10 size-sorted portfolios



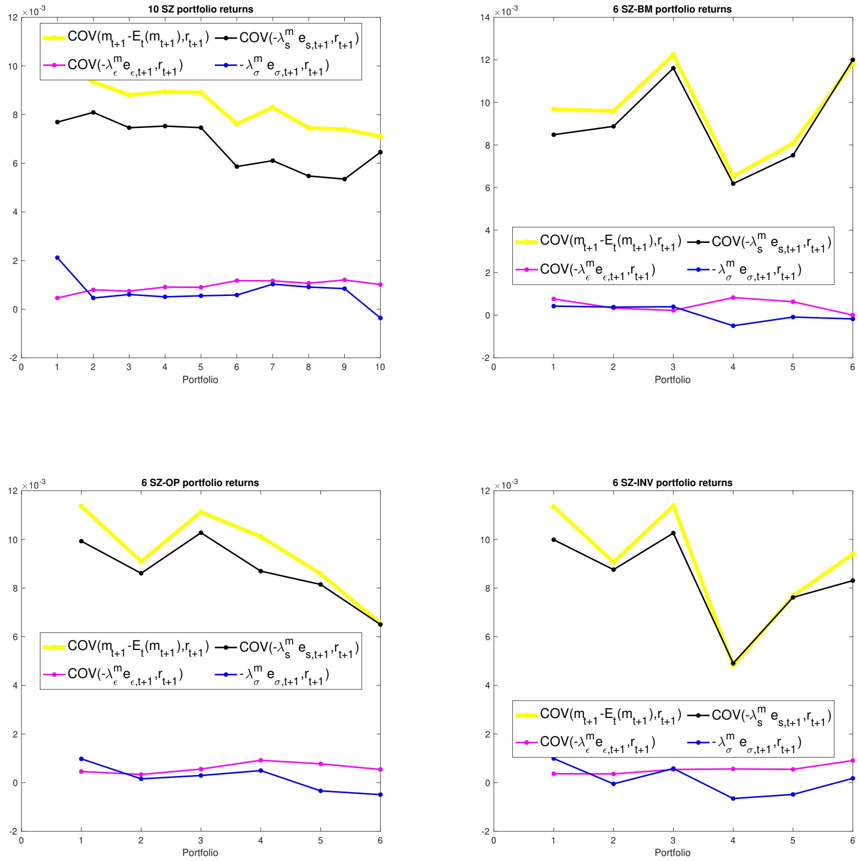
Notes: This figure presents model-implied average returns against realized average returns of 10 size-sorted portfolios.

Figure 6: Pricing of 6 SZ/BM & 6 SZ/OP & 6 SZ/INV portfolios



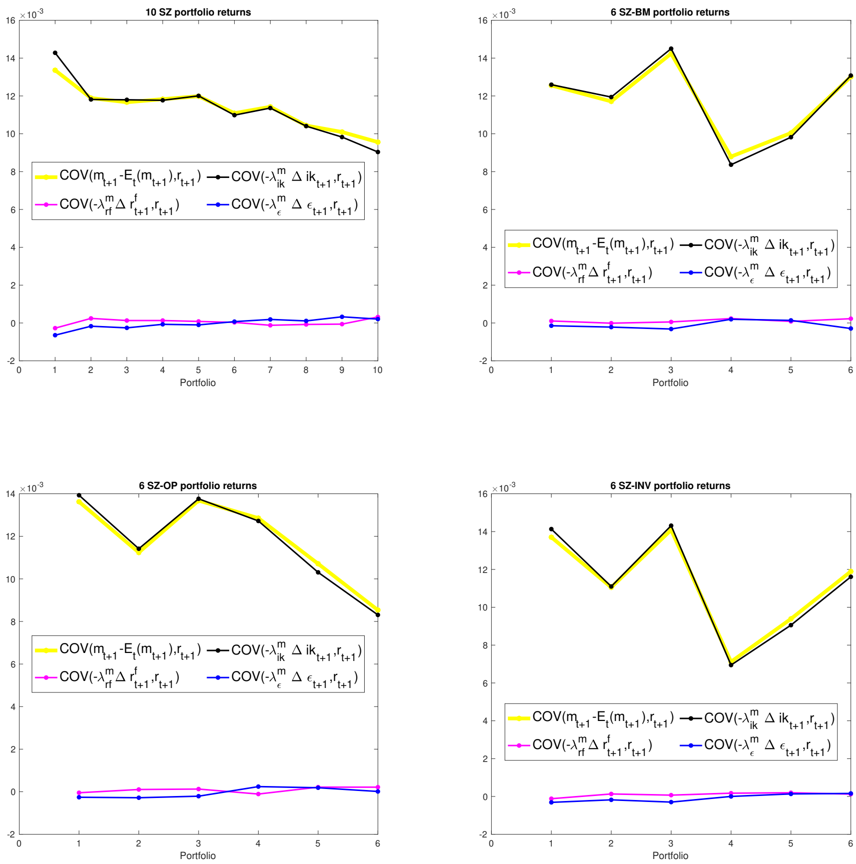
Notes: This figure presents model-implied average returns against realized average returns of 6 size-book-to-market-sorted portfolios, 6-size-investment-sorted portfolios, 6-size-profitability-sorted portfolios, and all 18 portfolios together.

Figure 7: Decomposition of covariances between unexpected innovations in the SDF and portfolio returns into primitive shocks



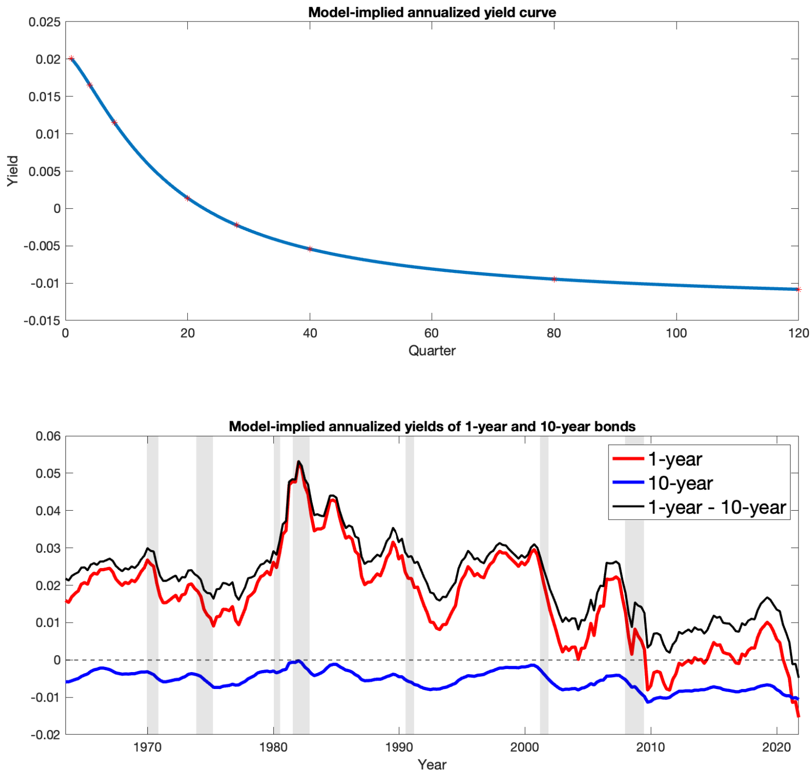
Notes: This figure presents a decomposition of covariances between unexpected innovations in the SDF and portfolio returns into primitive shocks.

Figure 8: Decomposition of covariances between unexpected innovations in the SDF and portfolio returns into observable shocks



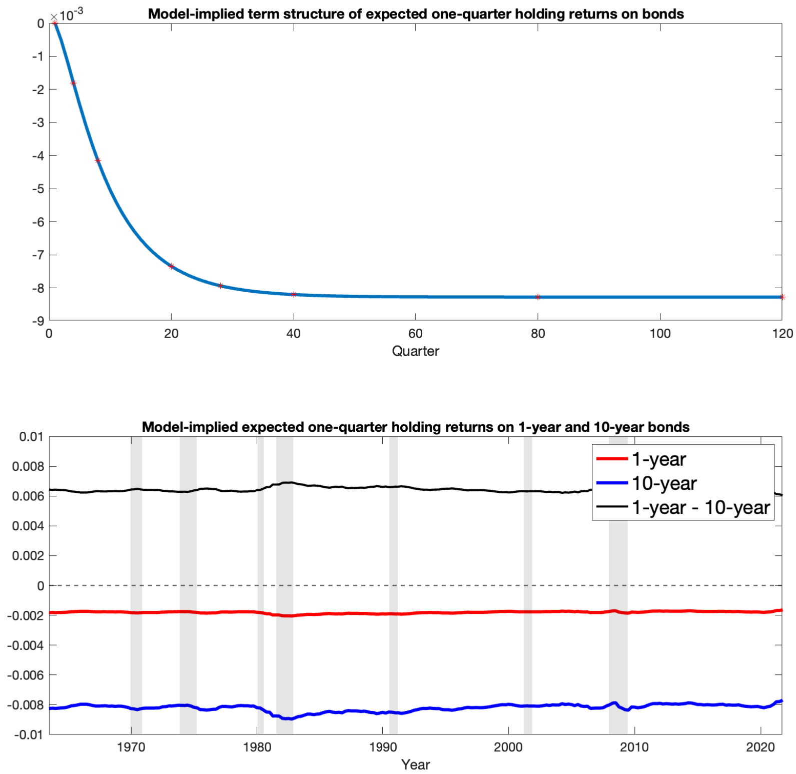
Notes: This figure presents a decomposition of covariances between unexpected innovations in the SDF and portfolio returns into observable shocks.

Figure 9: Model-implied annualized yield curve of bonds



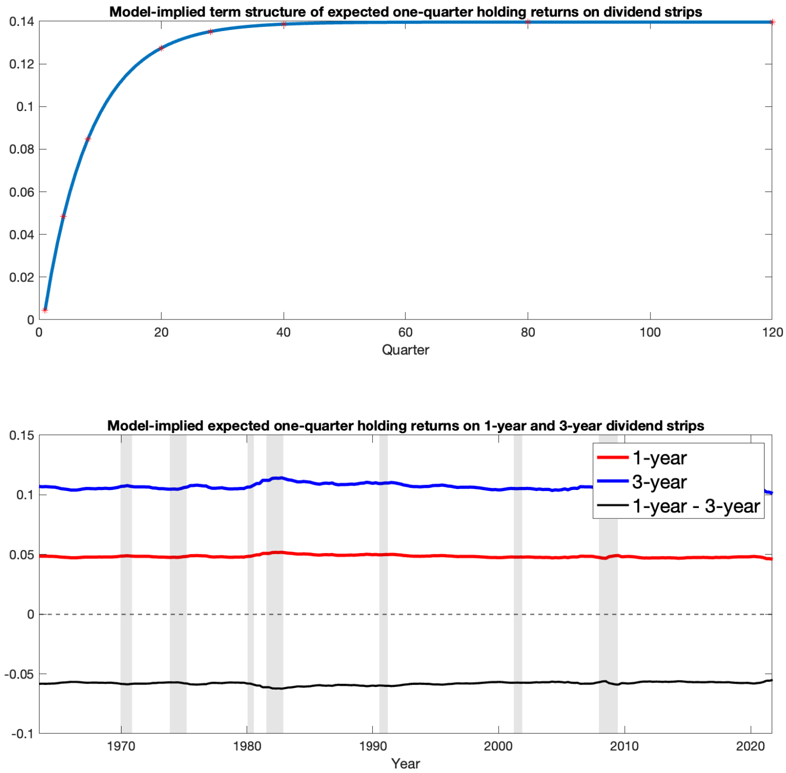
Notes: This figure presents model-implied yield curve of risk-free bonds and time variations in yield of 1-year and 10-year bonds.

Figure 10: Model-implied term structure of expected one-quarter holding returns on bonds



Notes: This figure presents model-implied expected one-quarter holding returns on risk-free bonds across maturities and time variations in expected one-quarter holding returns on 1-year and 10-year bonds.

Figure 11: Model-implied term structure of expected one-quarter holding returns on dividend strips



Notes: This figure presents model-implied expected one-quarter holding returns on dividend claims across maturities and time variations in expected one-quarter holding returns on 1-year and 3-year dividend claims.

Appendices

A Model Solution

A.1 Investment Euler equation

Denote a vector of state variables, $X_t = (K_t, \mathcal{E}_t)$. The producer's maximization problem is:

$$V(X_t) = \max_{\{I_t\}} \left\{ D_t + \mathbb{E}_t[M_{t+1}V(X_{t+1})] \right\}$$

subject to the following contraints,

$$\begin{aligned} D_t &= \varPi(\mathcal{E}_t, K_t) - \varPhi(I_t, K_t) \\ \varPi(\mathcal{E}_t, K_t) &= \mathcal{E}_t K_t \\ \varPhi(I_t, K_t) &= \frac{\kappa}{\eta+1} IK_t^{\eta+1} \varPi(\mathcal{E}_t, K_t) \\ K_{t+1} &= (1-\delta)K_t + I_t \end{aligned}$$

where $IK_t \equiv I_t/K_t$ denotes the investment-to-capital ratio, the investment rate.

The value function is then

$$V(X_t) = \max_{\{I_t\}} \left\{ \mathcal{E}_t K_t - \frac{\kappa}{\eta+1} IK_t^{\eta+1} \mathcal{E}_t K_t + \mathbb{E}_t[M_{t+1}V(X_{t+1})] \right\} \tag{A.1}$$

The first-order condition is given by

$$\frac{\partial V(X_t)}{\partial I_t} : \ \kappa IK_t^{\eta} \mathcal{E}_t = \mathbb{E}_t[M_{t+1}V_K(X_{t+1})] \tag{A.2}$$

where the RHS is the expected discounted marginal value of capital.

By the envelope theorem, the marginal value of capital is given by recursively

$$V_K(X_t) = \ \mathcal{E}_t + \kappa \frac{\eta}{\eta+1} IK_t^{\eta+1} \mathcal{E}_t + (1-\delta)\mathbb{E}_t[M_{t+1}V_K(X_{t+1})] \tag{A.3}$$

Combing (A.2) and (A.3), we have

$$V_K(X_t) = \ \mathcal{E}_t + \kappa \frac{\eta}{\eta+1} IK_t^{\eta+1} \mathcal{E}_t + (1-\delta)\kappa IK_t^{\eta} \mathcal{E}_t \tag{A.4}$$

which, substituted back in (A.2), yields the investment Euler equation (6) and the investment return (7),

$$\begin{aligned} 1 &= \mathbb{E}_t[M_t R^I_{t+1}] \\ R^I_{t+1} &= \frac{\mathcal{E}_{t+1}[1 + \kappa \frac{\eta}{\eta+1} IK_{t+1}^{\eta+1} + (1-\delta)\kappa IK_{t+1}^{\eta}]}{\mathcal{E}_t[\kappa IK_t^{\eta}]} \end{aligned}$$

One can prove that the ex-dividend value equals the marginal/average q times the capital stock at the end of current period, $P_t \equiv V(X_t) - D_t = (\kappa IK_t^{\eta}\mathcal{E}_t)K_{t+1} \equiv q_t K_{t+1}$. To show this, multiply the numerator and the denominator of the investment return by K_{t+1}. We obtain $R^I_{t+1} = (D_{t+1} + (\kappa IK_{t+1}^{\eta}\mathcal{E}_{t+1})K_{t+2})/(\kappa IK_t^{\eta}\mathcal{E}_t)K_{t+1}$, equivalent to $R^I_{t+1} = \frac{D_{t+1}+P_{t+1}}{P_t}$.

A.2 Log-linearization

To obtain approximate analytical solutions, we log-linearize the investment return as follows.

$$\begin{aligned} r^I_{t+1} &= \log\left(\frac{\mathcal{E}_{t+1}}{\mathcal{E}_t}\right) + \log\left(1 + \kappa\frac{\eta}{\eta+1} IK_{t+1}^{\eta+1} + (1-\delta)\kappa IK_{t+1}^{\eta}\right) - \log(\kappa IK_t^{\eta}) && \text{(A.5)} \\ &\approx (a_1 + b_1 ik_{t+1}) - (a_2 + b_2 ik_t) + \varDelta\varepsilon_{t+1} && \text{(A.6)} \end{aligned}$$

where $a_1 = \log\left(1 + \kappa\frac{\eta}{\eta+1}\exp[(\eta+1)ik^*] + (1-\delta)\kappa\exp(\eta ik^*)\right) - \frac{\kappa\eta\exp(\eta ik^*)[\exp(ik^*)+(1-\delta)]}{1+\kappa\frac{\eta}{\eta+1}\exp[(\eta+1)ik^*]+(1-\delta)\kappa\exp(\eta ik^*)}ik^*$, $b_1 = \frac{\kappa\eta\exp(\eta ik^*)[\exp(ik^*)+(1-\delta)]}{1+\kappa\frac{\eta}{\eta+1}\exp[(\eta+1)ik^*]+(1-\delta)\kappa\exp(\eta ik^*)}$, $a_2 = \log\kappa$, and $b_2 = \eta$.

To derive key variables in the model, we also need to log-linearize the dividend-price ratio.

$$\frac{D_t}{P_t} = \frac{\mathcal{E}_t K_t - \frac{\kappa}{\eta+1}IK_t^{\eta+1}\mathcal{E}_t K_t}{(\kappa IK_t^{\eta}\mathcal{E}_t)K_{t+1}} = \frac{1 - \frac{\kappa}{\eta+1}IK_t^{\eta+1}}{\kappa IK_t^{\eta}}\frac{K_t}{K_{t+1}} \tag{A.7}$$

which can be approximated as follows.

$$\begin{aligned} dp_t &= \log\left(1 - \frac{\kappa}{\eta+1}IK_t^{\eta+1}\right) - \log(\kappa IK_t^{\eta}) - \Delta k_{k+1} && \text{(A.8)}\\ &\approx (a_3 + b_3 ik_t) - (a_2 + b_2 ik_t) - (a_4 + b_4 ik_t) && \text{(A.9)}\\ &\equiv a_5 + b_5 ik_t && \text{(A.10)} \end{aligned}$$

where $a_3 = \log\left(1 - \frac{\kappa}{\eta+1}\exp[(\eta+1)ik^*]\right) - \frac{-\kappa\exp[(\eta+1)ik^*]}{1-\frac{\kappa}{\eta+1}\exp[(\eta+1)ik^*]}ik^*$, $b_3 = \frac{-\kappa\exp[(\eta+1)ik^*]}{1-\frac{\kappa}{\eta+1}\exp[(\eta+1)ik^*]}$, $a_4 = \log(e^{ik^*} + 1 - \delta) - \frac{e^{ik^*}}{e^{ik^*}+1-\delta}ik^*$, and $b_4 = \frac{e^{ik^*}}{e^{ik^*}+1-\delta}$, $a_5 = a_3 - a_2 - a_4$, $b_5 = b_3 - b_2 - b_4$.

The growth of the ex-dividend value is given by

$$\frac{P_{t+1}}{P_t} = \frac{q_{t+1}K_{t+2}}{q_t K_{t+1}} = \frac{\mathcal{E}_{t+1}[\kappa IK_{t+1}^{\eta}]}{\mathcal{E}_t[\kappa IK_t^{\eta}]}\frac{K_{t+2}}{K_{t+1}} = \left(\frac{IK_{t+1}}{IK_t}\right)^{\eta}\frac{\mathcal{E}_{t+1}}{\mathcal{E}_t}\frac{K_{t+2}}{K_{t+1}} \tag{A.11}$$

which can be approximated as

$$\begin{aligned} \Delta p_{t+1} &= \eta(ik_{t+1} - ik_t) + \Delta\varepsilon_{t+1} + \Delta k_{t+2}\\ &= a_4 + (b_4+\eta)ik_{t+1} - \eta ik_t + \Delta\varepsilon_{t+1} && \text{(A.12)} \end{aligned}$$

The sum of the log dividend-price ratio and the log growth of the ex-dividend value can be approximated by

$$\begin{aligned} dp_{t+1} + \Delta p_{t+1} &= a_5 + b_5 ik_{t+1} + a_4 + (b_4+\eta)ik_{t+1} - \eta ik_t + \Delta\varepsilon_{t+1}\\ &\equiv a_6 + (b_6+\eta)ik_{t+1} - \eta ik_t + \Delta\varepsilon_{t+1} && \text{(A.13)} \end{aligned}$$

where $a_6 \equiv a_4 + a_5$, $b_6 \equiv b_4 + b_5$.

The dividend growth is given by

$$\frac{D_{t+1}}{D_t} = \frac{DP_{t+1}}{DP_t}\frac{P_{t+1}}{P_t} \tag{A.14}$$

which can be approximated by

$$\begin{aligned} \Delta d_{t+1} &= dp_{t+1} - dp_t + \Delta p_{t+1}\\ &\approx b_5(ik_{t+1} - ik_t) + a_4 + (b_4+\eta)ik_{t+1} - \eta ik_t + \Delta\varepsilon_{t+1}\\ &\equiv a_4 + (b_6+\eta)ik_{t+1} - (b_5+\eta)ik_t + \Delta\varepsilon_{t+1} && \text{(A.15)} \end{aligned}$$

A.3 Solving the investment rate

We omit solutions to Case I, which is simpler to solve than Case II. In Case II, SDF and profitability are specified exogenously as in (39), (36), (37), (38).

$$\begin{aligned} m_{t+1} &= -\mu_m - \rho_s^m s_t - \rho_\sigma^m \sigma_t^2 - \lambda_\varepsilon^m \sigma_t e_{\varepsilon,t+1} - \lambda_s^m \sigma_t e_{s,t+1} - \lambda_\sigma^m \sigma_\sigma e_{\sigma,t+1}\\ \Delta\varepsilon_{t+1} &= \mu_\varepsilon + s_t + \sigma_t e_{\varepsilon,t+1}\\ s_{t+1} &= \mu_s + \rho_s s_t + \varphi_s \sigma_t e_{s,t+1}\\ \sigma_{t+1}^2 &= \mu_\sigma + \rho_\sigma \sigma_t^2 + \sigma_\sigma e_{\sigma,t+1} \end{aligned}$$

where $e_{\varepsilon,t}, e_{s,t}, e_{\sigma,t} \overset{i.i.d}{\sim} N(0,1)$ and are orthogonal to each other.

We conjecture the following policy function for the investment rate as in (40)

$$ik_t = \alpha + \beta\varepsilon_t + \phi\sigma_t^2$$

Now the investment return can be expressed as (41)

$$\begin{aligned}
r^I_{t+1} &\approx (a_1 - a_2) + b_1 ik_{t+1} - b_2 ik_t + \Delta\varepsilon_{t+1} \\
&= (a_1 - a_2) + b_1(\alpha + \beta s_{t+1} + \phi\sigma^2_{t+1}) - b_2(\alpha + \beta s_t + \phi\sigma_t^2) + (\mu_\varepsilon + s_t + \sigma_t e_{\varepsilon,t+1}) \\
&= [(a_1 - a_2) + (b_1 - b_2)\alpha + \mu_\varepsilon] + b_1\beta s_{t+1} + (1 - b_2\beta)s_t + b_1\phi\sigma^2_{t+1} - b_2\phi\sigma_t^2 + \sigma_t e_{\varepsilon,t+1} \\
&= [(a_1 - a_2) + (b_1 - b_2)\alpha + \mu_\varepsilon + b_1\beta\mu_s + b_1\phi\mu_\sigma] + [(b_1\rho_s - b_2)\beta + 1]s_t + [(b_1\rho_\sigma - b_2)\phi]\sigma_t^2 \\
&\quad + \sigma_t e_{\varepsilon,t+1} + (b_1\beta\varphi_s)\sigma_t e_{s,t+1} + (b_1\phi)\sigma_\sigma e_{\sigma,t+1}
\end{aligned}$$

Conditional moments of the investment return are

$$\mathbb{E}_t[r^I_{t+1}] = [(a_1 - a_2) + (b_1 - b_2)\alpha + \mu_\varepsilon + b_1\beta\mu_s + b_1\phi\mu_\sigma] + [(b_1\rho_s - b_2)\beta + 1]s_t + [(b_1\rho_\sigma - b_2)\phi]\sigma_t^2 \tag{A.16}$$

$$\mathbb{V}_t[r^I_{t+1}] = \sigma_t^2 + (b_1\beta\varphi_s)^2\sigma_t^2 + (b_1\phi)^2\sigma_\sigma^2 \tag{A.17}$$

$$COV_t(m_{t+1}, r^I_{t+1}) = -\lambda^m_\varepsilon\sigma_t^2 - \lambda^m_s(b_1\beta\varphi_s)\sigma_t^2 - \lambda^m_\sigma(b_1\phi)\sigma_\sigma^2 \tag{A.18}$$

Plugging all terms into (16), we have

$$\begin{aligned}
0 &= \mathbb{E}_t[m_{t+1}] + \mathbb{E}_t[r^I_{t+1}] + \frac{1}{2}\mathbb{V}_t[m_{t+1}] + \frac{1}{2}\mathbb{V}_t[r^I_{t+1}] + COV_t(m_{t+1}, r^I_{t+1}) \\
&= (-\mu_m - \rho^m_s s_t - \rho^m_\sigma\sigma_t^2) + \frac{1}{2}[(\lambda^m_s)^2\sigma_t^2 + (\lambda^m_\varepsilon)^2\sigma_t^2 + (\lambda^m_\sigma)^2\sigma_\sigma^2] \\
&\quad + [(a_1 - a_2) + (b_1 - b_2)\alpha + \mu_\varepsilon + b_1\beta\mu_s + b_1\phi\mu_\sigma] + [(b_1\rho_s - b_2)\beta + 1]s_t + [(b_1\rho_\sigma - b_2)\phi]\sigma_t^2 \\
&\quad + \frac{1}{2}[\sigma_t^2 + (b_1\beta\varphi_s)^2\sigma_t^2 + (b_1\phi)^2\sigma_\sigma^2] - \lambda^m_\varepsilon\sigma_t^2 - \lambda^m_s(b_1\beta\varphi_s)\sigma_t^2 - \lambda^m_\sigma(b_1\phi)\sigma_\sigma^2 \\
&= [(a_1 - a_2) + (b_1 - b_2)\alpha + \mu_\varepsilon + b_1\beta\mu_s + b_1\phi\mu_\sigma - \mu_m + \frac{1}{2}(\lambda^m_\sigma - b_1\phi)^2\sigma_\sigma^2] \\
&\quad + [(b_1\rho_s - b_2)\beta + 1 - \rho^m_s]s_t + [(b_1\rho_\sigma - b_2)\phi - \rho^m_\sigma + \frac{1}{2}(\lambda^m_\varepsilon - 1)^2 + \frac{1}{2}(\lambda^m_s - b_1\beta\varphi_s)^2]\sigma_t^2
\end{aligned}$$

Using the property that the Investment Euler equation holds at all times, we solve for the coefficients of the policy function of investment by method of undetermined coefficients.

We obtain coefficients of the optimal investment rate, (42), (43), and (44).

$$\begin{aligned}
\beta &= \frac{\rho^m_s - 1}{b_1\rho_s - b_2} \\
\phi &= \frac{\rho^m_\sigma - \frac{1}{2}(\lambda^m_\varepsilon - 1)^2 - \frac{1}{2}(\lambda^m_s - b_1\beta\varphi_s)^2}{b_1\rho_\sigma - b_2} \\
\alpha &= \frac{\mu_m - \frac{1}{2}(\lambda^m_\sigma - b_1\phi)^2\sigma_\sigma^2 - (a_1 - a_2) - \mu_\varepsilon - b_1\beta\mu_s - b_1\phi\mu_\sigma}{b_1 - b_2}
\end{aligned}$$

A.4 Term structure of interest rates

The price of a n-period risk-free zero-coupon bond is

$$B^n_t = E_t[M_{t+1}...M_{t+n}] = E_t[M_{t+1}B^{n-1}_{t+1}] \tag{A.19}$$

in the log form,

$$b^n_t = \mathbb{E}_t\left[\sum_{i=1}^n m_{t+i}\right] + \frac{1}{2}\mathbb{V}_t\left[\sum_{i=1}^n m_{t+i}\right] \tag{A.20}$$

and the yield of a n-period bond is

$$y_t^n = -\frac{b_t^n}{n} = -\frac{1}{n}\mathbb{E}_t\left[\sum_{i=1}^{n} m_{t+i}\right] - \frac{1}{2n}\mathbb{V}_t\left[\sum_{i=1}^{n} m_{t+i}\right] \tag{A.21}$$

Particularly, the 1-period risk-less bond yield, or the risk-free rate, is given by,

$$r_t^f = y_t^1 = -b_t^1 = -\mathbb{E}_t[m_{t+1}] - \frac{1}{2}\mathbb{V}_t[m_{t+1}]$$

With the SDF process specified in (39), we can express bond prices as functions of state variables. The 1-period bond price is given by

$$b_t^1 = [-\mu_m + \frac{1}{2}(\lambda_\sigma^m)^2\sigma_\sigma^2] - \rho_s^m s_t + [-\rho_\sigma^m + \frac{1}{2}(\lambda_\varepsilon^m)^2 + \frac{1}{2}(\lambda_s^m)^2]\sigma_t^2$$

Since the 1-period bond price and yield are linear in the state variable ε_t and σ_t^2, and we conjecture that the log n-period bond price has a linear form as in (58)

$$b_t^n = A_n + B_n s_t + C_n \sigma_t^2$$

We can find a recursive solution for A_n, B_n, shown by (59), (60), (61), by mathematical induction

$$\begin{aligned}
A_{n+1} &= A_n + B_n\mu_s + C_n\mu_\sigma - \mu_m + \frac{1}{2}(\lambda_\sigma^m - C_n)^2\sigma_\sigma^2 \\
B_{n+1} &= B_n\rho_s - \rho_s^m = \frac{\rho_s^m}{1-\rho_s}(\rho_s^{n+1} - 1) \\
C_{n+1} &= C_n\rho_\sigma - \rho_\sigma^m + \frac{1}{2}(\lambda_\varepsilon^m)^2 + \frac{1}{2}(\lambda_s^m - B_n\varphi_s)^2
\end{aligned}$$

Proof: First, note that when $n = 0$

$$b_t^0 = log1 = 0 \implies A_0 = 0,\ B_0 = 0,\ C_0 = 0$$

Given A_0, B_0, C_0, we can calculate and verify A_1, B_1, C_1

$$A_1 = [-\mu_m + \frac{1}{2}(\lambda_\sigma^m)^2\sigma_\sigma^2],\ B_1 = -\rho_s^m,\ C_1 = -\rho_\sigma^m + \frac{1}{2}(\lambda_\varepsilon^m)^2 + \frac{1}{2}(\lambda_s^m)^2$$

which are consistent with the price conjecture and coefficients above.

Second, implied by (A.19), we have,

$$\begin{aligned}
b_t^{n+1} &= \log\mathbb{E}_t[M_{t+1}B_{t+1}^n] = \log\mathbb{E}_t[\exp(m_{t+1} + b_{t+1}^n)] \\
&= \mathbb{E}_t[m_{t+1} + b_{t+1}^n] + \frac{1}{2}\mathbb{V}_t[m_{t+1} + b_{t+1}^n] \\
&= \mathbb{E}_t[m_{t+1}] + \mathbb{E}_t[b_{t+1}^n] + \frac{1}{2}\mathbb{V}_t[m_{t+1}] + \frac{1}{2}\mathbb{V}_t[b_{t+1}^n] + COV_t[m_{t+1}, b_{t+1}^n] \\
&= b_t^1 + \mathbb{E}_t[b_{t+1}^n] + \frac{1}{2}\mathbb{V}_t[b_{t+1}^n] + COV_t[m_{t+1}, b_{t+1}^n]
\end{aligned} \tag{A.22}$$

where the last equality is by the expression of the one-period bond price.

With the conjecture (58), we have

$$\begin{aligned}
b_t^{n+1} &= b_t^1 + \mathbb{E}_t[b_{t+1}^n] + \frac{1}{2}\mathbb{V}_t[b_{t+1}^n] + COV_t[m_{t+1}, b_{t+1}^n] \\
&= [-\mu_m + \frac{1}{2}(\lambda_\sigma^m)^2\sigma_\sigma^2] - \rho_s^m s_t + [-\rho_\sigma^m + \frac{1}{2}(\lambda_\varepsilon^m)^2 + \frac{1}{2}(\lambda_s^m)^2]\sigma_t^2 \\
&\quad + [A_n + B_n\mu_s + C_n\mu_\sigma] + B_n\rho_s s_t + C_n\rho_\sigma\sigma_t^2 + \frac{1}{2}(B_n\varphi_s)^2\sigma_t^2 + \frac{1}{2}(C_n)^2\sigma_\sigma^2 - \lambda_s^m(B_n\varphi_s)\sigma_t^2 - \lambda_\sigma^m(C_n)\sigma_\sigma^2 \\
&= [A_n + B_n\mu_s + C_n\mu_\sigma - \mu_m + \frac{1}{2}(\lambda_\sigma^m - C_n)^2\sigma_\sigma^2] + [B_n\rho_s - \rho_s^m]s_t + [C_n\rho_\sigma - \rho_\sigma^m + \frac{1}{2}(\lambda_\varepsilon^m)^2 + \frac{1}{2}(\lambda_s^m - B_n\varphi_s)^2]\sigma_t^2
\end{aligned}$$

Therefore, we verify the assumed linearity of log bond prices (58) and verify (59), (60), and (61) as the rescursive solution for A_n, B_n, C_n. □

Define the one-period holding return on a n-period bond as

$$r_{t+1}^{n} = b_{t+1}^{n-1} - b_t^n \tag{A.23}$$

Implied by (A.22), where n is replaced by $n-1$, the risk premium on an n-period bond over a 1-period bond (the term preimum) is given by (62)

$$\begin{aligned}\mathbb{E}_t[r_{t+1}^{bn}] - r_t^f + \frac{1}{2}\mathbb{V}_t[r_{t+1}^{bn}] &= \mathbb{E}_t[b_{t+1}^{n-1}] - b_t^n + b_t^1 + \frac{1}{2}\mathbb{V}_t[b_{t+1}^{n-1}] \\ &= -COV_t[m_{t+1}, b_{t+1}^{n-1}] = \lambda_s^m(B_{n-1}\varphi_s)\sigma_t^2 + \lambda_\sigma^m C_{n-1}\sigma_\sigma^2\end{aligned}$$

A.5 Term structure of equity premia

Let D_t^n denote the time-t price of a claim to the dividend paid in n periods. The price of the dividend claim follows the recursive relationship

$$D_t^n = \mathbb{E}_t[M_{t+1}D_{t+1}^{n-1}] \tag{A.24}$$

When $n = 1$, the price of the first dividend strip is simply given by ($D_{t+1}^0 = D_{t+1}$)

$$D_t^1 = \mathbb{E}_t[M_{t+1}D_{t+1}] \tag{A.25}$$

Scaled by the ex-dividend value, it can be solved as follows

$$\frac{D_t^1}{P_t} = \mathbb{E}_t\left[M_{t+1}\frac{D_{t+1}}{P_{t+1}}\frac{P_{t+1}}{P_t}\right] \tag{A.26}$$

Given

$$\begin{aligned}dp_{t+1} + \Delta p_{t+1} &= a_6 + (b_6+\eta)ik_{t+1} - \eta ik_t + \Delta\varepsilon_{t+1} \\ &= a_6 + (b_6+\eta)(\alpha + \beta s_{t+1} + \phi\sigma_{t+1}^2) - \eta(\alpha + \beta s_t + \phi\sigma_t^2) + \Delta(\mu_\varepsilon + s_t + \sigma_t e_{\varepsilon,t+1}) \\ &= [a_6 + b_6\alpha + \mu_\varepsilon + (b_6+\eta)\beta\mu_s + (b_6+\eta)\phi\mu_\sigma] + [(b_6+\eta)\beta\mu_s + 1 - \eta\beta]s_t + [(b_6+\eta)\phi\rho_\sigma]\sigma_t^2 \\ &\quad + \sigma_t e_{\varepsilon,t+1} + [(b_6+\eta)\beta\varphi_s]\sigma_t e_{s,t+1} + [(b_6+\eta)\phi]\sigma_\sigma e_{\sigma,t+1}\end{aligned}$$

We can calculate the price

$$\begin{aligned}d_t^1 - p_t &= \mathbb{E}_t(m_{t+1}) + \frac{1}{2}\mathbb{V}_t(m_{t+1}) + \mathbb{E}_t(dp_{t+1} + \Delta p_{t+1}) + \frac{1}{2}\mathbb{V}_t(dp_{t+1} + \Delta p_{t+1}) + COV_t(m_{t+1}, dp_{t+1} + \Delta p_{t+1}) \\ &= [-\mu_m + \frac{1}{2}(\lambda_\sigma^m)^2\sigma_\sigma^2] - \rho_s^m s_t + [-\rho_\sigma^m + \frac{1}{2}(\lambda_\varepsilon^m)^2 + \frac{1}{2}(\lambda_s^m)^2]\sigma_t^2 \\ &\quad + [a_6 + b_6\alpha + \mu_\varepsilon + (b_6+\eta)\beta\mu_s + (b_6+\eta)\phi\mu_\sigma] + [(b_6+\eta)\beta\mu_s + 1 - \eta\beta]s_t + [(b_6+\eta)\phi\rho_\sigma]\sigma_t^2 \\ &\quad + \frac{1}{2}\sigma_t^2 + \frac{1}{2}[(b_6+\eta)\beta\varphi_s]^2\sigma_t^2 + \frac{1}{2}[(b_6+\eta)\phi]^2\sigma_\sigma^2 - \lambda_\varepsilon^m\sigma_t^2 - \lambda_s^m[(b_6+\eta)\beta\varphi_s]\sigma_t^2 - \lambda_\sigma^m\frac{1}{2}[(b_6+\eta)\phi]\sigma_\sigma^2 \\ &= [a_6 + b_6\alpha + \mu_\varepsilon + (b_6+\eta)\beta\mu_s + (b_6+\eta)\phi\mu_\sigma - \mu_m + \frac{1}{2}(\lambda_\sigma^m - (b_6+\eta)\phi)^2\sigma_\sigma^2] \\ &\quad + [(b_6+\eta)\beta\rho_s - \eta\beta + 1 - \rho_s^m]s_t + [(b_6+\eta)\phi\rho_\sigma - \rho_\sigma^m + \frac{1}{2}(\lambda_\varepsilon^m - 1)^2 - \frac{1}{2}(\lambda_s^m - (b_6+\eta)\beta\varphi_s)^2]\sigma_t^2\end{aligned}$$

The log price of 1-period dividend claim scaled by the ex-dividend value is affine in state variables. Therefore we conjecture that the price of a n-period dividend claim scaled by the ex-dividend value is given by (63)

$$d_t^n - p_t = A_n' + B_n' s_t + C_n'\sigma_t^2$$

where coefficients are given by (64), (65), (66)

$$
\begin{aligned}
A'_{n+1} &= A'_n + a_4 + b_4\alpha + \mu_\varepsilon + [B'_n + (b_4+\eta)\beta]\mu_s + [C'_n + (b_4+\eta)\phi]\mu_\sigma \\
&\quad - \mu_m + \frac{1}{2}[\lambda^m_\sigma - (C'_n + (b_4+\eta)\phi)]^2\sigma^2_\sigma \\
B'_{n+1} &= [B'_n + (b_4+\eta)\beta]\rho_s - \eta\beta + 1 - \rho^m_s \\
C'_{n+1} &= [C'_n + (b_4+\eta)\phi]\rho_\sigma - \rho^m_\sigma + \frac{1}{2}(\lambda^m_\varepsilon - 1)^2 + \frac{1}{2}[\lambda^m_s - (B_n + (b_4+\eta)\beta)\varphi_s]^2
\end{aligned}
$$

Proof: First, note that when $n = 0$

$$
d^0_t - p_t = dp_t = (a_5 + b_5\alpha) + b_5\beta s_t + b_5\phi\sigma^2_t \implies A'_0 = a_5 + b_5\alpha,\ B'_0 = b_5\beta,\ C'_0 = b_5\phi
$$

Given A'_0, B'_0, C'_0, we can calculate and verify A'_1, B'_1, C'_1

$$
\begin{aligned}
A'_1 &= a_6 + b_6\alpha + \mu_\varepsilon + (b_6+\eta)\beta\mu_s + (b_6+\eta)\phi\mu_\sigma - \mu_m + \frac{1}{2}[\lambda^m_\sigma - (b_6+\eta)\phi]^2\sigma^2_\sigma \\
B'_1 &= (b_6+\eta)\beta\rho_s - \eta\beta + 1 - \rho^m_s \\
C'_1 &= (b_6+\eta)\phi\rho_\sigma - \rho^m_\sigma + \frac{1}{2}(\lambda^m_\varepsilon - 1)^2 + \frac{1}{2}[\lambda^m_s - (b_6+\eta)\beta\varphi_s]^2
\end{aligned}
$$

which are consistent with the price conjecture and coefficients above.

Second, using the recursive relation (A.24), we have

$$
\begin{aligned}
d^{n+1}_t - p_t &= \mathbb{E}_t[m_{t+1}] + \frac{1}{2}\mathbb{V}_t[m_{t+1}] + \mathbb{E}_t[d^n_{t+1} - p_{t+1} + \Delta p_{t+1}] + \frac{1}{2}\mathbb{V}_t[d^n_{t+1} - p_{t+1} + \Delta p_{t+1}] \\
&\quad + COV_t[m_{t+1}, d^n_{t+1} - p_{t+1} + \Delta p_{t+1}] \\
&= [-\mu_m + \frac{1}{2}(\lambda^m_\sigma)^2\sigma^2_\sigma] - \rho^m_s s_t + [-\rho^m_\sigma + \frac{1}{2}(\lambda^m_\varepsilon)^2 + \frac{1}{2}(\lambda^m_s)^2]\sigma^2_t \\
&\quad + [A'_n + a_4 + b_4\alpha + \mu_\varepsilon + (B'_n + (b_4+\eta)\beta)\mu_s + (C'_n + (b_4+\eta)\phi)\mu_\sigma] \\
&\quad + [(B'_n + (b_4+\eta)\beta)\rho_s - \eta\beta + 1]s_t + [(C'_n + (b_4+\eta)\phi)\rho_\sigma]\sigma^2_t \\
&\quad + \frac{1}{2}\sigma^2_t + \frac{1}{2}[(B'_n + (b_4+\eta)\beta)\varphi_s]^2\sigma^2_t + \frac{1}{2}[C'_n + (b_4+\eta)\phi]^2\sigma^2_\sigma \\
&\quad - \lambda^m_\varepsilon\sigma^2_t - \lambda^m_s[(B'_n + (b_4+\eta)\beta)\varphi_s]\sigma^2_t - \lambda^m_\sigma[C'_n + (b_4+\eta)\phi]\sigma^2_\sigma \\
&= [A'_n + a_4 + b_4\alpha + \mu_\varepsilon + (B'_n + (b_4+\eta)\beta)\mu_s + (C'_n + (b_4+\eta)\phi)\mu_\sigma - \mu_m + \frac{1}{2}[\lambda^m_\sigma - (C'_n + (b_4+\eta)\beta)]^2\sigma^2_\sigma] \\
&\quad + [(B'_n + (b_4+\eta)\beta)\rho_s - \eta\beta + 1 - \rho^m_s]s_t \\
&\quad + \big[(C'_n + (b_4+\eta)\phi)\rho_\sigma - \rho^m_\sigma + \frac{1}{2}(\lambda^m_\varepsilon - 1)^2 + \frac{1}{2}[\lambda^m_s - (B'_n + (b_4+\eta)\beta)\varphi_s]^2\big]\sigma^2_t
\end{aligned}
$$

from which we obtain the recursive formulae (64), (65), (66). □

Now we compute the one-period holding return on a n-periods dividend claim.

$$
R^n_{t+1} = \frac{D^{n-1}_{t+1}}{D^n_t} = \frac{D^{n-1}_{t+1}/K_{t+1}}{D^n_t/K_t}\frac{K_{t+1}}{K_t} \tag{A.27}
$$

$$
r^n_{t+1} = (d^{n-1}_{t+1} - k_{t+1}) - (d^n_t - k_t) + \Delta k_{t+1} \tag{A.28}
$$

The recursive relation above yields the risk premium on a n-period dividend claim in (67).

$$
\begin{aligned}
\mathbb{E}_t[r^n_{t+1}] + \frac{1}{2}\mathbb{V}_t[r^n_{t+1}] - r^f_t &= \mathbb{E}_t[d^{n-1}_{t+1} - k_{t+1}] + \frac{1}{2}\mathbb{V}_t[d^{n-1}_{t+1} - k_{t+1}] - (d^n_t - k_t) - r^f_t + \Delta k_{t+1} \\
&= -COV_t(m_{t+1}, d^{n-1}_{t+1} - k_{t+1}) \\
&= \lambda^m_\varepsilon\sigma^2_t + \lambda^m_s[(B'_{n-1} + (b_4+\eta)\beta)\varphi_s]\sigma^2_t + \lambda^m_\sigma[C'_{n-1} + (b_4+\eta)\phi]\sigma^2_\sigma
\end{aligned}
$$

A.6 Recovery of SDF

The expected profitability growth and the uncertainty of the profitability growth and their shocks can be recovered from data of the risk-free rate and the investment rate.

$$\begin{bmatrix} r_t^f \\ ik_t \end{bmatrix} = \begin{bmatrix} a \\ \alpha \end{bmatrix} + \begin{bmatrix} b & c \\ \beta & \phi \end{bmatrix} \begin{bmatrix} s_t \\ \sigma_t^2 \end{bmatrix}$$

where, for convenience, $a \equiv \mu_m - \frac{1}{2}(\lambda_\sigma^m)^2\sigma_\sigma^2,\ \ b \equiv \rho_s^m,\ \ c \equiv \rho_\sigma^m - \frac{1}{2}(\lambda_\varepsilon^m)^2 - \frac{1}{2}(\lambda_s^m)^2.$

The solution is given by

$$\begin{bmatrix} s_t \\ \sigma_t^2 \end{bmatrix} = \begin{bmatrix} b & c \\ \beta & \phi \end{bmatrix}^{-1} \begin{bmatrix} r_t^f - a \\ ik_t - \alpha \end{bmatrix} = \frac{1}{b\phi - \beta c} \begin{bmatrix} \phi & -c \\ -\beta & b \end{bmatrix} \begin{bmatrix} r_t^f - a \\ ik_t - \alpha \end{bmatrix}$$

In short-hand notations, state variables are recovered as

$$s_t = \eta_{s,0} + \eta_{s,1} r_t^f + \eta_{s,2} ik_t$$
$$\sigma_t^2 = \eta_{\sigma,0} + \eta_{\sigma,1} r_t^f + \eta_{\sigma,2} ik_t$$

where coefficients are given by

$$\eta_{s,0} = \frac{-\phi a + c\alpha}{b\phi - \beta c},\ \ \eta_{s,1} = \frac{\phi}{b\phi - \beta c},\ \ \eta_{s,2} = \frac{-c}{b\phi - \beta c} \tag{A.29}$$
$$\eta_{\sigma,0} = \frac{\beta a - b\alpha}{b\phi - \beta c},\ \ \ \eta_{\sigma,1} = \frac{-\beta}{b\phi - \beta c},\ \ \eta_{\sigma,2} = \frac{b}{b\phi - \beta c} \tag{A.30}$$

Their shocks can be recovered as

$$\varphi_s \sigma_t e_{s,t+1} = (1-\rho_s)(\eta_{s,0} - \bar{s}) + \eta_{s,1}(r_{t+1}^f - \rho_s r_t^f) + \eta_{s,2}(ik_{t+1} - \rho_s ik_t)$$
$$\sigma_\sigma e_{\sigma,t+1} = (1-\rho_\sigma)(\eta_{\sigma,0} - \bar{\sigma}^2) + \eta_{\sigma,1}(r_{t+1}^f - \rho_\sigma r_t^f) + \eta_{\sigma,2}(ik_{t+1} - \rho_\sigma ik_t)$$

The remaining shock to the profitability growth can be recovered from profitability growth.

$$\sigma_t e_{\varepsilon,t+1} = \Delta\varepsilon_{t+1} - s_t - \mu_\varepsilon$$

The SDF therefore can be recovered as

$$m_{t+1} = -\mu_m' - \rho_{r_f}^m r_t^f - \rho_{ik}^m ik_t - \lambda_{r_f}^m \Delta r_{t+1}^f - \lambda_{ik}^m \Delta ik_{t+1} - \lambda_\varepsilon^m \Delta\varepsilon_{t+1}$$

where coefficients are given by

$$\mu_m' = \mu_m - \lambda_\varepsilon^m \mu_\varepsilon + (\rho_s^m - \lambda_\varepsilon^\eta)\eta_{s,0} + \rho_\sigma^m \eta_{\sigma,0}$$
$$+ \lambda_s^m \frac{1}{\varphi_s}(1-\rho_s)(\eta_{s,0} - \bar{s}) + \lambda_\sigma^m (1-\rho_\sigma)(\eta_{\sigma,0} - \bar{\sigma}^2)$$
$$\rho_{r_f}^m = (\rho_s^m - \lambda_\varepsilon^m)\eta_{s,1} + \rho_\sigma^m \eta_{\sigma,1} + \lambda_s^m \frac{1}{\varphi_s}\eta_{s,1}(1-\rho_s) + \lambda_\sigma^m \eta_{\sigma,1}(1-\rho_\sigma)$$
$$\rho_{ik}^m = (\rho_s^m - \lambda_\varepsilon^m)\eta_{s,2} + \rho_\sigma^m \eta_{\sigma,2} + \lambda_s^m \frac{1}{\varphi_s}\eta_{s,2}(1-\rho_s) + \lambda_\sigma^m \eta_{\sigma,2}(1-\rho_\sigma)$$
$$\lambda_{r_f}^m = \lambda_s^m \frac{1}{\varphi_s}\eta_{s,1} + \lambda_\sigma^m \eta_{\sigma,1}$$
$$\lambda_{ik}^m = \lambda_s^m \frac{1}{\varphi_s}\eta_{s,2} + \lambda_\sigma^m \eta_{\sigma,2}$$

B Estimation

B.1 Target moments

Unconditional moments of state variables are given by

$$\begin{aligned}
\mathbb{V}(\sigma_t e_{s,t+1}) &= \mathbb{E}(\sigma_t^2 e_{s,t+1}^2) - \mathbb{E}(\sigma_t e_{s,t+1})^2 = \mathbb{E}(\mathbb{E}_t(\sigma_t^2 e_{s,t+1}^2)) - \mathbb{E}(\mathbb{E}_t(\sigma_t e_{s,t+1}))^2 \\
&= \mathbb{E}(\sigma_t^2 \mathbb{E}_t(e_{s,t+1}^2)) - \mathbb{E}(\sigma_t \mathbb{E}_t(e_{s,t+1}))^2 = \mathbb{E}(\sigma_t^2) = \bar{\sigma}^2 \\
\mathbb{V}(s_t) &= \rho_s^2 \mathbb{V}(s_{t-1}) + \mathbb{V}(\varphi_s \sigma_t e_{s,t+1}) \implies \mathbb{V}(s_t) = \varphi_s^2 \bar{\sigma}^2/(1-\rho_s^2) \\
\mathbb{V}(\sigma_t^2) &= \rho_\sigma^2 \mathbb{V}(\sigma_{t-1}^2) + \sigma_\sigma^2 \implies \mathbb{V}(\sigma_t^2) = \sigma_\sigma^2/(1-\rho_\sigma^2)
\end{aligned}$$

1. The maximum possible Sharpe ratio in the economy.

$$\frac{\mathbb{E}[R_t^e]}{\mathbb{V}[R_t^e]^{\frac{1}{2}}} \leq \frac{\mathbb{V}[M_t]^{\frac{1}{2}}}{\mathbb{E}[M_t]} = \left(\exp\left[\mathbb{V}[m_t]\right] - 1\right)^{\frac{1}{2}} \equiv maxSR$$

where the first inequality is derived from $0 = \mathbb{E}[M_t R_t^e]$ and the first equality is due to the property of the log-normality of SDF. As $e^x - 1 \approx x$ for small x, we may simply have

$$\begin{aligned}
maxSR \approx \mathbb{V}[m_t]^{\frac{1}{2}} &= [(\rho_s^m)^2 \mathbb{V}(s_t) + (\rho_\sigma^m)^2 \mathbb{V}(\sigma_t^2) + (\lambda_\varepsilon^m)^2 \mathbb{V}(\sigma_t e_{\varepsilon,t+1}) + (\lambda_s^m)^2 \mathbb{V}(\sigma_t e_{s,t+1}) + (\lambda_\sigma^m)^2 \sigma_\sigma^2]^{\frac{1}{2}} \\
&= [(\rho_s^m)^2 \frac{\varphi_s^2 \bar{\sigma}^2}{1-\rho_s^2} + (\rho_\sigma^m)^2 \frac{\sigma_\sigma^2}{1-\rho_\sigma^2} + (\lambda_\varepsilon^m)^2 \bar{\sigma}^2 + (\lambda_s^m)^2 \bar{\sigma}^2 + (\lambda_\sigma^m)^2 \sigma_\sigma^2]^{\frac{1}{2}}
\end{aligned}$$

2. The mean, variance, and autocovariance of the risk-free rate in the economy

$$\begin{aligned}
r_t^f &= \big[\mu_m - \frac{1}{2}(\lambda_\sigma^m)^2 \sigma_\sigma^2\big] + \rho_s^m s_t + \big[\rho_\sigma^m - \frac{1}{2}(\lambda_\varepsilon^m)^2 - \frac{1}{2}(\lambda_s^m)^2\big]\sigma_t^2 \\
\mathbb{E}(r_t^f) &= \big[\mu_m - \frac{1}{2}(\lambda_\sigma^m)^2 \sigma_\sigma^2\big] + \rho_s^m \bar{s} + \big[\rho_\sigma^m - \frac{1}{2}(\lambda_\varepsilon^m)^2 - \frac{1}{2}(\lambda_s^m)^2\big]\bar{\sigma}^2 \\
\mathbb{V}(r_t^f) &= (\rho_s^m)^2 \mathbb{V}(s_t) + \left(\rho_\sigma^m - \frac{1}{2}(\lambda_\varepsilon^m)^2 - \frac{1}{2}(\lambda_s^m)^2\right)^2 \mathbb{V}(\sigma_t^2) \\
&= (\rho_s^m)^2 \frac{\varphi_s^2 \bar{\sigma}^2}{1-\rho_s^2} + \left(\rho_\sigma^m - \frac{1}{2}(\lambda_\varepsilon^m)^2 - \frac{1}{2}(\lambda_s^m)^2\right)^2 \frac{\sigma_\sigma^2}{1-\rho_\sigma^2} \\
COV(r_t^f, r_{t-1}^f) &= COV(\rho_s^m s_t + \big[\rho_\sigma^m - \frac{1}{2}(\lambda_\varepsilon^m)^2 - \frac{1}{2}(\lambda_s^m)^2\big]\sigma_t^2,\ \rho_s^m s_{t-1} + \big[\rho_\sigma^m - \frac{1}{2}(\lambda_\varepsilon^m)^2 - \frac{1}{2}(\lambda_s^m)^2\big]\sigma_{t-1}^2) \\
&= (\rho_s^m)^2 \rho_s \mathbb{V}(s_t) + \left(\rho_\sigma^m - \frac{1}{2}(\lambda_\varepsilon^m)^2 - \frac{1}{2}(\lambda_s^m)^2\right)^2 \rho_\sigma \mathbb{V}(\sigma_t^2) \\
&= (\rho_s^m)^2 \rho_s \frac{\varphi_s^2 \bar{\sigma}^2}{1-\rho_s^2} + \left(\rho_\sigma^m - \frac{1}{2}(\lambda_\varepsilon^m)^2 - \frac{1}{2}(\lambda_s^m)^2\right)^2 \rho_\sigma \frac{\sigma_\sigma^2}{1-\rho_\sigma^2}
\end{aligned}$$

3. The mean, variance, and autocovariance of the investment rate

$$\begin{aligned}
ik_t &= \alpha + \beta s_t + \phi \sigma_t^2 \\
\mathbb{E}(ik_t) &= \alpha + \beta \mathbb{E}(s_t) + \phi \mathbb{E}(\sigma_t^2) \\
&= \alpha + \beta \bar{s} + \phi \bar{\sigma}^2 \\
\mathbb{V}(ik_t) &= \beta^2 \mathbb{V}(s_t) + \phi^2 \mathbb{V}(\sigma_t^2) \\
&= \beta^2 \frac{\varphi_s^2 \bar{\sigma}^2}{1-\rho_s^2} + \phi^2 \frac{\sigma_\sigma^2}{1-\rho_\sigma^2} \\
COV(ik_t, ik_{t-1}) &= COV(\beta s_t + \phi \sigma_t^2,\ \beta s_{t-1} + \phi \sigma_{t-1}^2) \\
&= \beta^2 \rho_s \frac{\varphi_s^2 \bar{\sigma}^2}{1-\rho_s^2} + \phi^2 \rho_\sigma \frac{\sigma_\sigma^2}{1-\rho_\sigma^2}
\end{aligned}$$

The covariance between the risk-free rate and the investment rate

$$
\begin{aligned}
COV(ik_t,\ r_t^f) &= COV(\beta s_t + \phi\sigma_t^2,\ \rho_s^m s_t + \big[\rho_\sigma^m - \frac{1}{2}(\lambda_\varepsilon^m)^2 - \frac{1}{2}(\lambda_s^m)^2\big]\sigma_t^2) \\
&= \beta\rho_s^m \mathbb{V}(s_t) + \phi\big[\rho_\sigma^m - \frac{1}{2}(\lambda_\varepsilon^m)^2 - \frac{1}{2}(\lambda_s^m)^2\big]\mathbb{V}(\sigma_t^2) \\
&= \beta\rho_s^m \frac{\varphi_s^2\bar{\sigma}^2}{1-\rho_s^2} + \phi\big[\rho_\sigma^m - \frac{1}{2}(\lambda_\varepsilon^m)^2 - \frac{1}{2}(\lambda_s^m)^2\big]\frac{\sigma_\sigma^2}{1-\rho_\sigma^2}
\end{aligned}
$$

4. The mean and variance of the average excess return

$$
\begin{aligned}
r_{t+1}^e =& r_{t+1}^I - r_t^f = r_{t+1}^I + \mathbb{E}_t[m_{t+1}] + \frac{1}{2}\mathbb{V}_t[m_{t+1}] \\
=& [(a_1 - a_2) + (b_1 - b_2)\alpha + \mu_\varepsilon + b_1\beta\mu_s + b_1\phi\mu_\sigma] + [(b_1\rho_s - b_2)\beta + 1]s_t + [(b_1\rho_\sigma - b_2)\phi]\sigma_t^2 \\
&+ \sigma_t e_{\varepsilon,t+1} + (b_1\beta\phi_s)\sigma_t e_{s,t+1} + (b_1\phi)\sigma_\sigma e_{\sigma,t+1} \\
&+ \big[-\mu_m + \frac{1}{2}(\lambda_\sigma^m)^2\sigma_\sigma^2\big] - \rho_s^m s_t + \big[-\rho_\sigma^m + \frac{1}{2}(\lambda_\varepsilon^m)^2 + \frac{1}{2}(\lambda_s^m)^2\big]\sigma_t^2 \\
=& [(a_1 - a_2) + (b_1 - b_2)\alpha + \mu_\varepsilon + b_1\beta\mu_s + b_1\phi\mu_\sigma - \mu_m + \frac{1}{2}(\lambda_\sigma^m)^2\sigma_\sigma^2] \\
&+ [(b_1\rho_s - b_2)\beta + 1 - \rho_s^m]s_t + [(b_1\rho_\sigma - b_2)\phi - \rho_\sigma^m + \frac{1}{2}(\lambda_\varepsilon^m)^2 + \frac{1}{2}(\lambda_s^m)^2\big]\sigma_t^2 \\
&+ \sigma_t e_{\varepsilon,t+1} + (b_1\beta\phi_s)\sigma_t e_{s,t+1} + (b_1\phi)\sigma_\sigma e_{\sigma,t+1} \\
=& [\frac{1}{2}(\lambda_\sigma^m)^2\sigma_\sigma^2 - \frac{1}{2}(\lambda_\sigma^m - b_1\phi)^2\sigma_\sigma^2] + [\frac{1}{2}(\lambda_\varepsilon^m)^2 + \frac{1}{2}(\lambda_s^m)^2 - \frac{1}{2}(\lambda_\varepsilon^m - 1)^2 - \frac{1}{2}(\lambda_s^m - b_1\beta\varphi_s)^2]\sigma_t^2 \\
&+ \sigma_t e_{\varepsilon,t+1} + (b_1\beta\phi_s)\sigma_t e_{s,t+1} + (b_1\phi)\sigma_\sigma e_{\sigma,t+1} \\
=& \frac{1}{2}\sigma_\sigma^2[(2\lambda_\sigma^m - b_1\phi)b_1\phi] + \frac{1}{2}\sigma_t^2[(2\lambda_\varepsilon^m - 1) + (2\lambda_s^m - b_1\beta\varphi_s)b_1\beta\varphi_s] \\
&+ \sigma_t e_{\varepsilon,t+1} + (b_1\beta\varphi_s)\sigma_t e_{s,t+1} + (b_1\phi)\sigma_\sigma e_{\sigma,t+1}
\end{aligned}
$$

whose moments are

$$
\begin{aligned}
\mathbb{E}(r_{t+1}^e) &= \frac{1}{2}\sigma_\sigma^2[(2\lambda_\sigma^m - b_1\phi)b_1\phi] + \frac{1}{2}\bar{\sigma}^2[(2\lambda_\varepsilon^m - 1) + (2\lambda_s^m - b_1\beta\varphi_s)b_1\beta\varphi_s] \\
\mathbb{V}(r_{t+1}^e) &= \left[\frac{1}{2}[(2\lambda_\varepsilon^m - 1) + (2\lambda_s^m - b_1\beta\varphi_s)b_1\beta\varphi_s]\right]^2 \frac{\sigma_\sigma^2}{1-\rho_\sigma^2} + \bar{\sigma}^2 + (b_1\beta\varphi_s)^2\bar{\sigma}^2 + (b_1\phi)^2\sigma_\sigma^2
\end{aligned}
$$

The covariance between the excess return and the investment rate

$$
\begin{aligned}
COV(ik_t,\ \mathbb{E}_t(r_{t+1}^e)) &= COV(\beta s_t + \phi\sigma_t^2,\ \frac{1}{2}\sigma_t^2[(2\lambda_\varepsilon^m - 1) + (2\lambda_s^m - b_1\beta\varphi_s)b_1\beta\varphi_s]) \\
&= \frac{1}{2}\phi[(2\lambda_\varepsilon^m - 1) + (2\lambda_s^m - b_1\beta\varphi_s)b_1\beta\varphi_s]\mathbb{V}(\sigma_t^2) \\
&= \frac{1}{2}\phi[(2\lambda_\varepsilon^m - 1) + (2\lambda_s^m - b_1\beta\varphi_s)b_1\beta\varphi_s]\frac{\sigma_\sigma^2}{1-\rho_\sigma^2}
\end{aligned}
$$

5. The mean and variance of the dividend-price ratio.

$$
\begin{aligned}
dp_t &= a_5 + b_5 ik_t \\
\mathbb{E}[dp_t] &= a_5 + b_5(\alpha + \beta\bar{s} + \phi\bar{\sigma}^2) \\
\mathbb{V}[dp_t] &= (b_5)^2\left[\beta^2\frac{\varphi_s^2\bar{\sigma}^2}{1-\rho_s^2} + \phi^2\ \frac{\sigma_\sigma^2}{1-\rho_\sigma^2}\right] \\
COV(dp_t, dp_{t-1}) &= COV(b_5(\beta s_t + \phi\sigma_t^2),\ b_5(\beta s_{t-1} + \phi\sigma_{t-1}^2)) \\
&= (b_5)^2\left[\beta^2\rho_s\frac{\varphi_s^2\bar{\sigma}^2}{1-\rho_s^2} + \phi^2\rho_\sigma\frac{\sigma_\sigma^2}{1-\rho_\sigma^2}\right]
\end{aligned}
$$

The covariance between dp_t and ik_t, r_t^f are given by

$$
\begin{aligned}
COV(dp_t,\ ik_t) &= b_5\left[\beta^2 \frac{\varphi_s^2\bar{\sigma}^2}{1-\rho_s^2} + \phi^2\ \frac{\sigma_\sigma^2}{1-\rho_\sigma^2}\right] \\
COV(dp_t,\ r_t^f) &= b_5\left[\beta\rho_s^m \frac{\varphi_s^2\bar{\sigma}^2}{1-\rho_s^2} + \phi\big[\rho_\sigma^m - \frac{1}{2}(\lambda_\varepsilon^m)^2 - \frac{1}{2}(\lambda_s^m)^2\big]\frac{\sigma_\sigma^2}{1-\rho_\sigma^2}\right] \\
COV(ik_t,\ \mathbb{E}_t(r_{t+1}^e)) &= b_5\left[\frac{1}{2}\phi[(2\lambda_\varepsilon^m - 1) + (2\lambda_s^m - b_1\beta\varphi_s)b_1\beta\varphi_s]\frac{\sigma_\sigma^2}{1-\rho_\sigma^2}\right]
\end{aligned}
$$

6. The mean and variance of the dividend growth.

$$
\begin{aligned}
\Delta d_{t+1} &= a_4 + (b_6+\eta)ik_{t+1} - (b_5+\eta)ik_t + \Delta\varepsilon_{t+1} \\
\mathbb{E}[\Delta d_{t+1}] &= a_4 + b_4\mathbb{E}(ik_t) + \mathbb{E}(\Delta\varepsilon_{t+1}) \\
\mathbb{V}[\Delta d_{t+1}] &=
\end{aligned}
$$

B.2 Estimation strategy

We choose the parameter vector Θ that minimizes weighted mean squared errors between the model-implied moments $X(\Theta)$ and actual moments of data X.

$$\hat{\Theta} = \arg\min_{\Theta} \big(X - X(\Theta)\big)' W \big(X - X(\Theta)\big)$$

We choose the following moments

$$\begin{cases}
\mathbb{E}(r_t^f) & = \big[\mu_m - \frac{1}{2}(\lambda_\sigma^m)^2\sigma_\sigma^2\big] + \rho_s^m \bar{s} + \big[\rho_\sigma^m - \frac{1}{2}(\lambda_\varepsilon^m)^2 - \frac{1}{2}(\lambda_s^m)^2\big]\bar{\sigma}^2 \\
\mathbb{V}(r_t^f) & = (\rho_s^m)^2 \frac{\varphi_s^2\bar{\sigma}^2}{1-\rho_s^2} + \left(\rho_\sigma^m - \frac{1}{2}(\lambda_\varepsilon^m)^2 - \frac{1}{2}(\lambda_s^m)^2\right)^2 \frac{\sigma_\sigma^2}{1-\rho_\sigma^2} \\
COV(r_t^f, r_{t-1}^f) & = (\rho_s^m)^2 \rho_s \frac{\varphi_s^2\bar{\sigma}^2}{1-\rho_s^2} + \left(\rho_\sigma^m - \frac{1}{2}(\lambda_\varepsilon^m)^2 - \frac{1}{2}(\lambda_s^m)^2\right)^2 \rho_\sigma \frac{\sigma_\sigma^2}{1-\rho_\sigma^2} \\
\mathbb{E}(ik_t) & = \alpha + \beta\bar{s} + \phi\bar{\sigma}^2 \\
\mathbb{V}(ik_t) & = \beta^2 \frac{\varphi_s^2\bar{\sigma}^2}{1-\rho_s^2} + \phi^2 \frac{\sigma_\sigma^2}{1-\rho_\sigma^2} \\
COV(ik_t, ik_{t-1}) & = \beta^2 \rho_s \frac{\varphi_s^2\bar{\sigma}^2}{1-\rho_s^2} + \phi^2 \rho_\sigma \frac{\sigma_\sigma^2}{1-\rho_\sigma^2} \\
\mathbb{E}(r_{t+1}^e) & = \frac{1}{2}\sigma_\sigma^2[(2\lambda_\sigma^m - b_1\phi)b_1\phi] + \frac{1}{2}\bar{\sigma}^2[(2\lambda_\varepsilon^m - 1) + (2\lambda_s^m - b_1\beta\varphi_s)b_1\beta\varphi_s] \\
\mathbb{V}(r_{t+1}^e) & = \left[\frac{1}{2}[(2\lambda_\varepsilon^m - 1) + (2\lambda_s^m - b_1\beta\varphi_s)b_1\beta\varphi_s]\right]^2 \frac{\sigma_\sigma^2}{1-\rho_\sigma^2} + \bar{\sigma}^2 + (b_1\beta\varphi_s)^2\bar{\sigma}^2 + (b_1\phi)^2\sigma_\sigma^2 \\
\mathbb{E}(\Delta\varepsilon_{t+1}) & = \mu_\varepsilon + \bar{s} \\
\mathbb{V}(\Delta\varepsilon_{t+1}) & = \frac{\varphi_s^2\bar{\sigma}^2}{1-\rho_s^2} + \bar{\sigma}^2 \\
COV(ik_t,\ r_t^f) & = \beta\rho_s^m \frac{\varphi_s^2\bar{\sigma}^2}{1-\rho_s^2} + \phi\big[\rho_\sigma^m - \frac{1}{2}(\lambda_\varepsilon^m)^2 - \frac{1}{2}(\lambda_s^m)^2\big] \frac{\sigma_\sigma^2}{1-\rho_\sigma^2} \\
COV(ik_t,\ r_{t+1}^e) & = \frac{1}{2}\phi[(2\lambda_\varepsilon^m - 1) + (2\lambda_s^m - b_1\beta\varphi_s)b_1\beta\varphi_s]\frac{\sigma_\sigma^2}{1-\rho_\sigma^2}
\end{cases}$$

where investment coefficients are given by

$$\beta = \frac{\rho_s^m - 1}{b_1\rho_s - b_2}, \quad \phi = \frac{\rho_\sigma^m - \frac{1}{2}(\lambda_\varepsilon^m - 1)^2 - \frac{1}{2}(\lambda_s^m - b_1\beta\varphi_s)^2}{b_1\rho_\sigma - b_2}$$

$$\alpha = \frac{\mu_m - \frac{1}{2}(\lambda_\sigma^m - b_1\phi)^2\sigma_\sigma^2 - (a_1 - a_2) - \mu_\varepsilon - b_1\beta\mu_s - b_1\phi\mu_\sigma}{b_1 - b_2}$$

and log-linearization constants are given by

$$a_1 = \log\big(1 + \kappa\frac{\eta}{\eta+1}\exp[(\eta+1)ik^*] + (1-\delta)\kappa\exp(\eta ik^*)\big) - \frac{\kappa\eta\exp(\eta ik^*)[\exp(ik^*) + (1-\delta)]}{1 + \kappa\frac{\eta}{\eta+1}\exp[(\eta+1)ik^*] + (1-\delta)\kappa\exp(\eta ik^*)} ik^*$$

$$b_1 = \frac{\kappa\eta\exp(\eta ik^*)[\exp(ik^*) + (1-\delta)]}{1 + \kappa\frac{\eta}{\eta+1}\exp[(\eta+1)ik^*] + (1-\delta)\kappa\exp(\eta ik^*)}, \quad a_2 = \log\kappa, \quad b_2 = \eta$$

C Transformation of productivity across states

In this section, we present a model in which the representation of production technology is flexible across states as in Cochrane (1993) and Belo (2010). Specifically, the producer is allowed to choose the state-contingent productivity level $\mathcal{E}_t$, subject to the constraint set defined by the following analytically tractable CES aggregator:

$$\mathbb{E}_t\left[\left(\frac{\mathcal{E}_{t+1}}{\Theta_{t+1}}\right)^{\alpha}\right]^{\frac{1}{\alpha}} \leq 1 \tag{C.1}$$

where Θ_{t+1} is the state-contingent natural productivity level and $\alpha > 1$ is a curvature parameter a controls the producer's ability to transform productivity across states.

With this novel technology, the producer chooses both the investment in the current period and the state-contingent productivity levels in the next period to maximize the cum-dividend value of the firm.

$$V(X_t) = \max_{\{I_t,\ \mathcal{E}_{t+1}\}} \left\{D_t + \mathbb{E}_t[M_{t+1}V(X_{t+1})]\right\}$$

where $X_t = (\Theta_t, \mathcal{E}_t, K_t)$, subject to the capital's law of motion (3), the flow of funds constraint (5), and the constrained set of productivity levels (C.1).

Form a Lagrangian and let λ_t be the Lagrange multiplier associated with the technology constraint.

$$\mathcal{L}_t = \mathcal{E}_t K_t - \frac{\kappa}{\eta+1} IK_t^{\eta+1}\mathcal{E}_t K_t + \mathbb{E}_t[M_{t+1}V(X_{t+1})] + \lambda_t\left(1 - \mathbb{E}_t\left[\left(\frac{\mathcal{E}_{t+1}}{\Theta_{t+1}}\right)^{\alpha}\right]^{\frac{1}{\alpha}}\right)$$

The first-order condition with respect to the investment remains unchanged. We have the same investment Euler equation (6) with the same investment return (7). The difference now is that we have a structure on the stochastic discount factor.

The first-order condition and the envelope condition with respect to the productivity $\mathcal{E}_{t+1}$ are given by

$$M_{t+1}V_{\mathcal{E}}(X_{t+1}) = \lambda_t \frac{\mathcal{E}_{t+1}^{\alpha-1}}{\Theta_{t+1}^{\alpha}} \tag{C.2}$$

$$V_{\mathcal{E}}(X_t) = K_t - \frac{\kappa}{\eta+1} IK_t^{\eta+1} K_t \tag{C.3}$$

Combining both conditions, we have

$$M_{t+1}K_{t+1}\left[1 - \frac{\kappa}{\eta+1} IK_{t+1}^{\eta+1}\right] = \lambda_t \frac{\mathcal{E}_{t+1}^{\alpha-1}}{\Theta_{t+1}^{\alpha}} \tag{C.4}$$

Taking expectation on both sides, we can substitute $\lambda_t = \mathbb{E}_t[M_{t+1}]K_{t+1}\left[1 - \frac{\kappa}{\eta+1} IK_{t+1}^{\eta+1}\right] / \mathbb{E}_t\left[\frac{\mathcal{E}_{t+1}^{\alpha-1}}{\Theta_{t+1}^{\alpha}}\right]$, leading to

$$\frac{M_{t+1}}{\mathbb{E}_t[M_{t+1}]} = \frac{\frac{\mathcal{E}_{t+1}^{\alpha-1}}{\Theta_{t+1}^{\alpha}}}{\mathbb{E}_t\left[\frac{\mathcal{E}_{t+1}^{\alpha-1}}{\Theta_{t+1}^{\alpha}}\right]} = \frac{\left(\frac{\mathcal{E}_{t+1}}{\mathcal{E}_t}\right)^{\alpha-1} / \left(\frac{\Theta_{t+1}}{\Theta_t}\right)^{\alpha}}{\mathbb{E}_t\left[\left(\frac{\mathcal{E}_{t+1}}{\mathcal{E}_t}\right)^{\alpha-1} / \left(\frac{\Theta_{t+1}}{\Theta_t}\right)^{\alpha}\right]} = \frac{\frac{\Delta\mathcal{E}_{t+1}^{\alpha-1}}{\Delta\Theta_{t+1}^{\alpha}}}{\mathbb{E}_t\left[\frac{\Delta\mathcal{E}_{t+1}^{\alpha-1}}{\Delta\Theta_{t+1}^{\alpha}}\right]} \tag{C.5}$$

This expression suggests that we can obtain the process of the SDF given processes of the natural productivity and the chosen productivity or we can obtain the process of the chosen productivity given processes of the natural productivity and the SDF.

Alternatively, apply the SDF (C.4) on the investment return, we have $\frac{\lambda_t}{K_{t+1}[1 - \frac{\kappa}{\eta+1} IK_{t+1}^{\eta+1}]} = \mathbb{E}_t\left[\frac{\mathcal{E}_{t+1}^{\alpha-1}}{\Theta_{t+1}^{\alpha}} R_{t+1}^I\right]^{-1}$, leading to

$$M_{t+1} = \mathbb{E}_t\left[\frac{\mathcal{E}_{t+1}^{\alpha-1}}{\Theta_{t+1}^{\alpha}} R_{t+1}^I\right]^{-1} \frac{\mathcal{E}_{t+1}^{\alpha-1}}{\Theta_{t+1}^{\alpha}} = \mathbb{E}_t\left[\frac{\Delta\mathcal{E}_{t+1}^{\alpha-1}}{\Delta\Theta_{t+1}^{\alpha}} R_{t+1}^I\right]^{-1} \frac{\Delta\mathcal{E}_{t+1}^{\alpha-1}}{\Delta\Theta_{t+1}^{\alpha}} \tag{C.6}$$

Now, as we do in the paper, we specify exogenously the process of the natural productivity and the SDF. Given these processes, we first solve for the process of the chosen productivity and then solve for the optimal investment.

C.1 Case I: $\Delta\theta_t$ is an AR(1)

We assume the log natural productivity growth is an AR(1) process with time-varying conditional volatility, which is also an AR(1). The SDF is driven by the natural productivity growth and the time-varying conditional volatility and subject to their shocks. Denote the log natural productivity growth as $s_t \equiv \Delta\theta_t$.

$$s_{t+1} = \mu_s + \rho_s s_t + \sigma_t e_{s,t+1} \tag{C.7}$$
$$\sigma_{t+1}^2 = \mu_\sigma + \rho_\sigma \sigma_t^2 + \sigma_\sigma e_{\sigma,t+1} \tag{C.8}$$
$$m_{t+1} = -\mu_m - \rho_s^m s_t - \rho_\sigma^m \sigma_t^2 - \lambda_s^m \sigma_t e_{s,t+1} - \lambda_\sigma^m \sigma_\sigma e_{\sigma,t+1} \tag{C.9}$$

where $\mu_s = (1-\rho_s)\bar{s} = (1-\rho_s)\Delta\bar{\theta}$ and $\mu_\sigma = (1-\rho_\sigma)\bar{\sigma}^2$. When $\rho_s = 0$, the natural productivity growth is i.i.d. When $\rho_s = 1$, the natural productivity growth is a random walk.

We first solve for the chosen productivity. We conjecture and verify that the chosen productivity has the following functional form

$$\Delta\varepsilon_{t+1} = \mu_\varepsilon + \rho_s^\varepsilon s_t + \rho_\sigma^\varepsilon \sigma_t^2 + \lambda_s^\varepsilon \sigma_t e_{s,t+1} + \lambda_\sigma^\varepsilon \sigma_\sigma e_{\sigma,t+1} \tag{C.10}$$

The last equality of expression (C.5) implies the following log-linear relations between the SDF and productivity growth.

$$m_{t+1} - \log \mathbb{E}_t[m_{t+1}] = [(\alpha-1)\Delta\varepsilon_{t+1} - \alpha\Delta\theta_{t+1}] - \log \mathbb{E}_t[\exp((\alpha-1)\Delta\varepsilon_{t+1} - \alpha\Delta\theta_{t+1})]$$
$$m_{t+1} - \mathbb{E}_t[m_{t+1}] - \frac{1}{2}\mathbb{V}_t[m_{t+1}] = [(\alpha-1)\Delta\varepsilon_{t+1} - \alpha\Delta\theta_{t+1}] - \mathbb{E}_t[(\alpha-1)\Delta\varepsilon_{t+1} - \alpha\Delta\theta_{t+1}] - \frac{1}{2}\mathbb{V}_t[(\alpha-1)\Delta\varepsilon_{t+1} - \alpha\Delta\theta_{t+1}]$$
$$\begin{aligned} m_{t+1} - \mathbb{E}_t[m_{t+1}] - \frac{1}{2}\mathbb{V}_t[m_{t+1}] &= (\alpha-1)(\Delta\varepsilon_{t+1} - \mathbb{E}_t[\Delta\varepsilon_{t+1}]) - \frac{1}{2}(\alpha-1)^2\mathbb{V}_t[\Delta\varepsilon_{t+1}] \\ &\quad - \alpha(\Delta\theta_{t+1} - \mathbb{E}_t[\Delta\theta_{t+1}]) - \frac{1}{2}\alpha^2\mathbb{V}_t[\Delta\theta_{t+1}] + (\alpha-1)\alpha COV_t(\Delta\varepsilon_{t+1}, \Delta\theta_{t+1}) \end{aligned} \tag{C.11}$$

Plugging in processes of the SDF, the natural productivity and the chosen productivity, we have the LHS and RHS of (C.11)

$$LHS = -\lambda_s^m \sigma_t e_{s,t+1} - \lambda_\sigma^m \sigma_\sigma e_{\sigma,t+1} - \frac{1}{2}[(\lambda_s^m)^2\sigma_t^2 + (\lambda_\sigma^m)^2\sigma_\sigma^2]$$
$$RHS = [(\alpha-1)\lambda_s^\varepsilon - \alpha]\sigma_t e_{s,t+1} + [(\alpha-1)\lambda_\sigma^\varepsilon]\sigma_\sigma e_{\sigma,t+1} - \frac{1}{2}[(\alpha-1)\lambda_s^\varepsilon - \alpha]^2\sigma_t^2 - \frac{1}{2}(\alpha-1)^2(\lambda_\sigma^\varepsilon)^2\sigma_\sigma^2$$

whose equality pins down λ_θ^s, $\lambda_\sigma^\varepsilon$

$$\lambda_s^\varepsilon = \frac{\alpha - \lambda_s^m}{\alpha - 1}, \quad \lambda_\sigma^\varepsilon = \frac{-\lambda_\sigma^m}{\alpha - 1}$$

We still need to pin down μ_ε, ρ_θ^ε, ρ_σ^ε using another condition. Since the productivity constraint (C.1) is always binding given that the value function is strictly increasing in productivity in each state of nature, we have

$$\begin{aligned} 1 &= \mathbb{E}_t[\exp(\alpha(\varepsilon_{t+1} - \theta_{t+1}))]^{\frac{1}{\alpha}} \\ 1 &= \mathbb{E}_t[\exp(\alpha(\Delta\varepsilon_{t+1} - \Delta\theta_{t+1}))]^{\frac{1}{\alpha}} \\ 0 &= \alpha\mathbb{E}_t[\Delta\varepsilon_{t+1} - \Delta\theta_{t+1}] + \frac{\alpha^2}{2}\mathbb{V}_t[\Delta\varepsilon_{t+1} - \Delta\theta_{t+1}] \\ 0 &= \mathbb{E}_t[\Delta\varepsilon_{t+1}] - \mathbb{E}_t[\Delta\theta_{t+1}] + \frac{\alpha}{2}\mathbb{V}_t[\Delta\varepsilon_{t+1}] + \frac{\alpha}{2}\mathbb{V}_t[\Delta\theta_{t+1}] - \alpha COV_t(\Delta\varepsilon_{t+1}, \Delta\theta_{t+1}) \end{aligned} \tag{C.12}$$

which yields

$$0 = [\mu_\varepsilon - \mu_s + \frac{\alpha}{2}(\lambda_\sigma^\varepsilon)^2\sigma_\sigma^2] + (\rho_s^\varepsilon - \rho_s)s_t + [\rho_\sigma^\varepsilon + \frac{\alpha}{2}(\lambda_s^m - 1)^2]\sigma_t^2$$

from which we obtain

$$\mu_\varepsilon = \mu_s - \frac{\alpha}{2}(\lambda_\sigma^\varepsilon)^2\sigma_\sigma^2, \;\; \rho_s^\varepsilon = \rho_s, \;\; \rho_\sigma^\varepsilon = -\frac{\alpha}{2}(\lambda_s^\varepsilon - 1)^2$$

In summary we have the coefficients of $\Delta\varepsilon_{t+1}$

$$\mu_\varepsilon = \mu_s - \frac{\alpha}{2}(\lambda_\sigma^\varepsilon)^2\sigma_\sigma^2 = \mu_s - \frac{\alpha}{2}\left(\frac{-\lambda_\sigma^m}{\alpha - 1}\right)^2\sigma_\sigma^2 \tag{C.13}$$

$$\rho_s^\varepsilon = \rho_s \tag{C.14}$$

$$\rho_\sigma^\varepsilon = -\frac{\alpha}{2}(\lambda_s^\varepsilon - 1)^2 = -\frac{\alpha}{2}\left(\frac{\alpha - \lambda_s^m}{\alpha - 1} - 1\right)^2 \tag{C.15}$$

$$\lambda_s^\varepsilon = \frac{\alpha - \lambda_s^m}{\alpha - 1} \tag{C.16}$$

$$\lambda_\sigma^\varepsilon = \frac{-\lambda_\sigma^m}{\alpha - 1} \tag{C.17}$$

Now we can solve for the optimal investment rate given the chosen productivity process using the investment Euler equation (16). We assume that the optimal investment rate has the following functional form

$$ik_t = \alpha + \beta s_t + \phi\sigma_t^2 \tag{C.18}$$

The investment return (10) now becomes

$$\begin{aligned}
r_{t+1}^I &= (a_1 - a_2) + (b_1 ik_{t+1} - b_2 ik_t) + \Delta\varepsilon_{t+1} \\
&= (a_1 - a_2) + b_1(\alpha + \beta s_{t+1} + \phi\sigma_{t+1}^2) - b_2(\alpha + \beta s_t + \phi\sigma_t^2) + \Delta\varepsilon_{t+1} \\
&= [(a_1 - a_2) + (b_1 - b_2)\alpha + b_1\beta\mu_s + b_1\phi\mu_\sigma + \mu_\varepsilon] + [(b_1\rho_s - b_2)\beta + \rho_s^\varepsilon]s_t + [(b_1\rho_\sigma - b_2)\phi + \rho_\sigma^\varepsilon]\sigma_t^2 \\
&\quad + (b_1\beta + \lambda_s^\varepsilon)\sigma_t e_{s,t+1} + (b_1\phi + \lambda_\sigma^\varepsilon)\sigma_\sigma e_{\sigma,t+1}
\end{aligned}$$

The investment Euler equation (16) yields

$$\begin{aligned}
0 &= \mathbb{E}_t[m_{t+1}] + \mathbb{E}_t[r_{t+1}^I] + \frac{1}{2}\mathbb{V}_t[m_{t+1}] + \frac{1}{2}\mathbb{V}_t[r_{t+1}^I] + COV_t(m_{t+1}, r_{t+1}^I) \\
&= (-\mu_m - \rho_s^m s_t - \rho_\sigma^m\sigma_t^2) + \frac{1}{2}[(\lambda_s^m)^2\sigma_t^2 + (\lambda_\sigma^m)^2\sigma_\sigma^2] \\
&\quad + [(a_1 - a_2) + (b_1 - b_2)\alpha + b_1\beta\mu_s + b_1\phi\mu_\sigma + \mu_\varepsilon] + [(b_1\rho_s - b_2)\beta + \rho_s^\varepsilon]s_t + [(b_1\rho_\sigma - b_2)\phi + \rho_\sigma^\varepsilon]\sigma_t^2 \\
&\quad + \frac{1}{2}(b_1\beta + \lambda_s^\varepsilon)^2\sigma_t^2 + \frac{1}{2}(b_1\phi + \lambda_\sigma^\varepsilon)^2\sigma_\sigma^2 - \lambda_s^m(b_1\beta + \lambda_s^\varepsilon)\sigma_t^2 - \lambda_\sigma^m(b_1\phi + \lambda_\sigma^\varepsilon)\sigma_\sigma^2 \\
&= [(a_1 - a_2) + (b_1 - b_2)\alpha + b_1\beta\mu_s + b_1\phi\mu_\sigma + \mu_\varepsilon - \mu_m + \frac{1}{2}[\lambda_\sigma^m - (b_1\phi + \lambda_\sigma^\varepsilon)]^2\sigma_\sigma^2] \\
&\quad + [(b_1\rho_s - b_2)\beta + \rho_s^\varepsilon - \rho_s^m]s_t + [(b_1\rho_\sigma - b_2)\phi + \rho_\sigma^\varepsilon - \rho_\sigma^m + \frac{1}{2}[\lambda_s^m - (b_1\beta + \lambda_s^\varepsilon)]^2]\sigma_t^2
\end{aligned}$$

Therefore we obtain

$$\alpha = \frac{\mu_m - (a_1 - a_2) - b_1\beta\mu_s - b_1\phi\mu_\sigma - \mu_\varepsilon - [\lambda_\sigma^m - (b_1\phi + \lambda_\sigma^\varepsilon)]^2\sigma_\sigma^2}{b_1 - b_2} \tag{C.19}$$

$$\beta = \frac{\rho_s^m - \rho_s^\varepsilon}{b_1\rho_s - b_2} \tag{C.20}$$

$$\phi = \frac{\rho_\sigma^m - \rho_\sigma^\varepsilon - \frac{1}{2}[\lambda_s^m - (b_1\beta + \lambda_s^\varepsilon)]^2}{b_1\rho_\sigma - b_2} \tag{C.21}$$

C.2 Case II: $\Delta\theta_t$ has a time-varying component

We assume that the log natural productivity growth is driven by a time-varying component, which is an AR(1), and subject to time-varying conditional volatility, which is also an AR(1). The SDF is driven by both state variables and subject to three sources of shocks.

$$\Delta\theta_{t+1} = \mu_\theta + s_t + \sigma_t e_{\theta,t+1} \tag{C.22}$$
$$s_{t+1} = \mu_s + \rho_s s_t + \varphi_s \sigma_t e_{s,t+1} \tag{C.23}$$
$$\sigma^2_{t+1} = \mu_\sigma + \rho_\sigma \sigma^2_t + \sigma_\sigma e_{\sigma,t+1} \tag{C.24}$$
$$m_{t+1} = -\mu_m - \rho^m_s s_t - \rho^m_\sigma \sigma^2_t - \lambda^m_\theta \sigma_t e_{\theta,t+1} - \lambda^m_s \sigma_t e_{s,t+1} - \lambda^m_\sigma \sigma_\sigma e_{\sigma,t+1} \tag{C.25}$$

where $\mu_s = (1-\rho_s)\bar{s}$ and $\mu_\sigma = (1-\rho_\sigma)\bar{\sigma}^2$.

Now we first solve for the chosen productivity. We conjecture and verify that the chosen productivity has the following functional form

$$\Delta\varepsilon_{t+1} = \mu_\varepsilon + \rho^\varepsilon_s s_t + \ \rho^\varepsilon_\sigma \sigma^2_t + \lambda^\varepsilon_\theta \sigma_t e_{\theta,t+1} + \lambda^\varepsilon_s \sigma_t e_{s,t+1} + \lambda^\varepsilon_\sigma \sigma_\sigma e_{\sigma,t+1} \tag{C.26}$$

Again using (C.5), where

$$LHS = -\lambda^m_\theta \sigma_t e_{\theta,t+1} - \lambda^m_s \sigma_t e_{s,t+1} - \lambda^m_\sigma \sigma_\sigma e_{\sigma,t+1} - \frac{1}{2}[(\lambda^m_\theta)^2\sigma^2_t + (\lambda^m_s)^2\sigma^2_t + (\lambda^m_\sigma)^2\sigma^2_\sigma]$$
$$RHS = [(\alpha-1)\lambda^\varepsilon_\theta - \alpha]\sigma_t e_{\theta,t+1} + (\alpha-1)\lambda^\varepsilon_s \sigma_t e_{s,t+1} + (\alpha-1)\lambda^\varepsilon_\sigma \sigma_\sigma e_{\sigma,t+1}$$
$$+ \sigma^2_t[-\frac{1}{2}(\alpha-1)^2(\lambda^\varepsilon_\theta)^2 - \frac{1}{2}(\alpha-1)^2(\lambda^\varepsilon_s)^2 - \frac{1}{2}\alpha^2 + (\alpha-1)\alpha\lambda^\varepsilon_\theta] - \frac{1}{2}(\alpha-1)^2(\lambda^\varepsilon_\sigma)^2\sigma^2_\sigma$$

from which we obtain

$$\lambda^\varepsilon_\theta = \frac{\alpha - \lambda^m_\theta}{\alpha-1},\ \lambda^\varepsilon_s = \frac{-\lambda^m_s}{\alpha-1},\ \lambda^\varepsilon_\sigma = \frac{-\lambda^m_\sigma}{\alpha-1}$$

Using the binding constraint (C.1), we have

$$0 = (\mu_\varepsilon + \rho^\varepsilon_s s_t + \ \rho^\varepsilon_\sigma \sigma^2_t) - (\mu_\theta + s_t) + \frac{\alpha}{2}[(\lambda^\varepsilon_\theta)^2\sigma^2_t + (\lambda^\varepsilon_s)^2\sigma^2_t + (\lambda^\varepsilon_\sigma)^2\sigma^2_\sigma] + \frac{\alpha}{2}\sigma^2_t - \alpha\lambda^\varepsilon_\theta\sigma^2_t$$

from which we obtain

$$\mu_\varepsilon = \mu_\theta - \frac{\alpha}{2}(\lambda^\varepsilon_\sigma)^2\sigma^2_\sigma,\ \rho^\varepsilon_s = 1,\ \rho^\varepsilon_\sigma = -\frac{\alpha}{2}(\lambda^\varepsilon_\theta - 1)^2 - \frac{\alpha}{2}(\lambda^\varepsilon_s)^2$$

In summary, we have the coefficients the chosen productivity process

$$\mu_\varepsilon = \mu_\theta - \frac{\alpha}{2}(\lambda^\varepsilon_\sigma)^2\sigma^2_\sigma = \mu_\theta - \frac{\alpha}{2}\left(\frac{-\lambda^m_\sigma}{\alpha-1}\right)^2\sigma^2_\sigma \tag{C.27}$$
$$\rho^\varepsilon_s = 1 \tag{C.28}$$
$$\rho^\varepsilon_\sigma = -\frac{\alpha}{2}(\lambda^\varepsilon_\theta - 1)^2 - \frac{\alpha}{2}(\lambda^\varepsilon_s)^2 = -\frac{\alpha}{2}\left(\frac{1-\lambda^m_\theta}{\alpha-1}\right)^2 - \frac{\alpha}{2}\left(\frac{-\lambda^m_s}{\alpha-1}\right)^2 \tag{C.29}$$
$$\lambda^\varepsilon_\theta = \frac{\alpha-\lambda^m_\theta}{\alpha-1} \tag{C.30}$$
$$\lambda^\varepsilon_s = \frac{-\lambda^m_s}{\alpha-1} \tag{C.31}$$
$$\lambda^\varepsilon_\sigma = \frac{-\lambda^m_\sigma}{\alpha-1} \tag{C.32}$$

Now we proceed to solve for the optimal investment. We conjecture

$$ik_t = \alpha + \beta s_t + \phi\sigma^2_t \tag{C.33}$$

The investment return (10) now becomes

$$\begin{aligned}
r^I_{t+1} &= (a_1 - a_2) + (b_1 ik_{t+1} - b_2 ik_t) + \varDelta\varepsilon_{t+1} \\
&= (a_1 - a_2) + b_1(\alpha + \beta s_{t+1} + \phi\sigma^2_{t+1}) - b_2(\alpha + \beta s_t + \phi\sigma^2_t) + \varDelta\varepsilon_{t+1} \\
&= [(a_1 - a_2) + (b_1 - b_2)\alpha + b_1\beta\mu_s + b_1\phi\mu_\sigma + \mu_\varepsilon] + [(b_1\rho_s - b_2)\beta + \rho^\varepsilon_s]s_t + [(b_1\rho_\sigma - b_2)\phi + \rho^\varepsilon_\sigma]\sigma^2_t \\
&\quad + (\lambda^\varepsilon_\theta)\sigma_t e_{\theta,t+1} + (b_1\beta\varphi_s + \lambda^\varepsilon_s)\sigma_t e_{s,t+1} + (b_1\phi + \lambda^\varepsilon_\sigma)\sigma_\sigma e_{\sigma,t+1}
\end{aligned}$$

The investment Euler equation (16) yields

$$\begin{aligned}
0 &= \mathbb{E}_t[m_{t+1}] + \mathbb{E}_t[r^I_{t+1}] + \frac{1}{2}\mathbb{V}_t[m_{t+1}] + \frac{1}{2}\mathbb{V}_t[r^I_{t+1}] + COV_t(m_{t+1}, r^I_{t+1}) \\
&= (-\mu_m - \rho^m_s s_t - \rho^m_\sigma \sigma^2_t) + \frac{1}{2}[(\lambda^m_\theta)^2\sigma^2_t + (\lambda^m_s)^2\sigma^2_t + (\lambda^m_\sigma)^2\sigma^2_\sigma] \\
&\quad + [(a_1 - a_2) + (b_1 - b_2)\alpha + b_1\beta\mu_s + b_1\phi\mu_\sigma + \mu_\varepsilon] + [(b_1\rho_s - b_2)\beta + \rho^\varepsilon_s]s_t + [(b_1\rho_\sigma - b_2)\phi + \rho^\varepsilon_\sigma]\sigma^2_t \\
&\quad + \frac{1}{2}(\lambda^\varepsilon_\theta)^2\sigma^2_t + \frac{1}{2}(b_1\beta + \lambda^\varepsilon_s)^2\sigma^2_t + \frac{1}{2}(b_1\phi + \lambda^\varepsilon_\sigma)^2\sigma^2_\sigma - \lambda^m_\theta\lambda^\varepsilon_\theta\sigma^2_t - \lambda^m_s(b_1\beta + \lambda^\varepsilon_s)\sigma^2_t - \lambda^m_\sigma(b_1\phi + \lambda^\varepsilon_\sigma)\sigma^2_\sigma \\
&= [(a_1 - a_2) + (b_1 - b_2)\alpha + b_1\beta\mu_s + b_1\phi\mu_\sigma + \mu_\varepsilon - \mu_m + \frac{1}{2}[\lambda^m_\sigma - (b_1\phi + \lambda^\varepsilon_\sigma)]^2\sigma^2_\sigma] \\
&\quad + [(b_1\rho_s - b_2)\beta + \rho^\varepsilon_s - \rho^m_s]s_t + [(b_1\rho_\sigma - b_2)\phi + \rho^\varepsilon_\sigma - \rho^m_\sigma + \frac{1}{2}[\lambda^m_\theta - \lambda^\varepsilon_\theta]^2 + \frac{1}{2}[\lambda^m_s - (b_1\beta + \lambda^\varepsilon_s)]^2]\sigma^2_t
\end{aligned}$$

We obtain investment coefficients

$$\alpha = \frac{\mu_m - (a_1 - a_2) - b_1\beta\mu_s - b_1\phi\mu_\sigma - \mu_\varepsilon - [\lambda^m_\sigma - (b_1\phi + \lambda^\varepsilon_\sigma)]^2\sigma^2_\sigma}{b_1 - b_2} \tag{C.34}$$

$$\beta = \frac{\rho^m_s - \rho^\varepsilon_s}{b_1\rho_s - b_2} \tag{C.35}$$

$$\phi = \frac{\rho^m_\sigma - \rho^\varepsilon_\sigma - \frac{1}{2}[\lambda^m_\theta - \lambda^\varepsilon_\theta]^2 - \frac{1}{2}[\lambda^m_s - (b_1\beta + \lambda^\varepsilon_s)]^2}{b_1\rho_\sigma - b_2} \tag{C.36}$$

C.3 Micro-foundation of productivity transformation

Consider an economy populated by a continuum of firms indexed by $i \in [0,1]$ in perfectly competitive markets. They are heterogenous in their productivity process $A_{i,t}$.

$$A_{i,t} = X_t Z_{i,t} \tag{C.37}$$

which contains an aggregate component X_t and an idiosyncratic component $Z_{i,t}$.

Define aggregate productivity as $A_t \equiv (\int A_{i,t}K_{i,t}di)/(\int K_{i,t}di)$. The aggregate productivity growth is given by

$$\frac{A_{t+1}}{A_t} = \frac{(\int A_{i,t+1}K_{i,t+1}di)/(\int K_{i,t+1}di)}{(\int A_{i,t}K_{i,t}di)/(\int K_{i,t}di)} = \frac{\int A_{i,t+1}K_{i,t+1}di}{\int A_{i,t}K_{i,t}di} \Big/ \frac{\int K_{i,t+1}di}{\int K_{i,t}di} \tag{C.38}$$

The second component is simply the aggregate capital growth K_{t+1}/K_t. The first component can be decomposed into

$$\begin{aligned}\frac{\int A_{i,t+1}K_{i,t+1}di}{\int A_{i,t}K_{i,t}di} &= \frac{\int X_{t+1}Z_{i,t+1}K_{i,t+1}di}{\int X_t Z_{i,t}K_{i,t}di} = \frac{X_{t+1}}{X_t}\frac{\int Z_{i,t+1}K_{i,t+1}di}{\int Z_{i,t}K_{i,t}di} \\ &= \frac{X_{t+1}}{X_t}\left[\frac{\int(1-\delta_i)Z_{i,t+1}K_{i,t}di}{\int Z_{i,t}K_{i,t}di} + \frac{\int Z_{i,t+1}I_{i,t}di}{\int Z_{i,t}K_{i,t}di}\right]\end{aligned} \tag{C.39}$$

We have

$$\frac{A_{t+1}}{A_t} = \frac{X_{t+1}}{X_t}\left[\frac{\int Z_{i,t+1}K_{i,t+1}di}{\int Z_{i,t}K_{i,t}di}\right] \Big/ \frac{K_{t+1}}{K_t} \tag{C.40}$$

where $\frac{X_{t+1}}{X_t}$ corresponds to $\frac{\Theta_{t+1}}{\Theta_t}$ and $\frac{A_{t+1}}{A_t}$ corresponds to $\frac{\mathcal{E}_{t+1}}{\mathcal{E}_t}$.

The underlying productivity can be measured as

$$\frac{X_{t+1}}{X_t} = \frac{A_{t+1}}{A_t}\frac{K_{t+1}}{K_t} \Big/ \left[\frac{\int Z_{i,t+1}K_{i,t+1}di}{\int Z_{i,t}K_{i,t}di}\right] = \frac{Y_{t+1}}{Y_t} \Big/ \left[\frac{\int Z_{i,t+1}K_{i,t+1}di}{\int Z_{i,t}K_{i,t}di}\right] \tag{C.41}$$

The expression (C.5)

$$\frac{M_{t+1}}{\mathbb{E}_t[M_{t+1}]} = \frac{\frac{\Delta\mathcal{E}_{t+1}^{\alpha-1}}{\Delta\Theta_{t+1}^{\alpha}}}{\mathbb{E}_t\left[\frac{\Delta\mathcal{E}_{t+1}^{\alpha-1}}{\Delta\Theta_{t+1}^{\alpha}}\right]}$$

suggests an SDF M_{t+1}^* for excess returns.

$$\begin{aligned}M_{t+1}^* = \frac{\Delta\mathcal{E}_{t+1}^{\alpha-1}}{\Delta\Theta_{t+1}^{\alpha}} = \frac{\Delta A_{t+1}^{\alpha-1}}{\Delta X_{t+1}^{\alpha}} &= \frac{\Delta A_{t+1}^{\alpha-1}}{\left(\Delta A_{t+1}\Delta K_{t+1} \Big/ \left[\frac{\int Z_{i,t+1}K_{i,t+1}di}{\int Z_{i,t}K_{i,t}di}\right]\right)^{\alpha}} \\ &= \Delta A_{t+1}^{-1}\Delta K_{t+1}^{-\alpha}\left[\frac{\int Z_{i,t+1}K_{i,t+1}di}{\int Z_{i,t}K_{i,t}di}\right]^{\alpha} \qquad (C.42) \\ &= \Delta Y_{t+1}^{-1}\Delta K_{t+1}^{1-\alpha}\left[\frac{\int Z_{i,t+1}K_{i,t+1}di}{\int Z_{i,t}K_{i,t}di}\right]^{\alpha} \qquad (C.43)\end{aligned}$$