Optimal Allocation to Private Equity

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Abstract

We study the portfolio problem of an investor (LP) that invests in stocks, bonds, and private equity (PE) funds. The LP repeatedly commits capital to PE funds. This capital is only gradually contributed and eventually distributed back to the LP, requiring the LP to hold a liquidity buffer for its uncalled commitments. Despite being riskier, PE investments are not monotonically declining in risk aversion. Instead, there are two qualitatively different investment strategies with intuitive heuristics. We introduce a secondary market for PE partnership interests to study optimal trading in this market and implications for the LP's optimal investments.

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The number of US publicly traded companies is declining and an increasing amount of capital is invested in private equity (PE) and other non-public assets.[1] Typical PE investors, so-called limited partners (LPs), are pension funds, sovereign wealth funds, university endowments, and other institutional investors. Despite their increasing allocations to PE, it remains unclear how LPs should optimally manage their PE investments as part of their broader portfolio allocations. We study this problem.

We introduce PE investments into a standard portfolio model. PE investments are risky, illiquid, and long-term. They are risky because PE investments earn a risky return and generate risky distributions to the LP. They are illiquid because the LP must hold them to maturity, and the LP cannot liquidate (or collateralize) its PE investments to convert them into current consumption, although this is relaxed in the extension with a secondary market. They are long-term because the LP's commitments are only gradually called and invested into underlying private assets by PE funds, and the eventual distributions back to the LP only arrive after a substantial time. Thus, the LP can only adjust its PE exposure slowly, and it may need to hold a liquidity reserve to meet future capital calls arising from its current uncalled commitments. A main contribution of our analysis is to present a tractable model that captures these aspects of PE investments, particularly the LP's repeated capital commitments to an arbitrary number of PE funds, and to solve for the LP's optimal investment policies.

The particular nature of PE investments has interesting implications for the LP's optimal allocations and investment policies. Unsurprisingly, the illiquid and long-term nature of PE investments can lead the LP to reduce its PE allocation relative to the first-best allocation. Perhaps more surprisingly, the optimal PE allocation is not monotonically decreasing in the LP's risk aversion. With reasonable parameter values, the LP's PE allocation is largely constant, and a more risk averse LP may, in fact, choose a higher PE allocation than a less risk averse

[1] Doidge, Karolyi, and Stulz (2017) discuss the declining number of listed companies, and Bain & Company (2019) show the increasing allocations to alternative investments.

LP. In contrast, risk aversion significantly affects the LP's allocations to traded stocks and bonds.

Depending on risk aversion, there are two distinct investment strategies. A conservative LP, with higher risk aversion (in our specification, a relative risk aversion of $\gamma = 3$), holds relatively more liquid reserves of stocks and bonds than illiquid PE investments. This LP tends to be unconstrained and to remain close to an interior optimum, leaving it largely unaffected by the illiquid and long-term nature of PE investments. In response to a positive shock to the net asset value (NAV) of its PE investments, the conservative LP reduces new PE commitments to gradually rebalance its portfolio back towards the interior optimum, basically treating PE investments as another traded stock. In contrast, an aggressive LP with lower risk aversion ($\gamma = 1$) faces a binding liquidity constraint. It would prefer greater PE exposure, but larger PE commitments would require a greater liquidity reserve, which is costly. In response to a positive shock to the NAV of its PE investments, the aggressive LP does not rebalance its portfolio, and instead it enjoys a temporary increase in its PE exposure without a corresponding increase in its liquidity reserve.

We extend the model with a secondary market where LPs can trade PE partnership interests to investigate two effects: First, an aggressive LP assigns higher valuations to mature PE funds than a conservative LP. Mature funds provide PE exposure with relatively less liquidity requirements, which is valuable for a constrained LP, and there are gains from trade when conservative LPs can sell partnership interests in mature PE funds to aggressive LPs. Second, the presence of a secondary market where an LP can liquidate its PE investments relaxes its liquidity constraint. In times of stress an LP may choose to liquidate a fraction of its PE investments, even at a discount. And knowing that this liquidity is available, ex-post, allows the LP to hold a more aggressive allocation, ex-ante. We quantify these effects and find that the gains from a single trade, while positive, are modest. However, a secondary market that provides liquidity in times of stress substantially affects the aggressive LP's optimal investments even though the LP only rarely actually transacts in this market.

A large literature investigates the portfolio and asset pricing implications of illiquidity in general, and we do not attempt to summarize this literature. Our analysis is in the spirit of Longstaff (2001) who uses numerical methods to study a portfolio problem with illiquid assets that can only be traded in limited amounts.

In the context of PE specifically, Ang, Papanikolaou, and Westerfield (2014) analyze an investor's optimal investments with an illiquid asset that can only be rebalanced infrequently, at Poisson arrival times, to capture a situation where the investor must search for a counterparty. They find that this form of illiquidity increases the investor's effective risk aversion. Our analysis does not capture the arrival of trading opportunities. Instead, we focus on the ongoing commitments and long-term dynamics of PE investments, which have distinct implications. For example, in our model, LPs with higher risk aversion are unaffected by the illiquidity of PE investments, and it does not increase their effective risk aversion. Sorensen, Wang, and Yang (2014) model an LP that rebalances stocks and bonds while holding an illiquid asset to maturity in order to analyze the costs of the unspanned risks of the illiquid asset, taking the investment in the illiquid asset as given. **?** extend this analysis with a secondary market for partnership interests. **?** evaluate the "endowment model" used by some institutions that invest in alternative assets with lock-up periods. **?** study commitment risk for an investor in one, two, or an infinite number of PE funds using a Markov switching model to capture the state of the economy. Finally, there is an extensive literature about the risks and returns of PE investments, and Korteweg (2019) surveys this literature.

1 Model

We extend a standard portfolio model with illiquid PE investments that work as follows. Each period (in our specification, each year) a limited partner (LP) decides how much capital to commit to new PE funds. This capital is not immediately invested, however. Instead, it is gradually contributed to (or drawn down or called by) the PE funds over a period of several years. When capital is contributed (or drawn or called), it is paid by the LP to the PE funds, which invest the capital in underlying private assets (typically, equity stakes in portfolio companies). After holding these private assets for an extended period, typically several years, the private assets are sold (exited) by the PE funds. The proceeds from these sales, net of the PE funds' fees (management fees and carried interest), are paid out (distributed) to the LP.

The LP thus maintains a stock of uncalled commitments (drawdown obligations) representing the capital that the LP has already committed but that has not yet been contributed to the PE funds. These uncalled commitments are gradually called by the PE funds and then invested in underlying private assets. The LP's PE exposure arises from its claim to the future distributions from the currently held private assets, and the NAV is the value of this claim, which constitutes the LP's illiquid wealth.[2] Each period the LP can decide to commit additional capital to new PE funds and increase its stock of uncalled commitments. Once committed, however, the LP cannot decide to reduce its uncalled commitments. Uncalled commitments only decline gradually, as capital is called and invested by the PE funds.

PE funds are only open for new commitments at inception and typically have ten-year lives, although funds are routinely extended. New PE funds are continuously raised, and LPs diversify their PE investments both across different contemporaneous funds and over time by committing capital to new funds on an ongoing basis, resulting in a diversified portfolio of staggered PE fund commitments. In this case, distributions received from earlier fund commitments, when available, can pay contributions to later commitments, and our model is consistent with this practice.

[2]The NAV, denoted P_t below, is the price that PE investments would trade at, if they were traded. In the liquid model introduced below, we allow for such trades.

1.1 Preferences and Timing

The analysis is based on a standard, discrete-time, infinite-horizon, partial-equilibrium, portfolio model with i.i.d. returns and an LP with power utility (CRRA) preferences:

$$\mathrm{E}_t\left[\sum_{s=t}^{\infty}\delta^{s-t}\frac{C_s^{1-\gamma}}{1-\gamma}\right] \tag{1}$$

The LP's consumption is C_t, its time preference is δ, and γ is its relative risk aversion.

Like Ang et al. (2014) we assume a relative risk aversion greater than or equal to one, $\gamma \geq 1$, implying an elasticity of intertemporal substitution less than or equal to one. The LP has negative infinite utility in states with zero consumption. The LP is "bankrupt" when it is unable to consume, and the LP will avoid such states at all costs.[3] Below, we emphasize the results for an LP with $\gamma = 1$, which we call an aggressive LP, and for an LP with $\gamma = 3$, which we call a conservative LP.

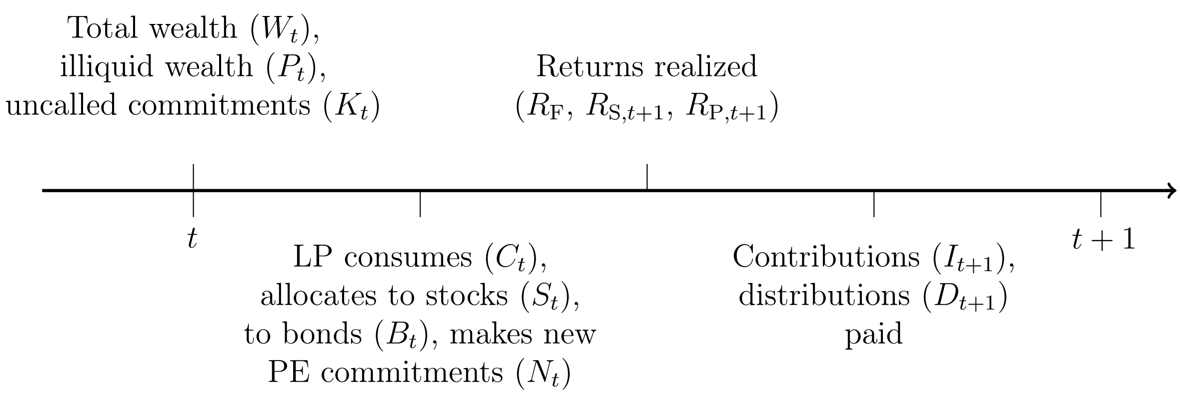


Figure 1: Timing of events during the period from time t to time $t+1$.

The timing of the model is illustrated in Figure 1. At time t, the LP has total wealth W_t, illiquid wealth P_t, and uncalled commitments K_t. After observing these three state variables, the LP consumes and invests. The LP consumes C_t out of its liquid wealth, $W_t - P_t$. Its

[3] Maurin, Robinson, and Stromberg (2020) and Banal-Estañol, Ippolito, and Vicente (2017) formally model LPs' liquidity constraints, distortions when LPs can default, and the penalties for defaulting.

remaining liquid wealth, $W_t - P_t - C_t$, is allocated to traded stocks, S_t, and bonds, B_t:

$$B_t = W_t - P_t - C_t - S_t \tag{2}$$

The LP decides the amount of capital to commit to new PE funds, N_t. New PE commitments increase the LP's total stock of uncalled commitments, $N_t + K_t$, but they do not directly affect the LP's wealth or budget constraint in the period when they are made.

After the LP has consumed and made its portfolio decisions, returns are realized and contributions and distributions are paid. In practice, returns, contributions, and distributions are earned and paid throughout the period. For simplicity, we assume that returns are realized first and then contributions and distributions are paid. This assumption emphasizes the long-term nature of PE investments, because new PE commitments cannot be contributed and provide PE exposure to the LP in the period when they are made. Rather, new PE commitments are contributed and invested in private assets with at least a one-period lag.

Conventionally, returns earned from time t to $t+1$ have subscript $t+1$, and the returns to stocks and PE investments during this period are denoted $R_{\mathrm{S},t+1}$ and $R_{\mathrm{P},t+1}$, respectively. Bonds earn the risk-free rate, R_{F}. All returns are gross returns. Correspondingly, distributions and contributions paid from time t to $t+1$ are denoted D_{t+1} and I_{t+1}, respectively. The LP pays contributions, I_{t+1}, out of its liquid wealth to the PE funds who invest them in private assets. Distributions, D_{t+1}, are paid from the PE funds to the LP as liquid wealth.

The three state variables update as follows. The LP's uncalled commitments, K_t, increase by the amount of new PE commitments, N_t, and decrease by the amount contributed, I_{t+1}:

$$K_{t+1} = K_t + N_t - I_{t+1} \tag{3}$$

The NAV of the PE investments, P_t, earns the risky return, $R_{\mathrm{P},t+1}$, and then it increases by the amount contributed to the PE funds, I_{t+1}, and it decreases by the amount distributed from the PE funds, D_{t+1}:

$$P_{t+1} = R_{\mathrm{P},t+1}P_t + I_{t+1} - D_{t+1} \tag{4}$$

The LP's end-of-period wealth, W_{t+1}, is the combined value of its PE investments, stocks, and bonds. Contributions and distributions made during the period do not affect the end-of-period wealth since they only reallocate wealth between liquid and illiquid investments:

$$W_{t+1} = R_{\mathrm{P},t+1}P_t + R_{\mathrm{S},t+1}S_t + R_{\mathrm{F}}(W_t - P_t - C_t - S_t) \tag{5}$$

1.2 Linear Fund Dynamics

In practice, an LP invests in many PE funds, each with its own amounts of uncalled commitments, contributions, distributions, and NAVs. Modeling each fund separately, however, is difficult due to the resulting high-dimensional state space and the curse of dimensionality in dynamic programming. Motivated by Takahashi and Alexander (2002), we assume linear fund dynamics where each fund's contributions and distributions are linear in the fund's uncalled commitments and NAV.[4] An implication of linear fund dynamics is that the LP's aggregate contributions and distributions across all its PE funds are linear in the aggregate uncalled commitments and aggregate NAV. Aggregate uncalled commitments and aggregate NAV are therefore sufficient statistics for the LP's portfolio of PE investments, in an arbitrary number of PE funds, which reduces the state space to three dimensions (two, in the normalized problem below) and avoids the numerical difficulties that arise when specifying each fund separately.

Formally, let PE funds be indexed by u, and let the set of PE funds that the LP has a partner-

[4] Our model simplifies Takahashi and Alexander (2002) in two ways. Their contribution rate varies over time, and our rate is fixed by λ_{N} and λ_{K}. Moreover, our distribution rate is fixed by λ_{D}, and their distribution rate approaches 100% as the fund winds down.

ship interest in at time t, including the new PE funds that the LP commits capital to at time t, be denoted U_t.[5] In practice, multiple LPs invest in a PE fund, and the uncalled commitments, contributions, distributions, and NAVs defined below represent the LP's share of each fund.

In its first year, contributions to a new fund u are the fraction λ_{N} of the fund's newly committed capital, $N_{u,t}$:

$$I_{u,t+1} = \lambda_{\mathrm{N}} N_{u,t} \tag{6}$$

In subsequent years, contributions are the fraction, λ_{K}, of the fund's remaining uncalled commitments, $K_{u,t}$:

$$I_{u,t+1} = \lambda_{\mathrm{K}} K_{u,t} \tag{7}$$

Since $N_t = \sum_{u \in U_t} N_{u,t}$ is the LP's aggregate new PE commitments, and $K_t = \sum_{u \in U_t} K_{u,t}$ is the aggregate existing commitments, the LP's aggregate contributions, I_{t+1}, are:

$$I_{t+1} = \sum_{u \in U_t} I_{u,t+1} = \lambda_{\mathrm{N}} N_t + \lambda_{\mathrm{K}} K_t \tag{8}$$

Hence, aggregate contributions, I_t, are linear in aggregate new commitments, N_t, and aggregate existing commitments, K_t. Our model allows for different intensities of contributions for new and existing commitments, but in our baseline specification $\lambda_{\mathrm{N}} = \lambda_{\mathrm{K}} = 30\%$ for simplicity.[6]

Distributions are modeled similarly. Fund u's distributions, $D_{u,t+1}$, are the fraction, λ_{D}, of fund

[5] When fund u has a ten-year life, the LP has a partnership interest in this fund starting the year of the LP's initial commitment and extending through the following nine years until the fund terminates. Note also that a new PE fund, u, that the LP commits capital to at time t starts out with zero NAV, so $P_{u,t} = 0$.

[6] For comparison, Takahashi and Alexander (2002) consider a contribution rate of 25% when the fund is one year old, 33.3% when it is two years old, and 50% after that.

u's NAV, which is $R_{u,t+1}P_{u,t}$, where $R_{u,t+1}$ is fund u's gross return, so:

$$D_{u,t+1} = \lambda_{\mathrm{D}} R_{u,t+1} P_{u,t} \tag{9}$$

The LP's aggregate distributions across all its PE funds is denoted D_{t+1} and equals:

$$D_{t+1} = \sum_{u \in U_t} D_{u,t+1} = \lambda_{\mathrm{D}} R_{\mathrm{P},t+1} P_t \tag{10}$$

Here, $P_t = \sum_{u \in U_t} P_{u,t}$ is the LP's aggregate NAV, and $R_{\mathrm{P},t+1} = \sum_{u \in U_t} R_{u,t+1} \frac{P_{u,t}}{P_t}$ is the value-weighted average return of the LP's PE funds. In our specification, the intensity of distributions is $\lambda_{\mathrm{D}} = 40\%$ of the fund's remaining NAV. Even though distributions are a deterministic fraction of the NAV, the fund's NAV is stochastic, because private assets earn risky returns, and the LP receives a risky flow of distributions from its PE investments.

The above derivations are important for the generality of our analysis. In the general problem, the LP invests in an arbitrary number of contemporaneous and staggered PE funds, but specifying each fund individually is intractable, as mentioned. The above discussions shows, however, that linear fund dynamics imply that the aggregate NAV and aggregate uncalled commitments are sufficient statistics for the LP's portfolio of PE investments. Hence, the solution to the LP's general problem can be found by solving a reduced problem with a more tractable state space. Without loss of generality, we therefore focus on the LP's reduced problem below.[7]

A limitation of linear fund dynamics is that PE funds never completely end, as also discussed by

[7] An additional detail of the specification is how the idiosyncratic risk of the value-weighted PE return, R_{P}, depends on the LP's PE investments, P. Here, we assume that the LP has partnership interests in a diversified number of PE funds, and that changes in the amount of the LP's PE investments arise from proportional changes in the stakes in these funds, leaving the idiosyncratic risk constant. Alternatively, changes in the LP's PE investments can be associated with changes in the number of PE funds, in which case a lower amount of PE investments would be associated with an increase in the idiosyncratic risk of R_{P} due to less diversification. In principle, our model can accommodate a dependency between the idiosyncratic risk of R_{P} and P, although we have not pursued this extension. Robinson and Sensoy (2016) find that most of the volatility in PE investments can be diversified.

Takahashi and Alexander (2002). In practice, PE funds have ten-year lives, although these lives are routinely extended.[8] In our specification the remaining economic value in the PE funds' later years is minimal, mitigating this limitation of the linear fund dynamics. Specifically, 83.2% of a fund's committed capital is contributed during its first five years, and 97.2% is contributed after ten years. Moreover, in steady state, the LP holds a balanced portfolio of younger and older PE funds, and specification errors in the dynamics will largely average out across the funds when aggregating their contributions and distributions.[9]

Management fees are implicit in this setup. In practice, PE funds charge annual management fees of 0.5%–2% of the total committed capital, which means that this capital is contributed by the LP to the PE funds, but this capital is not used to acquire underlying private assets. Below, we extend our model to explicit management fees, but in most of the analysis management fees are implicit, and the contributed capital, I_t, is the combined amount invested in underlying private assets and paid in management fees. In this case, the NAV, P_t, includes the value of management fees, and the PE return, $R_{\mathrm{P},t}$, is net of both management fees and carried interest.[10] An advantage of implicit management fees is that the analysis of illiquidity is more transparent, because the optimal allocations can be compared to the first-best benchmark when PE is liquid and freely traded. With liquid PE there is no notion of committed capital, however, and the liquid model does not accommodate explicit management fees. Another advantage of implicit management fees is that the return to PE investments, $R_{\mathrm{P},t}$, is the return net of both

[8] In their analysis of long-term PE performance, **?** report that it is common for PE funds to have activity fifteen years after inception.

[9] To illustrate this averaging, assume that PE funds live for three periods, and let the dynamics of contributions be specified by λ_0, λ_1, and λ_2, which are the relative amounts of a fund's total committed capital that is contributed during each period. All committed capital is eventually contributed, so $\lambda_0 + \lambda_1 + \lambda_2 = 1$. A diversified LP with an equal amount, κ, committed to funds at each of the three ages has aggregate contributions of $\lambda_0\kappa + \lambda_1\kappa + \lambda_2\kappa = \kappa$. It follows that regardless of the specification of the dynamics, i.e., the particular choice of λ, the resulting amount of aggregate contributions is unaffected.

[10] Actual accounting for management fees can be complicated. PE funds with deal-by-deal carry, sometimes known as the US Market Standard, typically attribute management fees to individual deals and gross up the valuations of these deals with the attributed management fees. Another PE fund compensation structure, sometimes known as the UK Market Standard, has the general partner borrowing the amounts that correspond to the fees and repaying these amounts out of a priority share of the fund's profits, which is consistent with our model of implicit management fees.

management fees and carried interest, which is the return that is typically reported in empirical studies. In Section 4, we model management fees explicitly, which requires a reinterpretation of the contributed capital, I_t, the NAV, P_t, and the PE return, $R_{\mathrm{P},t}$. The main finding is that the optimal policies are largely similar with implicit and explicit management fees and that this modeling choice is not critical for our results.

The implied dynamics of capital calls and distributions are shown in Figures 2 to 5, which show impulse responses of an initial \$100 PE commitment in year 0, followed by no further commitments. Figure 2 shows annual capital calls. In year one, \$30 is called ($\lambda_{\mathrm{N}} = 30\%$ times $N_0 = \$100$). In year two, \$21 is called ($\lambda_{\mathrm{K}} = 30\%$ times $K_1 = \$70$). In the following years, corresponding declining amounts of capital are called.

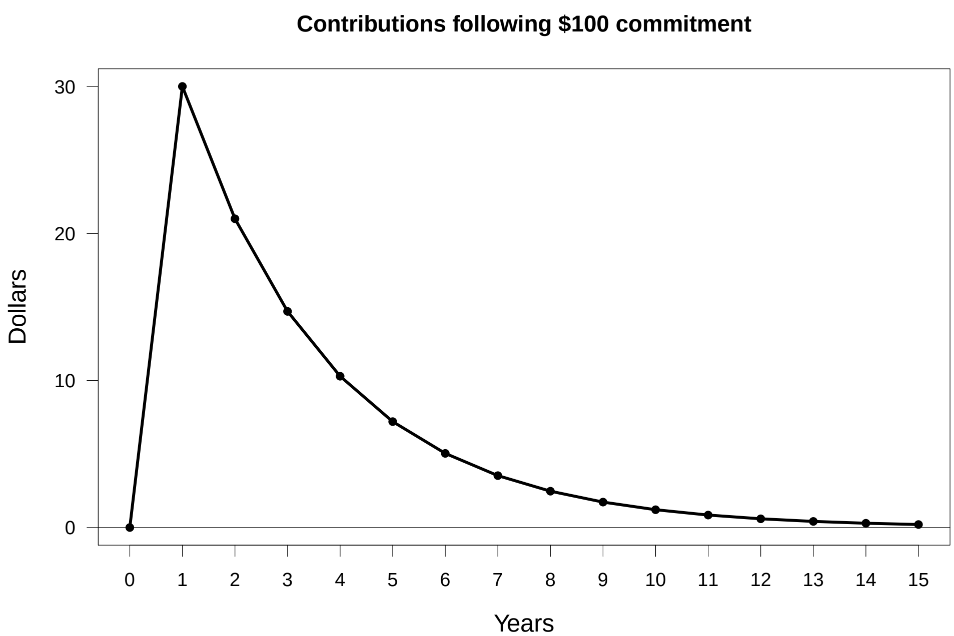


Figure 2: Impulse response function of contributions following a \$100 commitment in year 0.

Figure 3 shows the dynamics of distributions. Distributions depend on NAVs, which are risky, and the figure shows average distributions along with the 5th and 95th percentiles (distributional assumptions are provided below). The largest annual distributions arrive three to five years after the initial commitment, reflecting the long-term nature of PE investments.

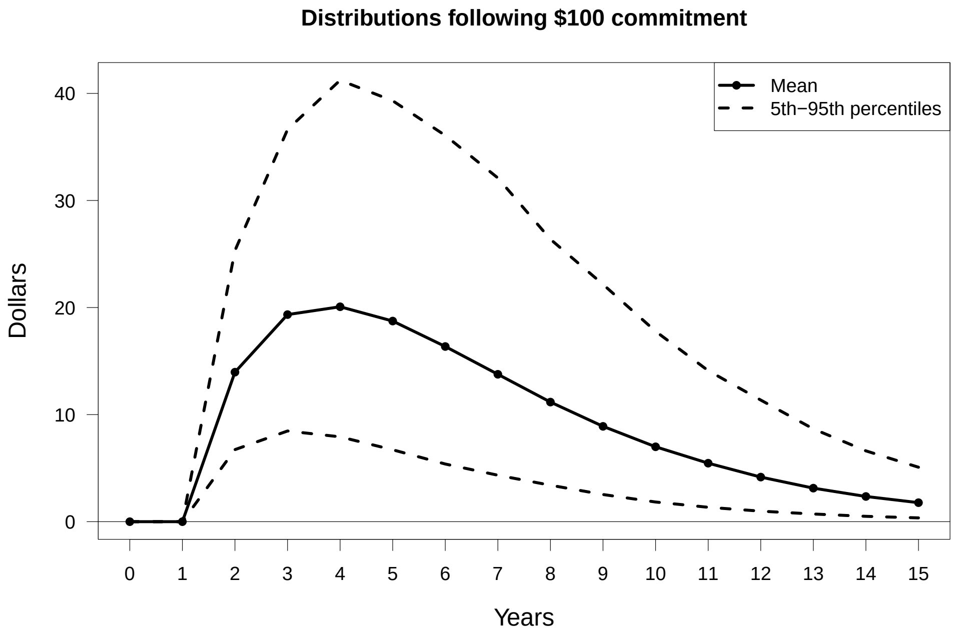


Figure 3: Impulse response function of distributions following a $100 commitment in year 0. The solid line shows average annual distributions. The dashed lines show the 95^{th} and 5^{th} percentiles of the distribution of annual distributions.

Figure 4 shows the cumulative net cash flow, i.e., cumulative distributions minus cumulative contributions. The cumulative cash flow exhibits the well-known "J-curve" dynamics, where it is initially negative, meaning that the LP initially contributes more capital than it receives. With average cash flows the LP breaks even around year six, and then the cumulative cash flow becomes positive and converges to its final value as the fund winds down. The average cumulative net cash flow from an initial $100 investment is around $50, implying an average multiple (TVPI or MOIC) around 1.5.

Figure 5 shows the dynamics of the fund's NAV. Interestingly, $100 of initial commitments only result in a maximal PE exposure around $45, on average, and this maximal exposure only arises about three years after the initial commitment. A challenge when managing PE investments is thus that actual PE exposure adjusts slowly, only reaches its maximum level several years after a new commitment is made, and that this maximal level of exposure is only a fraction of the amount of committed capital.

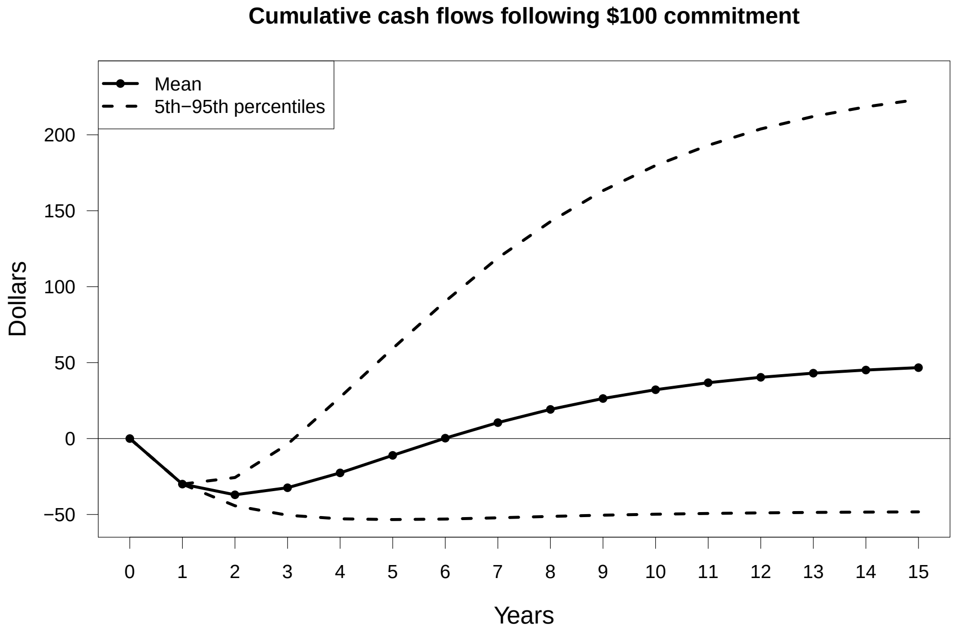


Figure 4: Impulse response function of cumulative net cash flows, the sum of past and current net cash flows, following a \$100 commitment in year 0. The solid line shows average cumulative cash flows. The dashed lines show the 95^{th} and 5^{th} percentiles.

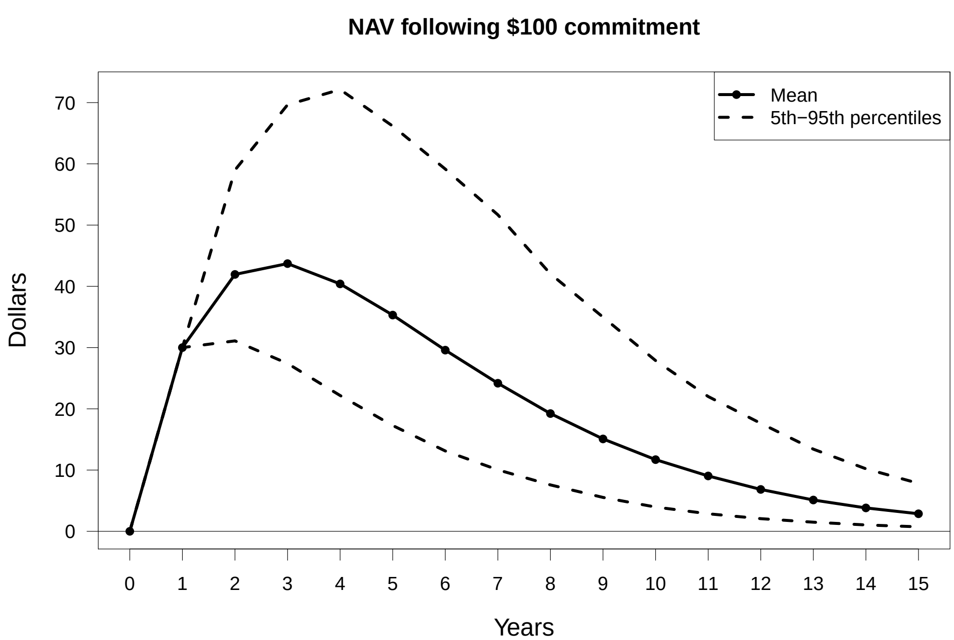


Figure 5: Impulse response function of NAV (or illiquid wealth) following a \$100 commitment in year 0. The solid line shows the average NAV. The dashed lines show the 95^{th} and 5^{th} percentiles.

1.3 Distributional Assumptions and Parameters

The returns to PE and stocks are denoted $R_{\mathrm{P},t}$ and $R_{\mathrm{S},t}$ respectively. These returns are the two sources of uncertainty in the model, and they follow standard i.i.d log-normal distributions. Let $r_{\mathrm{P},t} = \ln R_{\mathrm{P},t}$ and $r_{\mathrm{S},t} = \ln R_{\mathrm{S},t}$. Then, $r_{\mathrm{P},t}$ and $r_{\mathrm{S},t}$ are normal distributed with means μ_{P} and μ_{S}, variances σ^2_{P} and σ^2_{S}, and covariance $\rho\sigma_{\mathrm{P}}\sigma_{\mathrm{S}}$, where ρ is the correlation between the log-returns:

$$(r_{\mathrm{P},t}, r_{\mathrm{S},t}) \sim \mathrm{N}(\mu, \Sigma)$$

Table 1 shows parameter values for our baseline specification. The log risk-free rate is $r_{\mathrm{F}} = 2\%$. The log-return on stocks has expectation $\mu_{\mathrm{S}} = 6\%$ and volatility $\sigma_{\mathrm{S}} = 20\%$. The parameters for PE investments, ρ, μ_{P} and σ_{P}, imply that PE has $\alpha = 3\%$ and $\beta = 1.6$, as defined by the log-linear CAPM:

$$\ln \mathrm{E}[R_{\mathrm{P}}] - r_{\mathrm{F}} = \alpha + \beta\left(\ln \mathrm{E}[R_{\mathrm{S}}] - r_{\mathrm{F}}\right) \tag{11}$$

with $\beta = \rho\sigma_{\mathrm{P}}/\sigma_{\mathrm{S}}$.

Our parameters differ slightly from those in Ang et al. (2014), who assume a risk-free rate of 4%, expected log-return on stocks of 12%, and volatility of 15%. To isolate the effect of illiquidity, Ang et al. (2014) choose ρ, μ_{P}, and σ_{P} so PE investments have the same risk and return as the stock market. In our specification, PE investments are positively correlated with stocks, are more volatile than stocks, and have a positive alpha. For robustness we also solve the model for other specifications of alpha, beta, and the idiosyncratic risk, as reported in Appendix C.

Table 1: This table reports parameter values used as inputs of the model. It also includes important statistics that are implied by those parameter values.

Parameter/Statistic	Expression	Value
Draw-downs of new commitments	λ_{N}	30%
Draw-downs of older commitments	λ_{K}	30%
Distribution intensity	λ_{D}	40%
Log of risk-free rate	r_{F}	2%
Expected log-return to stocks	μ_{S}	6%
Expected log-return to PE	μ_{P}	6.6%
Volatility of log-return to stocks	σ_{S}	20%
Volatility of log-return to PE	σ_{P}	40%
Alpha of PE	α	3%
Beta of PE	β	1.6
Subjective discount factor	δ	95%
Implied:		
Log of expected return to stocks	$\ln \mathrm{E}[R_{\mathrm{S}}]$	8%
Log of expected return to PE	$\ln \mathrm{E}[R_{\mathrm{P}}]$	14.6%
Sharpe ratio of stocks	$\left(\ln \mathrm{E}[R_{\mathrm{S}}] - r_{\mathrm{F}}\right)/\sigma_{\mathrm{S}}$	0.3
Sharpe ratio of PE	$\left(\ln \mathrm{E}[R_{\mathrm{P}}] - r_{\mathrm{F}}\right)/\sigma_{\mathrm{P}}$	0.315
Correlation of log-returns to stocks and PE	ρ	0.8

1.4 Reduced Portfolio Problem

The LP's preferences, liquidity constraint, and the dynamics of contributions and distributions define the LP's reduced problem:

$$
\begin{aligned}
& V(W,P,K) = \max_{(C,N,S)} \left\{ C^{1-\gamma} + \delta \mathrm{E}\left[V(W',P',K')^{1-\gamma} \right] \right\}^{\frac{1}{1-\gamma}} \\
& \text{subject to} \\
& W' = R_{\mathrm{P}} P + R_{\mathrm{S}} S + R_{\mathrm{F}} \left(W - P - C - S \right) \\
& P' = \left(1 - \lambda_{\mathrm{D}}\right) R_{\mathrm{P}} P + \lambda_{\mathrm{N}} N + \lambda_{\mathrm{K}} K \\
& K' = (1-\lambda_{\mathrm{K}}) K + (1-\lambda_{\mathrm{N}}) N \\
& 0 \leq C \leq W - P \\
& N \geq 0
\end{aligned}
\tag{12}
$$

The problem has three state variables: W is total wealth, P is aggregate illiquid wealth (NAV), and K is aggregate stock of uncalled commitments. Each period illiquid wealth (NAV), P, earns the return R_{P}, decreases by the distributed amount, $\lambda_{\mathrm{D}} R_{\mathrm{P}} P$, and increases by the contributed capital, $\lambda_{\mathrm{N}} N + \lambda_{\mathrm{K}} K$. The end-of-period stock of uncalled commitments, K, increases by new commitments, N, and decreases by the amount of contributed capital, $\lambda_{\mathrm{N}} N + \lambda_{\mathrm{K}} K$.

The LP maximizes the value function over three controls: First, it chooses new commitments, $N \geq 0$. Second, the LP chooses consumption, $C \geq 0$, and since the LP can only consume out of liquid wealth, $C \leq W - P$. If it exhausts its liquid wealth and is unable to consume, it has negative infinite utility. Third, the LP chooses its stock allocation, S, and the remaining wealth, $W - C - P - S$, is invested in bonds.

The value function is homogeneous in wealth, and we guess the functional form:

$$
V(W,P,K) = v(p,k) W \tag{13}
$$

where $k = K/W$ and $p = P/W$ are the LP's uncalled commitments and illiquid wealth normalized by total wealth. Substituting into the Bellman equation (12), we verify the guess for $v(k, p)$ solving the normalized problem:

$$
\begin{aligned}
& v(p,k) = \max_{(c,n,\omega_{\mathrm{S}})} \left\{ c^{1-\gamma} + \delta \mathrm{E}\left[\left[(1-c) R_{\mathrm{W}} v(p',k') \right]^{1-\gamma} \right] \right\}^{\frac{1}{1-\gamma}} \\
& \text{subject to} \\
& R_{\mathrm{W}} = \omega_{\mathrm{P}}(R_{\mathrm{P}} - R_{\mathrm{F}}) + \omega_{\mathrm{S}}(R_{\mathrm{S}} - R_{\mathrm{F}}) + R_{\mathrm{F}} \\
& p' = \left[(1-\lambda_{\mathrm{D}}) R_{\mathrm{P}} p + \lambda_{\mathrm{N}} n + \lambda_{\mathrm{K}} k \right] / \left[(1-c) R_{\mathrm{W}} \right] \\
& k' = \left[k + n - \lambda_{\mathrm{N}} n - \lambda_{\mathrm{K}} k \right] / \left[(1-c) R_{\mathrm{W}} \right] \\
& 0 \le c \le 1 - p \\
& n \ge 0
\end{aligned}
\tag{14}
$$

The normalized control variables are $c = C/W$, $s = S/W$, and $n = N/W$. Furthermore, it is convenient to define the portfolio weight $\omega_{\mathrm{S}} = S/(W - C)$, $\omega_{\mathrm{P}} = P/(W - C)$, and $\omega_{\mathrm{B}} = 1 - \omega_{\mathrm{S}} - \omega_{\mathrm{P}}$. These portfolio weights are normalized by the LP's post-consumption wealth in order to sum to one.[11]

1.5 Liquidity Constraint

The LP ensures that there is zero probability that it exhausts its liquid wealth, or, equivalently, that its illiquid wealth, P, exceeds its total wealth, W. Since $p = P/W$, the optimal policy satisfies the liquidity constraint:

$$
\mathrm{Prob}(p > 1) = 0 \tag{15}
$$

This liquidity constraint is the main friction in our analysis. One immediate implication of the constraint and the unbounded support of the log-returns to stocks and PE investments

[11] Below, the optimal consumption ratio, c, is 4%–5% with minimal variation. Hence, the LP's PE allocation is approximately $\omega_{\mathrm{P}} \approx 1/0.95 \times p$.

is that the LP wants to hold a liquidity reserve of safe assets, in our case risk-free bonds, at least as large as its total stock of uncalled commitments. Another immediate implication of the liquidity constraint is that the LP will not leverage its allocations to stocks and PE by holding a negative amount of bonds, because a large negative shock then leaves the LP with insufficient liquid wealth. Moreover, the LP will not short stocks, because a large positive shock then exhausts its liquid wealth. As in Longstaff (2001), the LP optimally acts as if subject to no-leverage and no-shorting constraints although these constraints are not explicitly imposed.

This liquidity constraint, however, is not straightforward to evaluate empirically. In practice LPs hold reserves in a variety of assets (the bonds in our model can be interpreted as a general reserve asset). Moreover, LPs hold other illiquid assets—such as direct investments in private companies, real estate, and natural resources—which have their own liquidity requirements and implications for the LP's liquidity reserve. Hence, a simple comparison of an LP's bond holdings to its stock of uncalled commitments may not be an adequate test of the liquidity constraint. One case where this comparison is meaningful is for the Return-Enhanced Investment Grade Notes (REIGNs) issued by KKR in 2019. These long-term notes have payouts backed by investments in PE funds as their sole illiquid asset, and they have a liquidity reserve expressly to cover the uncalled commitments of these PE funds. Consistent with our liquidity constraint, the policy for REIGNs states that "... the Liquidity Reserve is required to equal the sum of all scheduled fixed coupon payments on the Notes and all future drawdown obligations." (Kroll Bond Rating Agency, 2019).

Proposition 1 (proof in Internet Appendix) shows the dynamic implications of the liquidity constraint. In the dynamic model the LP may need to hold a reserve of bonds, depending on its new and existing commitments, to ensure that it can make current and future contributions, and the proposition captures this effect.

Proposition 1. *Any optimal strategy satisfies the following inequality:*

$$B_t \quad \geq \quad a_{\mathrm{K}} K_t + a_{\mathrm{N}} N_t \tag{16}$$

The coefficients a_{K} and a_{N} are given by:

$$a_{\mathrm{K}} = \frac{\lambda_{\mathrm{K}}}{\lambda_{\mathrm{K}} + R_{\mathrm{F}} - 1} \qquad a_{\mathrm{N}} = \frac{\frac{\lambda_{\mathrm{K}}}{R_{\mathrm{F}}} + \lambda_{\mathrm{N}}\left(1 - \frac{1}{R_{\mathrm{F}}}\right)}{\lambda_{\mathrm{K}} + R_{\mathrm{F}} - 1}$$

Proposition 1 gives the required liquidity reserve that ensures that the LP can be certain to meet future contributions. The reserve is linear in the LP's new and existing commitments, with the coefficients, a_{K} and a_{N}, both between zero and one. Economically, the proposition captures the appreciation of the liquidity reserve due to the bonds earning the risk-free rate, which relaxes the liquidity constraint. Both a_{K} and a_{N} are decreasing in R_{F}, and they attain their maximal values, $a_{\mathrm{K}} = a_{\mathrm{N}} = 1$, when $R_{\mathrm{F}} = 1$. In this case, the risk free rate is zero, and the required liquidity reserve is simply $B_t \geq K_t + N_t$. Faster capital calls increase the size of the required liquidity reserve, because the reserve has less time to appreciate before it is needed to fund contributions. Hence, a_{K} and a_{N} increase in λ_{K}, and a_{N} increases in λ_{N}. Similarly, $a_{\mathrm{N}} - a_{\mathrm{K}}$ has the same sign as $\lambda_{\mathrm{N}} - \lambda_{\mathrm{K}}$, because when new commitments are called faster than existing commitments, the new commitments have less time to appreciate and therefore require a larger reserve. Finally, the liquidity reserve is independent of the rate of distributions, λ_{D}, because distributions are risky, and the LP cannot rely on them to pay future contributions.

2 Optimal Allocation to Private Equity

We use a standard value-function iteration algorithm (Ljungqvist and Sargent, 2018) to solve the Bellman equation of the normalized problem in equation (14) for different levels of risk aversion. We simulate the model 1000 times for initial convergence ("burn-in") followed by

10,000 simulations to recover the joint distribution of returns, state variables, and choice variables. Appendix A describes the details of the numerical procedure. Panel A of Table 2 shows the resulting distributions of the state variables, choice variables, and portfolio returns. For comparison, Panels B and C also show the solutions to the liquid model and the model with a secondary market, discussed below.

The average allocations are shown in Figure 6. Unsurprisingly, LPs with higher risk aversion choose smaller stock allocations and larger bond allocations. Perhaps more surprisingly, the optimal PE allocation is not monotonically decreasing with risk aversion. It follows an inverse-U shape and largely remains in the range of 15%–25%. To illustrate, the conservative LP, with $\gamma = 3$, has an average PE allocation of $\omega_{\mathrm{P}} = 16.2\%$, and its new capital commitments are $n = 5.3\%$, on average, bringing the total amount of uncalled commitments of $n+k = 17.5\%$ each period. The aggressive LP, with $\gamma = 1$, has a PE allocation of $\omega_{\mathrm{P}} = 23.6\%$, new commitments of $n = 7.6\%$, and total uncalled commitments of $n + k = 25.9\%$, on average, each period. Their liquid holdings differ more substantially. The conservative LP, on average, holds $w_{\mathrm{S}} = 24.4\%$ in stocks and $\omega_{\mathrm{B}} = 59.4\%$ in bonds. The aggressive LP holds $\omega_{\mathrm{S}} = 50.9\%$ in stocks, almost twice as much as the conservative LP, and it holds only $\omega_{\mathrm{B}} = 25.6\%$ in bonds. Despite PE investments being substantially more risky than both stocks and bonds, the PE allocation is much less sensitive to the LP's risk aversion.

These allocations can be compared to the first-best allocations, which are the optimal allocations if PE were liquid and could be freely traded at a price equal to the NAV, effectively making PE investments another traded stock. This liquid model is standard, it is described in Appendix B, and the optimal allocations are given in Panel B of Table 2 and shown in Figure 7. The LP's PE allocation now declines in risk aversion, as expected. Moreover, for more risk averse LP's, the first-best allocations from the liquid model largely coincide with the allocations with illiquid PE investments. To illustrate, for the conservative LP the average PE allocation is $\omega_{\mathrm{P}} = 16.2\%$ which effectively equals its first-best allocation of $\omega_{\mathrm{P}} = 15.9\%$. Intuitively, the

Table 2: Summary statistics of state and choice variables implied by the LP's optimal policies. This table compares the distribution of state variables, choice variables, and portfolio returns resulting from the optimal policies of two types of LP, aggressive and conservative, across three different models. The reduced model is given in Section 1, the liquid model is in Appendix B, and the secondary market is discussed in Section 3. All figures in the table are percentages.

Panel A: Reduced Model

	Aggressive ($\gamma = 1$)					Conservative ($\gamma = 3$)				
Variable	Mean	S.D.	p5	p50	p95	Mean	S.D.	p5	p50	p95
p	22.4	2.9	18.0	22.1	27.6	15.5	2.6	12.0	15.2	20.4
k	18.3	3.4	12.9	18.1	24.0	12.3	2.9	7.0	12.3	17.0
n	7.6	3.3	1.8	7.7	12.9	5.3	3.2	0.4	5.0	10.8
$k+n$	25.9	0.3	25.5	25.9	26.2	17.5	3.7	10.0	17.8	23.3
ω_{P}	23.6	3.1	19.0	23.3	29.1	16.2	2.7	12.5	15.8	21.3
ω_{S}	50.9	3.0	45.8	50.9	55.1	24.4	4.1	16.7	25.0	30.1
ω_{B}	25.6	0.3	25.1	25.6	25.9	59.4	1.4	57.4	59.2	62.0
$R_{\mathrm{W}} - R_{\mathrm{F}}$	6.4	21.4	-22.7	3.3	45.5	3.7	12.6	-13.2	1.8	27.0
c	5.1	0.0	5.0	5.1	5.1	4.1	0.0	4.1	4.1	4.1

Panel B: Liquid Model (first-best)

	Aggressive ($\gamma = 1$)					Conservative ($\gamma = 3$)				
Variable	Mean	S.D.	p5	p50	p95	Mean	S.D.	p5	p50	p95
p	55.6	0.0	55.6	55.6	55.6	15.2	0.0	15.2	15.2	15.2
ω_{P}	58.5	0.0	58.5	58.5	58.5	15.9	0.0	15.9	15.9	15.9
ω_{S}	41.5	0.0	41.5	41.5	41.5	24.8	0.0	24.8	24.8	24.8
ω_{B}	0.0	0.0	0.0	0.0	0.0	59.3	0.0	59.3	59.3	59.3
$R_{\mathrm{W}} - R_{\mathrm{F}}$	10.5	35.9	-36.6	4.4	77.1	3.6	12.4	-13.3	1.9	26.8
c	5.0	0.0	5.0	5.0	5.0	4.1	0.0	4.1	4.1	4.1

Panel C: Secondary Market

	Aggressive ($\gamma = 1$)					Conservative ($\gamma = 3$)				
Variable	Mean	S.D.	p5	p50	p95	Mean	S.D.	p5	p50	p95
p	53.9	4.4	48.3	53.2	61.6	15.5	2.6	12.1	15.1	20.2
k	46.1	13.6	26.8	44.7	70.6	12.2	3.1	6.6	12.4	17.1
n	18.5	12.5	0.0	18.4	39.0	5.3	3.4	0.7	4.8	11.6
$k+n$	64.6	4.9	59.3	64.0	71.5	17.5	4.0	9.7	17.7	23.4
ω_{P}	56.7	4.6	50.1	56.0	64.8	16.2	2.7	12.6	15.8	21.0
ω_{S}	36.9	4.9	28.8	37.7	42.5	24.4	4.1	17.0	25.0	29.9
ω_{B}	6.4	0.9	5.8	6.3	7.0	59.4	1.4	57.5	59.2	62.0
$R_{\mathrm{W}} - R_{\mathrm{F}}$	10.1	34.0	-34.0	4.3	73.8	3.7	12.6	-13.1	1.8	26.8
c	5.0	0.0	5.0	5.0	5.1	4.1	0.0	4.1	4.1	4.1

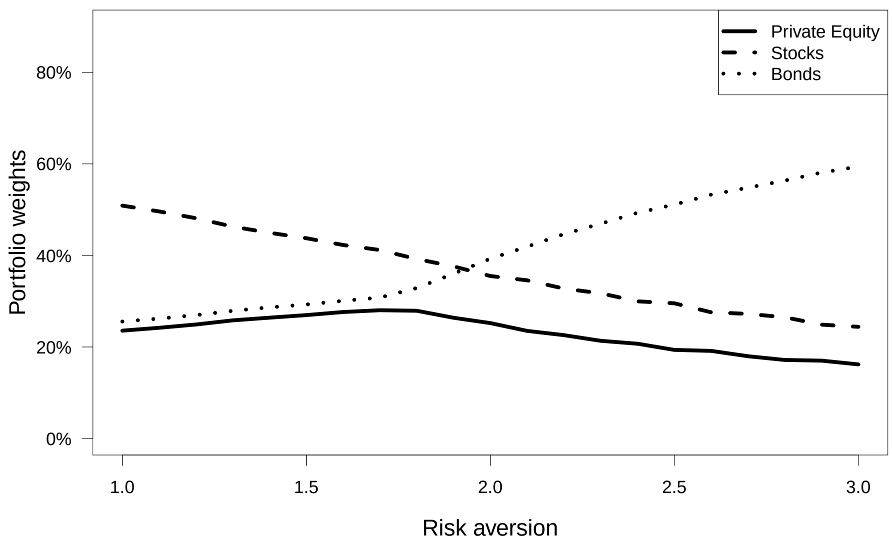


Figure 6: Average portfolio weights resulting from the model. This figure plots the average allocation of LPs solving our investment problem where PE is illiquid and requires commitment. Average allocations are computed in three steps. First, we solve numerically the Bellman equation (14). At any point in the state space, that solution provides the corresponding portfolio weights in stocks $\omega_{\mathrm{S}}^{*} = s^{*}/(1-c^{*})$ and PE $\omega_{\mathrm{P}}^{*} = p/(1-c^{*})$, while $1-\omega_{\mathrm{S}}^{*}-\omega_{\mathrm{P}}^{*}$ is the risk-free allocation. Second, we simulate the evolution of those variables under optimal strategies. Third, we compute their averages. This procedure is repeated for relative risk aversion $\gamma \in \{1, 1.1, 1.2, \ldots, 2.9, 3\}$.

liquidity constraint tends to be slack for more risk averse LPs, and they can effectively ignore the illiquidity of PE investments in their portfolio allocations.

In contrast, for less risk averse LPs the liquidity constraint tends to bind, and these LPs allocate substantially less capital to PE and substantially more to bonds than their first-best allocations. For the aggressive LP, the average PE allocation is $\omega_{\mathrm{P}} = 23.6\%$ and its first-best allocation is $\omega_{\mathrm{P}} = 58.5\%$.

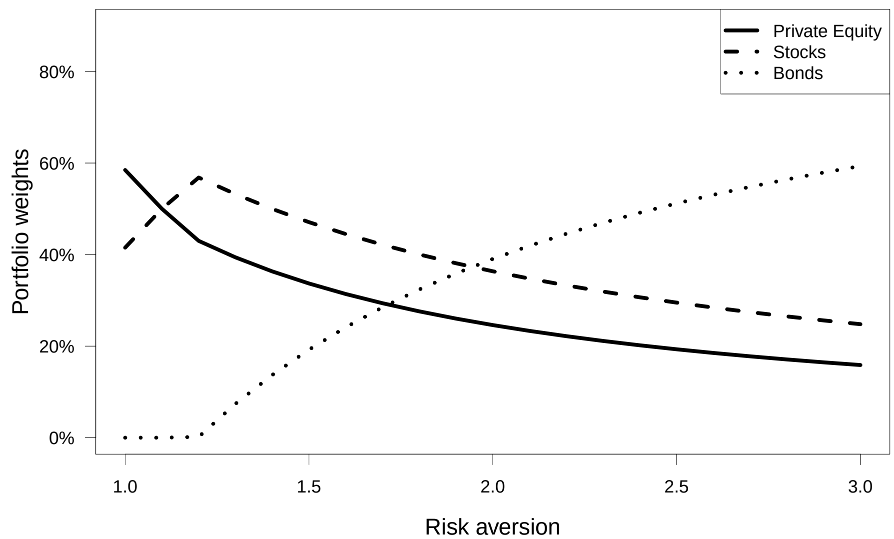


Figure 7: Optimal portfolio weights with liquid PE. This figure plots the optimal portfolio weights of investors operating in perfectly liquid markets. Optimal portfolio weights in stocks and PE are respectively ω_{S} and ω_{P} solving Bellman equation (B.2). The residual weight goes to risk-free bonds. Markets are perfectly liquid because PE can be traded freely like stocks and bonds. We consider investors with relative risk aversion coefficient $\gamma \in \{1, 1.1, 1.2, \ldots, 2.9, 3\}$.

Because the conservative LP's portfolio is largely unaffected by the illiquidity of PE investments, its returns are also largely unaffected, and it earns an average return of 3.7% both with illiquid and liquid PE investments. In contrast, the aggressive LP earns an average return of just 6.4% with illiquid PE investments, which is substantially below the 10.6% average return this LP would earn from its first-best allocation.

These allocations and returns are calculated for the baseline specification where PE has a beta of 1.6 and an alpha of 3%. We solve the model for other specifications, and Appendix C shows the optimal allocations and policies. A consistent finding across these specifications is that PE allocations are less sensitive to the LP's risk aversion than the allocations to stocks and bonds, despite PE being substantially more risky.

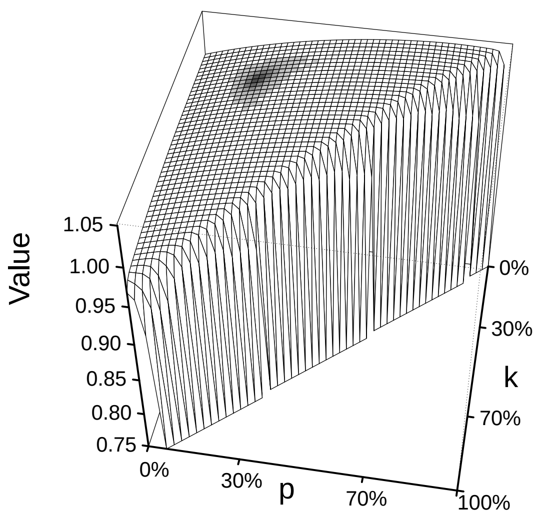
2.1 Value Function

Figure 8 shows the LP's normalized value function, $v(p, k)$, as a function of the two normalized state variables: illiquid wealth as a fraction of total wealth, $p = P/W$, and uncalled commitments as a fraction of total wealth, $k = K/W$. Returns are risky and transitions in the state space are stochastic. In Figure 8 the state space is divided into cells with sides of 2%, and the shading indicates how frequently each cell is visited during the simulations. Cells that are never visited are blank. Cells that are visited at least once (and less than 10%) are increasingly darker. A single cell is visited in more than 10% of the simulations, which is when then conservative LP holds $14\% < p < 16\%$ and $12\% < k < 14\%$, and it is shaded black.

For both the aggressive ($\gamma = 1$) and the conservative LP ($\gamma = 3$) the value function is decreasing in uncalled commitments, k, when uncalled commitments increases above the shaded area of the state space. This follows naturally from the first-order condition for k, because an LP can freely commit additional capital to PE, and the LP will continue to commit new capital as long as the value function is increasing.

Importantly, in Figure 8 the value function increases in illiquid wealth, p, for the aggressive LP ($\gamma = 1$), but it decreases in p for the conservative LP ($\gamma = 3$), above the shaded part of the state space. An aggressive LP would prefer to increase its PE exposure, but the binding liquidity constraint makes it costly to commit additional capital because it also requires an increase in the LP's liquidity reserve. In contrast, the conservative LP's liquidity constraint tends to be slack, and its portfolio moves around an interior optimum. Its first-best bond allocation exceeds

a. Aggressive ($\gamma = 1$) b. Conservative ($\gamma = 3$)



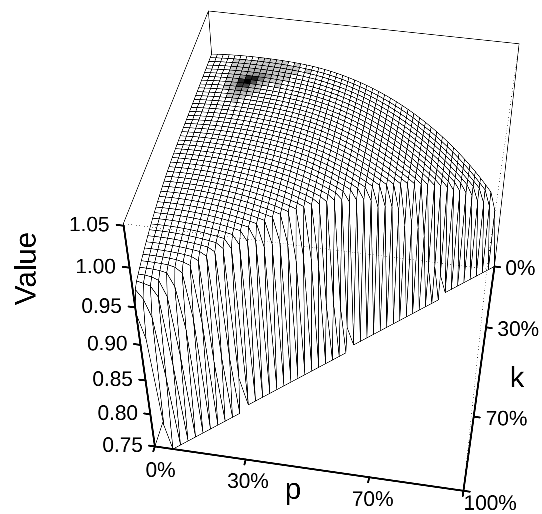


Figure 8: Value function per unit of wealth. This figure compares the value function $v(p, k)$ of the normalized Bellman equation (14) for investors with low risk-aversion ($\gamma = 1$) and investors with high risk-aversion ($\gamma = 3$). In the figure, each function is rescaled and expressed in units of value at $(p, k) = (0, 0)$. That is, we plot $v(p, k)/v(0, 0)$ for the two types of investors. We only display functional values above a certain threshold, which is 0.75 for the low risk-aversion case and 0.7 for the high risk-aversion case. The length of each axis is kept constant across the two cases, so the shapes of the functions are directly comparable. The shade on the surface of the function indicates how often investors reach a certain area of the state space. Darker areas indicate states that are reached more often. Shades are constructed using a Monte Carlo simulation where investors behave optimally.

the amount needed for its liquidity reserve, and it can freely commit additional capital to PE until it reaches its first-best allocation, given by the liquid model.

2.2 Optimal New Commitments

The aggressive and conservative LPs have different optimal policies for new PE commitments, as illustrated in Figure 9. A conservative LP ($\gamma = 3$) tends to be close to an interior optimum, and in response to a positive shock to the NAV of its PE investments this LP reduces its new PE commitments to rebalance its PE exposure back towards the interior optimum. In contrast, an aggressive LP prefers a greater PE exposure but it cannot commit additional capital due to the binding liquidity constraint. The aggressive LP therefore does not reduce its new PE commitments in response to a positive shock to the NAV of its existing PE investments. Instead, it enjoys a temporary increase in its PE exposure, and it continues to make new commitments at the same maximally sustainable rate, as seen in Figure 9.

Figure 9 shows that the optimal policies for new PE commitments consist of multiple segments, each of which appears to be close to linear. The shading shows, however, that the state mostly moves on a single such segment, although it occasionally drops to the bottom segment with zero new commitments. In Table 2 the aggressive LP's average (standard deviation) of new commitments, n, is 7.6% (3.3%) (and n equals zero for 0.5% of the simulations). For the conservative LP, n is 5.3% (3.2%) (and n equals zero for 1.5% of the simulations).

Since the state remains mostly on a single linear segment, we can locally approximate the optimal policy by regressing the optimal new commitments, $n^*(p, k)$, on the two normalized state variables. Table 3 shows that regressions with and without an interaction term have similar coefficients, and that the R^2 only increases marginally when the interaction term is included, confirming that the segment is close to linear and is well approximated by the linear regression. Hence, we focus on specifications (1) and (3), without interaction terms, because their coeffi-

a. Aggressive ($\gamma = 1$) b. Conservative ($\gamma = 3$)

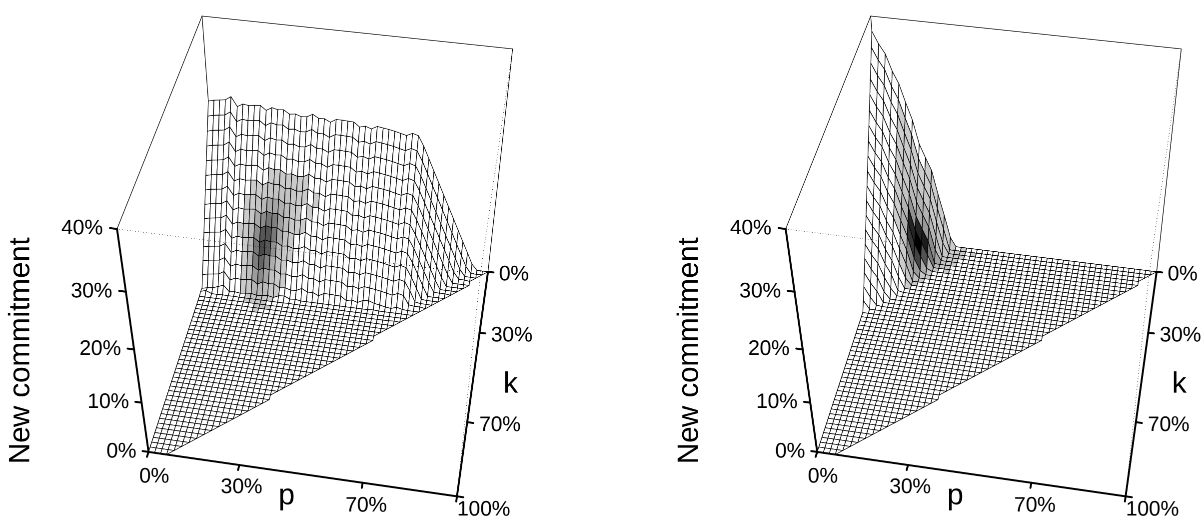


Figure 9: Optimal new commitment. This figure compares optimal commitment strategy for investors with low risk-aversion ($\gamma = 1$) and investors with high risk-aversion ($\gamma = 3$). For each type, the plots show optimal new commitment as a share of wealth $n^*(p, k)$ obtained solving the Bellman equation (14). The shade on the surface of the function indicates how often investors reach a certain area of the state space. Darker areas indicate states that are reached more often. Shades are constructed using a Monte Carlo simulation where investors behave optimally.

cients are easier to interpret. Figures C.5 and C.6 in the appendix show the optimal policies for other parameter choices, and Table C.7 contains the corresponding regression coefficients. In Table 3 the coefficient for PE exposure (or NAV), p, is consistently close to zero for the

Table 3: This table reports coefficients β_0, β_1, β_2, and β_3 from OLS estimations of the following model:

$$n^*(p,k) = \beta_0 + \beta_1 p + \beta_2 k + \beta_3 (p \times k) + \epsilon$$

In this equation, ϵ is an error term, while $n^*(p,k)$ is the optimal commitment function obtained solving problem (14), and plotted in figure 9 for two different levels of risk-aversion. The model is estimated twice for the case of low risk-aversion ($\gamma = 1$) and twice for the case of high risk-aversion ($\gamma = 3$). For each level of risk-aversion, the first estimation imposes $\beta_3 = 0$, while the second estimation ignores that restriction. The data for each estimation is taken from a Monte Carlo simulation where the evolution of the state variables is simulated under optimal policies. We run the simulation for 11000 periods, discarding the first 1000 periods and eventual observations for which $n^* = 0$.

	Aggressive ($\gamma = 1$)		Conservative ($\gamma = 3$)	
	(1)	(2)	(3)	(4)
Illiquid wealth (p)	-0.02	-0.07	-1.45	-1.53
Commitments (k)	-0.99	-1.04	-0.98	-1.09
Interaction ($p \times k$)		0.27		0.71
Constant	0.26	0.27	0.40	0.41
R^2	1.00	1.00	0.98	0.98

aggressive LP, and it is negative for the conservative LP (see Table C.7 for the variation across other specifications). The coefficient for uncalled commitments, k, is close to -1.0 for both LPs. Hence, the optimal policy for new PE commitments is approximately:

$$n^*(p,k) + k \approx \text{constant} - \pi_{\mathrm{P}} p \tag{17}$$

where π_{P} is the negative of the regression coefficient on p. For example, in specification (3) in Table 3, it equals $\pi_{\mathrm{P}} = 1.45$. The LPs optimally determine their new commitments, n, to target a level certain of total uncalled commitments, $n + k$. For the aggressive LP, the optimal policy for new commitments is simple. The coefficient on p is close to zero, so π_{P} is also close to zero, and the aggressive LP targets a fixed level of uncalled commitments, $n + k \approx$ constant, regardless of the performance of its other investments. For the conservative LP, the coefficient π_{P} is positive, and this LP optimally reduces the target level of uncalled commitments in response to a positive shock its PE investments.

2.3 Optimal Stock and Bond Allocations

a. Aggressive ($\gamma = 1$) b. Conservative ($\gamma = 3$)

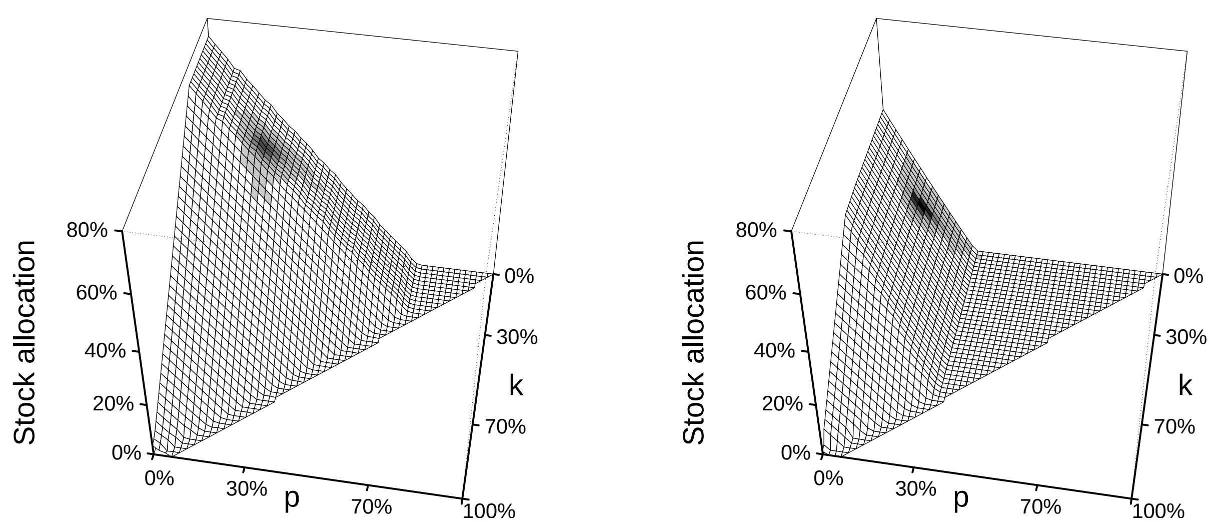


Figure 10: Optimal stock allocation. This figure compares optimal stock allocation for investors with low risk-aversion ($\gamma = 1$) and investors with high risk-aversion ($\gamma = 3$). For each type, the plots show optimal portfolio weight in stocks $\omega^*_{\mathrm{S}}(p, k)$ obtained solving the Bellman equation (14). The shade on the surface of the function indicates how often investors reach a certain area of the state space. Darker areas indicate states that are reached more often. Shades are constructed using a Monte Carlo simulation where investors behave optimally.

Figure 10 shows optimal stock allocations as functions of the state variables: illiquid wealth, p, and uncalled commitments, k. Table 2 shows that the aggressive LP's average (standard deviation) stock allocation is $\omega_{\mathrm{S}} = 50.9\%$ (3.0%), its average bond allocation is $\omega_{\mathrm{B}} = 25.6\%$ (0.3%), and its average PE allocation is $\omega_{\mathrm{P}} = 23.6\%$ (3.1%). For the conservative LP, the average stock allocation is 24.4% (4.1%), average bond allocation is 59.4% (1.4%), and average PE allocation is 16.2% (2.7%). Table 4 shows the coefficients from a regression of the optimal stock allocation on the state variables in the shaded part of the state space. As above, the optimal allocation is well approximated by a linear regression, and we focus on specifications

(1) and (3) without interaction terms.

In Table 4, the coefficients on uncalled commitments, k, are close to zero, and the LPs' optimal stock allocations are largely unaffected by their aggregate uncalled PE commitments. This is natural. The LPs commit new capital, n, each period to bring their total uncalled commitments, $k + n$, to a target level. Small deviations in the LPs' uncalled commitments, k, are therefore immaterial, because the LPs can undo these deviations by adjusting their new commitments, n, to reach the same target level. Only large positive shocks to uncalled commitments can affect the LPs' optimal allocations, because new commitments cannot be reduced below zero in response to such shocks. Hence, the LPs' optimal stock allocation is approximately:

$$\omega_{\mathrm{S}}^{*}(p,k) \approx \text{constant} - \phi_{\mathrm{P}}\omega_{\mathrm{P}} \tag{18}$$

In this expression, the PE portfolio weight, ω_{P}, equals $p/(1-c)$, the coefficient ϕ_{P} is the negative of the regression coefficient on p multiplied by $1-c$, where c is the LP's average consumption-to-wealth ratio. The approximation then follows from the low volatility of the optimal consumption-to-wealth ratio seen in Table 2. For example, in specification (3) of Table 4, $\phi_{\mathrm{P}} = 1.59 \times (1 - 0.041) = 1.53$.

The conservative LP hedges a larger PE exposure by reducing its stock allocation. Across the specifications in Table 4, the conservative LP's coefficient for PE allocation, ϕ_{P}, is close to the β of PE. Intuitively, the conservative LP is unconstrained, and when the beta of PE equals 1.6, as in our baseline specification, the conservative LP optimally responds to a 1% increase in its PE allocation, ω_{P}, by reducing its stock allocation by 1.6% to maintain a constant risk exposure across its total portfolio.

In contrast, the aggressive LP has a ϕ_{P} close to one. This LP's liquidity constraint binds, and it maintains a maximally sustainable level of uncalled commitments along with a liquidity reserve of bonds that is consistent with this level. Since $\omega_{\mathrm{B}}^{*}(p,k) = 1 - \omega_{\mathrm{S}}^{*}(p,k) - \omega_{\mathrm{P}}$, rearranging the expression shows that the aggressive LP's optimal bond allocation, ω_{B}, is approximately constant:

$$\omega_{\mathrm{S}}^*(p,k) + \omega_{\mathrm{P}} \approx \text{constant} \Leftrightarrow \omega_{\mathrm{B}}^*(p,k) \approx 1 - \text{constant} \tag{19}$$

Table 4: This table reports coefficients β_0, β_1, β_2, and β_3 from OLS estimations of the following model:

$$\omega_{\mathrm{S}}^*(p,k) = \beta_0 + \beta_1 p + \beta_2 k + \beta_3 (p \times k) + \epsilon$$

In this equation, ϵ is an error term, while $\omega_{\mathrm{S}}^*(p,k)$ is the optimal stock allocation obtained solving problem (14), and plotted in figure 10 for two different levels of risk-aversion. The model is estimated twice for the case of low risk-aversion ($\gamma = 1$) and twice for the case of high risk-aversion ($\gamma = 3$). For each level of risk-aversion, the first estimation imposes $\beta_3 = 0$, while the second estimation ignores that restriction. The data for each estimation is taken from a Monte Carlo simulation where the evolution of the state variables is simulated under optimal policies. We run the simulation for 11000 periods, discarding the first 1000 periods and eventual observations for which $\omega_{\mathrm{S}}^* = 0$.

	Aggressive ($\gamma = 1$)		Conservative ($\gamma = 3$)	
	(1)	(2)	(3)	(4)
Illiquid wealth (p)	−1.04	−1.01	−1.59	−1.56
Commitments (k)	−0.03	0.00	0.00	0.05
Interaction ($p \times k$)		−0.13		−0.31
Constant	0.75	0.74	0.49	0.49
R^2	0.99	0.99	1.00	1.00

2.4 Optimal Consumption

Figure 12 shows optimal consumption for the conservative and aggressive LPs. A standard result for the liquid model with i.i.d. shocks and CRRA preferences is that optimal consumption is a constant fraction of total wealth. Figure 12 confirms that with illiquid PE investments the optimal consumption rate remains close to constant in the shaded part of the state space. The aggressive LP's average (standard deviation) consumption ratio, c, is 5.1% (0.020%), and the conservative LP's is 4.1% (0.001%). Brown, , Kang, and Weisbenner (2014) study the payout policies of university endowments and report typical payout rates of 4%–6%, which is consistent with the optimal consumption rate in our model.

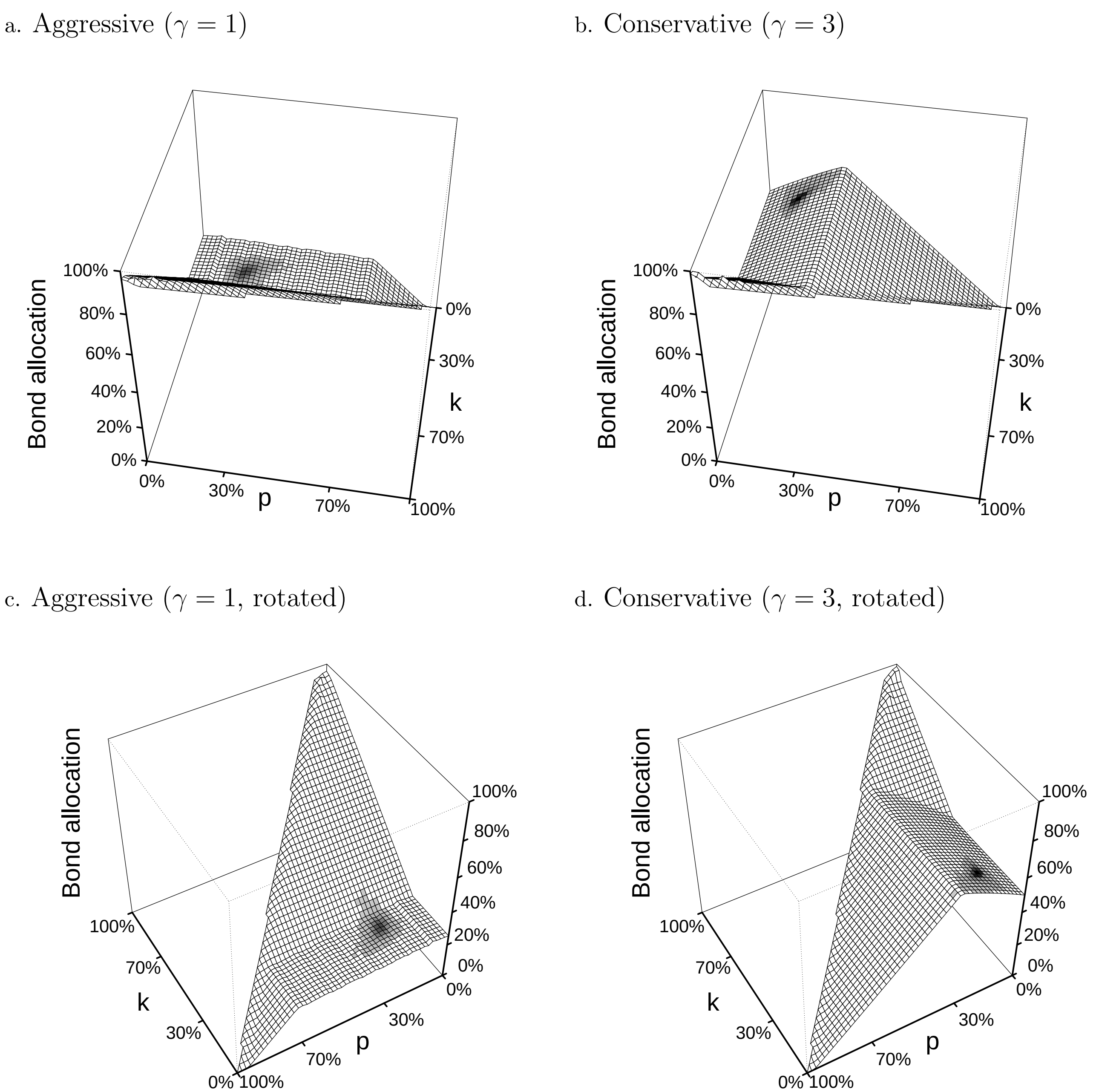


Figure 11: Optimal bond allocation. This figure compares optimal bond allocation for investors with low risk-aversion ($\gamma = 1$) and investors with high risk-aversion ($\gamma = 3$). For each type, the plots show optimal portfolio weight in bonds $\omega^*_{\mathrm{B}}(p, k) = 1 - \omega^*_{\mathrm{S}}(p, k) - \omega^*_{\mathrm{P}}(p, k)$ obtained solving the Bellman equation (14). The shade on the surface of the function indicates how often investors reach a certain area of the state space. Darker areas indicate states that are reached more often. Shades are constructed using a Monte Carlo simulation where investors behave optimally. The two plots at the bottom of the figure are rotated versions of those at the top.

Table 5: This table reports coefficients β_0, β_1, β_2, and β_3 from OLS estimations of the following model:

$$\omega^*_{\mathrm{B}}(p,k) = \beta_0 + \beta_1 p + \beta_2 k + \beta_3 (p \times k) + \epsilon$$

In this equation, ϵ is an error term, while $\omega^*_{\mathrm{B}}(p,k) = 1 - \omega^*_{\mathrm{S}}(p,k) - \omega^*_{\mathrm{P}}(p,k)$ is the optimal bond allocation obtained solving problem (14), and plotted in figure 11 for two different levels of risk-aversion. The model is estimated twice for the case of low risk-aversion ($\gamma = 1$) and twice for the case of high risk-aversion ($\gamma = 3$). For each level of risk-aversion, the first estimation imposes $\beta_3 = 0$, while the second estimation ignores that restriction. The data for each estimation is taken from a Monte Carlo simulation where the evolution of the state variables is simulated under optimal policies. We run the simulation for 11000 periods, discarding the first 1000 periods and eventual observations for which $\omega^*_{\mathrm{S}} = 0$.

	Aggressive ($\gamma = 1$)		Conservative ($\gamma = 3$)	
	(1)	(2)	(3)	(4)
Illiquid wealth (p)	-0.02	-0.04	0.55	0.52
Commitments (k)	0.03	0.00	0.00	-0.05
Interaction ($p \times k$)		0.13		0.31
Constant	0.25	0.26	0.51	0.51
R^2	0.15	0.16	1.00	1.00

a. Aggressive ($\gamma = 1$)

b. Conservative ($\gamma = 3$)

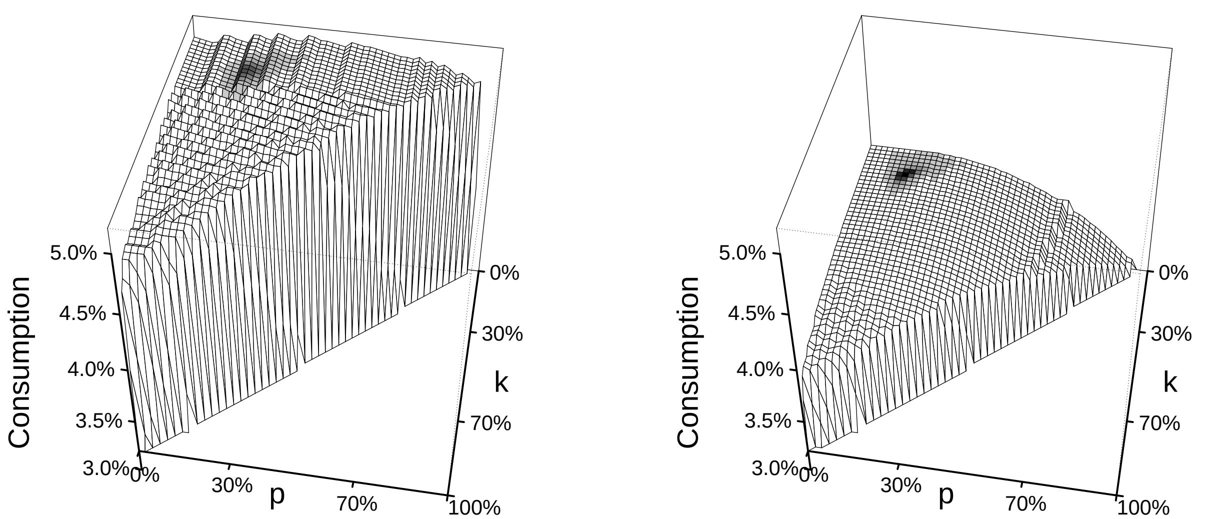


Figure 12: Optimal consumption as a fraction of wealth. This figure compares optimal consumption strategy for investors with low risk-aversion ($\gamma = 1$) and investors with high risk-aversion ($\gamma = 3$). For each type, the plots show optimal consumption to wealth ratio $c^*(p,k)$ obtained solving the Bellman equation (14). The shade on the surface of the function indicates how often investors reach a certain area of the state space. Darker areas indicate states that are reached more often. Shades are constructed using a Monte Carlo simulation where investors behave optimally.

3 Secondary Market

We now introduce a secondary market for PE investments where LPs can trade partnership interests in PE fund.[12] This creates two effects: First, there are gains from trade when unconstrained LPs can sell partnership interests in mature PE funds to constrained LPs. Mature PE funds are partly invested, with positive NAVs and fewer uncalled commitments, allowing a constrained LP to increase its PE exposure with a relatively smaller increase in its liquidity needs. Second, following a negative shock to the value of its liquid investments, a stressed LP may decide to liquidate a fraction of its PE investments, even at a discount, to raise liquidity. Moreover, access to liquidity, ex-post, relaxes the LP's liquidity constraint, ex-ante, allowing it to hold more risky allocations.

3.1 Gains from Trade

An aggressive constrained LP assigns a higher valuation to a mature PE fund than a conservative unconstrained LP. The aggressive LP prefers greater PE exposure, but it faces a binding liquidity constraint, and a mature fund provides PE exposure with a relatively smaller amount of uncalled commitments. In contrast, a conservative LP assigns no special value to mature PE funds compared to simply committing additional capital to new PE funds, even though these new commitments imply a higher level of uncalled commitments.

To quantify the gains from trade, Table 6 shows the effects of a transfer of PE investments from a conservative to an aggressive LP. In this calculation, the two LPs have the same amounts of wealth, so the transfer represents the same percentage change in their PE investments. The transfer is compensated by a payment of liquid wealth that leaves both LPs' total wealth

[12] The secondary market for partnership interests is distinct from the market where PE funds trade underlying private assets, typically by buying and selling equity stakes in portfolio companies, which is also called a "secondary market." **?** model valuation with a secondary market for partnership interests. Hege and Nuti (2011), Albuquerque, Cassel, Phalippou, and Schroth (2018), and Nadauld, Sensoy, Vorkink, and Weisbach (2018) report empirical evidence of pricing and discounts for secondary transactions in the secondary market for partnership interests.

unchanged. The two LPs start with the average levels of NAV, p, and uncalled commitments, k, that are implied by their respective optimal investment policies. For the conservative LP, these levels are $p_{\mathrm{C}} = 15.5\%$ and $k_{\mathrm{C}} = 12.3\%$ (see Table 2). For the aggressive LP, they are $p_{\mathrm{A}} = 22.4\%$ and $k_{\mathrm{A}} = 18.3\%$. The transaction transfers an amount of NAV, denoted Δp, and an amount of uncalled commitments, denoted Δk, from the conservative to the aggressive LP. The aggressive LP pays Δp of liquid wealth to the conservative LP, leaving their total wealths unchanged. After the transaction, the aggressive LP's NAV is $p_{\mathrm{A}} + \Delta p$, and its uncalled commitments are $k_{\mathrm{A}} + \Delta k$. The conservative LP has $p_{\mathrm{C}} - \Delta p$ and $k_{\mathrm{C}} - \Delta k$.

Table 6 reports normalized value functions, $v(p, k)$, for different levels of Δp and Δk. The first column of Table 6 shows the initial values, $v(p, k)$, before the transaction, i.e., with $\Delta p = \Delta k = 0\%$. The following columns show the effects of transacting PE funds with decreasing maturities, corresponding to increasing amounts of uncalled commitments, Δk, and decreasing amounts of NAV, Δp. Table 6 shows the two value functions and their sum. An increase in the sum implies that the transfer increases the two LPs' aggregate utility.

Even when the value function for the conservative LP declines, as long as the sum increases, there exists an additional transfer of liquid wealth that leaves both LPs strictly better off.

Table 6 shows that the conservative LP is largely unaffected by the transaction. This LP is unconstrained, with an excess liquidity reserve, and it compensates for the reduction in p by increasing its new commitments to rebuild its PE exposure. In the short term, it also increases its stock allocation to rebalance its risk exposure. The transfer of PE funds therefore leaves the conservative LP with only a minor and temporary deviation from its optimal risk exposures, and its value function remains effectively unchanged.

In contrast, the transaction benefits the aggressive LP by temporarily increasing its PE exposure. The benefit is larger for a more mature fund, with greater NAV and less uncalled commitments. The benefit declines for younger funds, and a transaction of just uncalled commitments, $\Delta p = 0\%$, does not benefit the aggressive LP.

In Table 6 the greatest increase in total valuations is 0.402%, and the gains from a single transaction are therefore modest. The gains are larger when the LPs transact repeatedly. Anticipating such repeated transactions, however, an unconstrained LP would increase its PE commitments to have extra PE funds to sell to a constrained LP, as these PE funds mature. In turn, a constrained LP would reduce its own commitments to new funds and instead manage its PE exposure by acquiring mature PE funds from an unconstrained LP, further relaxing its liquidity constraint. Formally, analyzing such strategic interactions among LPs is interesting, but it is well beyond the scope of this paper.

Table 6: This table quantifies the net welfare gains from a single transfer of PE partnership interests of $(\Delta p, \Delta k)$ from a conservative to an aggressive LP. The two LPs start with the same amounts of wealth and the average levels of p and k that are implied by their optimal investment policies.

Δp Δk	No trade	10% 0%	8% 2%	5% 5%	2% 8%	0% 10%
$v_{\mathrm{A}}(p_{\mathrm{A}} + \Delta p, k_{\mathrm{A}} + \Delta k)$	.0624	.0629	.0628	.0627	.0625	.0624
$v_{\mathrm{C}}(p_{\mathrm{C}} - \Delta p, k_{\mathrm{C}} - \Delta k)$	.0372	.0371	.0372	.0372	.0372	.0372
$v_{\mathrm{A}} + v_{\mathrm{C}}$	.0996	.1000	.0999	.0998	.0997	.0996
Gains from Trade	0.000%	0.402%	0.331%	0.221%	0.090%	0.000%

3.2 Insuring Liquidity Shocks

A secondary market for partnership interests can also insure LPs against liquidity shocks. To quantify this effect, we extend the model to allow the LP to sell, but not buy, a fraction of its PE investments each period. The extension is summarized below, and more details are in Appendix D. In practice, an LP must sell its currently held PE funds, with their actual mix of NAV and uncalled commitments. We capture this mix by restricting the LP to sell NAV and uncalled commitments in the same proportions as in the LP's current PE investments. Each period, t, the LP can liquidate a fraction, $0 \leq f_t \leq 1$, of its PE investments. When $f_t = 0$ the LP does not transact in the secondary market. When $f_t > 0$ the sale of PE investments reduces the LP's NAV by $f_t P_t$ and reduces the uncalled commitments by $f_t K_t$. The fraction f_t is an additional choice variable in the LP's problem. It is convenient to model the transaction using the non-normalized NAV, P_t, and non-normalized uncalled commitments, K_t (rather than the normalized p_t and k_t). Due to the discounts, described below, the LP's total wealth decreases when it transacts in the secondary market, complicating the resulting changes in the normalized state variables.

Liquidating PE investments at times of stress is costly, and they are liquidated at a discount, which represents a transfer of wealth from the selling to the buying LPs.[13] We take this discount as given. LPs that provide liquidity must be relatively unconstrained. Albuquerque et al. (2018) report that buying LPs are mostly pension funds, endowments, and banks, which are arguably relatively conservative and unconstrained investors. As shown above, unconstrained LPs have lower valuations of mature funds than constrained LPs, which would naturally creates a discount for stressed sales of PE investments in the secondary market.

The discount has two parts. The LP sells $f_t P_t$ of NAV in return for $(1-\psi_\mathrm{P}) f_t P_t$ of liquid wealth, where $\psi_\mathrm{P} \geq 0$ is the discount to NAV. The sale also reduces the LP's uncalled commitments by $f_t K_t$, which costs the LP $\psi_\mathrm{K} f_t K_t$, where $\psi_\mathrm{K} \geq 0$ is the cost of disposing of uncalled commitments. In practice, reported discounts for secondary transactions are typically (one minus) the transaction price divided by the NAV, which confounds these two parts of the discount. Empirically disentangling the two parts would require comparing transactions of, otherwise similar, funds with relatively more and less uncalled commitments.

In terms of timing, we assume that the LP transacts in the secondary market after contributions and distributions are paid and before new commitments are made. Uncalled commitments are now the sum of the LP's remaining existing commitments, $(1-f_t)K_t$, and its new commitments, N_t. The LP's illiquid wealth is the remaining NAV, $(1-f_t)P_t$. End-of-period wealth is now

[13] Because the discount is a transfer between the transacting LPs, it is not relevant for the analysis of gains from trade in the previous section.

also net of the discount for transacting in the secondary market, $f_t\left(\psi_{\mathrm{P}}P_t + \psi_{\mathrm{K}}K_t\right)$.

In our baseline specification, $\psi_{\mathrm{K}} = 5\%$ and $\psi_{\mathrm{P}} = 20\%$. Nadauld et al. (2018) report average discounts to NAV from 9.0% to 13.8% (although, they do not separate the part of the discount due to uncalled commitments). Since our transactions occur at times of liquidity stress, we assume higher than average discounts. Higher discounts are also a conservative choice, giving a lower bound on the effects of a secondary market. Below, we find that it allows the aggressive LP to effectively return to its first-best allocations, and smaller discounts would only strengthen this finding.

The Bellman equation of the extended model is given in Appendix D. We solve the Bellman equation using the numerical methods from the original model (see Appendix A for details), and Proposition 2 generalizes the liquidity constraint with a secondary market.

Proposition 2. *Suppose that the LP can sell PE interests in the secondary market with liquidity costs ψ_{P} and ψ_{K} both between 0 and 1. Suppose also that $\lambda_{\mathrm{N}} = \lambda_{\mathrm{K}}$. Then, any optimal strategy satisfies the following inequality at all times:*

$$B_t \;\; \geq \;\; a_{\mathrm{SM}}\Big((1 - f_t)K_t + N_t\Big)$$

In this expression, B_t is savings in bonds, and the coefficient a_{SM} is defined as follows:

$$a_{\mathrm{SM}} \;\; = \;\; \frac{\psi_{\mathrm{K}}(1 - \lambda_{\mathrm{K}}) + \psi_{\mathrm{P}}\lambda_{\mathrm{K}}}{R_{\mathrm{F}}}$$

Proof in Internet Appendix.

In Proposition 2, with the secondary market, B_t equals $W_t - C_t - S_t - P_t + f_t(1-\psi_{\mathrm{P}})P_t - f_t\psi_{\mathrm{K}}K_t$. When ψ_{K} and ψ_{P} are less than one, which is natural, the coefficient a_{SM} is below the coefficient

a_{K} from proposition 1. Intuitively, a_{SM} is increasing in ψ_{K} and ψ_{P}, and the effect of λ_{K} on a_{SM} depends on the relative magnitudes of ψ_{K} and ψ_{P}.

Figure 13 shows the optimal allocations with a secondary market, and we compare these allocations to the optimal allocations without a secondary market, in Figure 6, and to the first-best allocations in Figure 7. The conservative LP has similar allocations in all three figures, because this LP already effectively holds its first-best allocation even when PE is illiquid.

In contrast, the secondary market allows the aggressive LP to hold allocations substantially closer to the first-best. Its average NAV increases from $\omega_{\mathrm{P}} = 23.6\%$ without a secondary market to $\omega_{\mathrm{P}} = 56.7\%$ with a secondary market (first-best $\omega_{\mathrm{P}} = 58.5\%$). Its stock allocation changes from $\omega_{\mathrm{S}} = 50.9\%$ without a secondary market to $\omega_{\mathrm{S}} = 36.9\%$ (first-best $\omega_{\mathrm{S}} = 41.5\%$). And its bond allocation changes from $\omega_{\mathrm{B}} = 25.6\%$ without a secondary market to $\omega_{\mathrm{B}} = 6.4\%$ (first-best $\omega_{\mathrm{B}} = 0.0\%$).

The positive allocation of 6.4% to bonds may be surprising, given that the aggressive LP's first-best bond allocation is zero. After a decline in the values of stocks and PE investments, the aggressive LP's PE investments consist of relatively more uncalled commitments, K_t, than NAV, P_t. Therefore, the discount to uncalled commitments becomes relatively more important, making it more costly for the aggressive LP to raise liquid wealth in the secondary market, net of this discount. In this situation, the LP can be unable to fund its contributions forcing it to sell its PE investments, at a discount, but with a low NAV this sale effectively requires the LP to pay a buyer to accept the LP's unfunded liabilities, and the bond reserve allows the LP make this payment and remain solvent.

The value functions with a secondary market are shown in Figure 14. The aggressive LP is now also close to the interior optimum, and its liquidity constraint tends to be slack. The secondary market, through its insurance function, increases the aggressive LP's utility by 11%.

Figure 15 shows optimal policies for trading in the secondary market, $f^*(p,k)$. As above, the state space is divided into cells, and the shading indicates how frequently a cell is visited during the simulations.[14] Cells that are not visited during the simulations are blank. Figure 15 shows that the conservative LP does not transact in the secondary market. The aggressive LP sometimes, albeit rarely, liquidates a positive fraction of its PE investments in response to a negative shock to its liquid wealth, which corresponds to a positive shock to the normalized state variables, p and k.

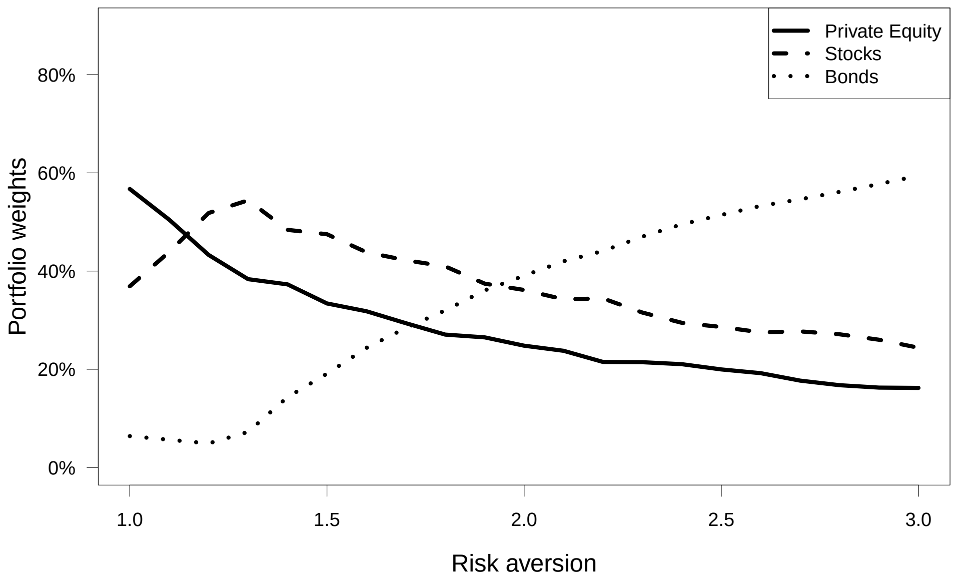


Figure 13: Average portfolio weights with secondary market. This figure plots the average allocation of LPs solving the extended model with secondary market. Average allocations are computed in three steps. First, we solve numerically the Bellman equation of the extended model (see appendix D). At any point in the state space, that solution provides the corresponding portfolio weights in stocks, private equity, and risk-free bonds. Second, we simulate the evolution of those portfolio weights under optimal strategies. Third, we compute their averages. This procedure is repeated for risk-aversion $\gamma \in \{1, 1.1, 1.2, \ldots, 2.9, 3\}$.

The Internet Appendix reports optimal allocations and policies for additional specifications. The results are robust. In general, the conservative LP is unaffected by the introduction of a secondary market. The aggressive LP benefits from the secondary market's insurance function, which allows the aggressive LP to approach its first-best allocation, even when the secondary market is characterized by relatively large discounts.

[14] With a secondary market, the LPs visit a larger part of the state space, and we increase the size of the cells to 5% times 5% to reduce the computational workload.

a. Aggressive ($\gamma = 1$) b. Conservative ($\gamma = 3$)

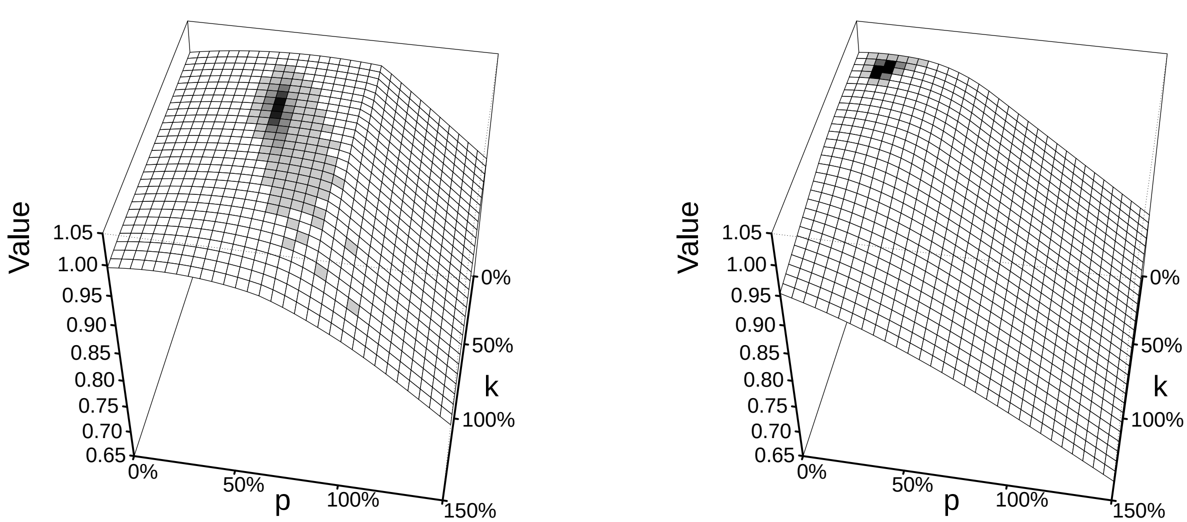
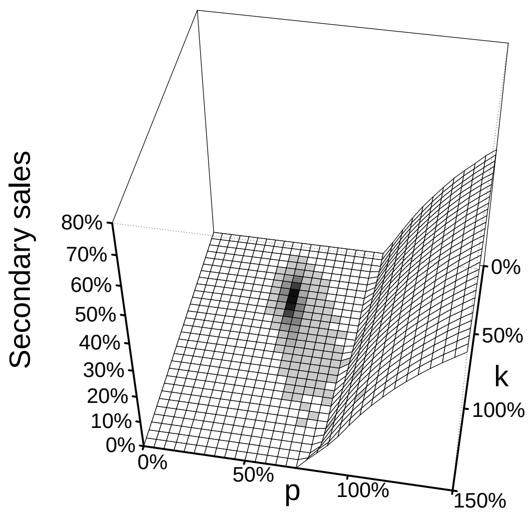
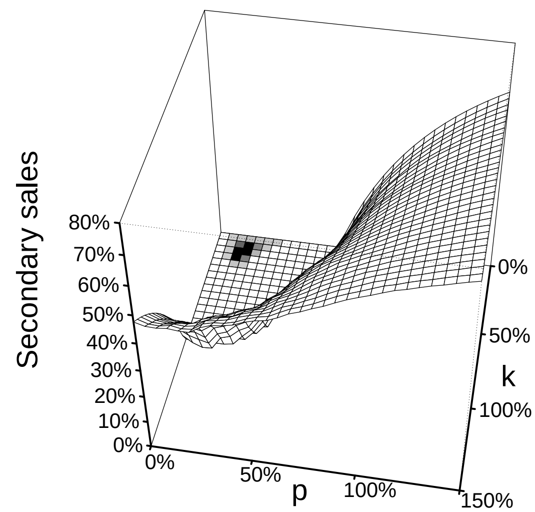


Figure 14: Value function with secondary market. This figure compares the value function of investors with low risk-aversion ($\gamma = 1$) and investors with high risk-aversion ($\gamma = 3$) from the model with secondary market. In the figure, each function is rescaled and expressed in units of value at $(p, k) = (0, 0)$. The shade on the surface of the function indicates how often investors reach a certain area of the state space. Darker areas indicate states that are reached more often. Shades are constructed using a Monte Carlo simulation where investors behave optimally.

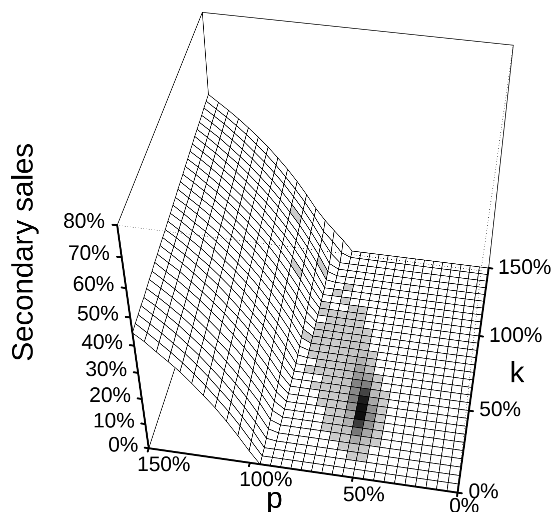
a. Aggressive ($\gamma = 1$)



b. Conservative ($\gamma = 3$)



c. Aggressive ($\gamma = 1$, rotated)



d. Conservative ($\gamma = 3$, rotated)

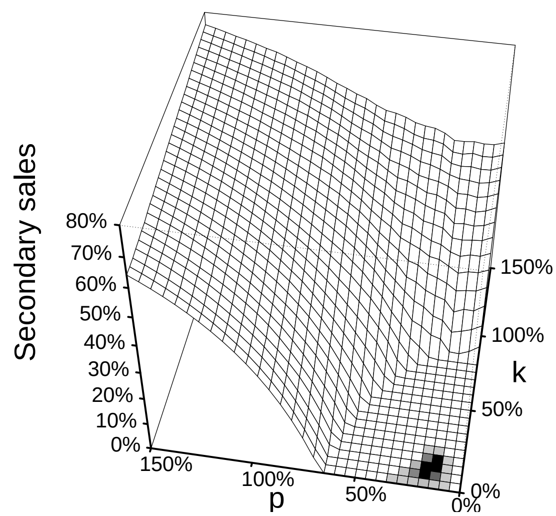


Figure 15: Optimal sales in the secondary market. This figure compares the optimal share f of PE investments sold in the secondary market at different points in the state space by investors with low risk-aversion ($\gamma = 1$) and investors with high risk-aversion ($\gamma = 3$). The shade on the surface of the function indicates how often investors reach a certain area of the state space. Darker areas indicate states that are reached more often. Shades are constructed using a Monte Carlo simulation where investors behave optimally. The two plots at the bottom of the figure are rotated versions of those at the top.

4 Conclusion

We present the first formal analysis of an LP's investment problem with ongoing commitments to an arbitrary number of private equity (PE) funds. Our model captures three aspects of PE investments: they are risky, illiquid, and long-term investments. PE investments are risky because they earn an uncertain return. They are illiquid because after committing capital to PE funds, the LP must hold this investment to maturity, and the LP cannot liquidate (or collateralize) its PE investments to convert them into current consumption, although this is relaxed somewhat with a secondary market. PE investments are long-term because the LP's committed capital is not immediately invested into private assets. Rather, the LP maintains a stock of uncalled commitments, which are gradually contributed to the PE funds, and the LP's need to pay future contributions creates the main liquidity friction in our analysis. We show that linear fund dynamics substantially simplify the problem, because the LP's aggregate uncalled commitments and aggregate uncalled NAV become summary statistics for the LP's entire portfolio of PE investment in the LP's problem.

Depending on the LP's risk aversion, we find two distinct investment strategies. A conservative LP with a high risk aversion (here, a relative risk aversion of $\gamma = 3$) is unconstrained. It's first best portfolio allocation places a sufficient amount of capital in safe assets, and the LP's liquidity constraint is not binding. The LP is close to its first-best interior optimum, and it effectively treats PE investing as another traded stock. In response to a positive shock to the value of its PE investments it reduces its allocation to traded stocks, and it reduces commitments to new PE funds to rebalance and return to the optimal portfolio.

In contrast, an aggressive LP with a lower risk aversion ($\gamma = 1$) faces a binding liquidity constraint. The aggressive LP does not rebalance its portfolio to maintain constant risk exposures. It has a substantially larger allocation to the risk free asset and a lower allocation to stocks than its first-best allocation. Each period it determines the amount of new commitments to target a given level of total uncalled commitments.

We extend the analysis with a secondary market for PE partnership interests and analyze two effects. There are gains from trade when mature PE funds are traded from unconstrained to constrained LPs, because mature funds provide greater PE exposure relative to the required liquidity reserve. Moreover, a secondary market can insure a constrained LP against negative shocks to the value of its liquid investments by providing liquidity, ex-post, to the stressed LP. In turn, anticipating that it will be able to liquidate its PE investments, an aggressive LP will hold a greater PE allocation, ex-ante. In our specification, the gains from a single trade are economically small. In contrast, insuring the aggressive LP from liquidity shocks has a large effect, and it effectively allows the aggressive LP to hold its first-best portfolio.

Our model also allows for different specifications of management fees, which slightly affects the interpretation of the state variables and the return to PE investments (either net or gross of management fees). Most of our analysis assumes implicit management fees, because the resulting portfolio allocations can be compared directly to the first-best allocations from a liquid model. Another benefit of implicit fees is that the PE return in the model is net of fees, which is easier to calibrate because this is the return that is usually reported in industry studies and empirical research. We also analyze the model with explicit management fees, but after the appropriate adjustments, the solution and optimal policies are very similar, and this modeling choice appears unimportant for our analysis.

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A Numerical Methods

In this section, we describe the numerical procedure used to solve the portfolio problems in the main text. We take the reduced model as a working example.

We use a standard value function iteration algorithm to solve the following problem:

$$
\begin{aligned}
& v(p,k) = \max_{(c,n,\omega_{\mathrm{S}})} \left\{ c^{1-\gamma} + \delta \mathrm{E}\left[[(1-c) R_{\mathrm{W}} v(k',p')]^{1-\gamma} \right] \right\}^{\frac{1}{1-\gamma}} \\
& \text{subject to} \\
& R_{\mathrm{W}} = \omega_{\mathrm{P}} R_{\mathrm{P}} + \omega_{\mathrm{S}} R_{\mathrm{S}} + (1 - \omega_{\mathrm{P}} - \omega_{\mathrm{S}}) R_{\mathrm{F}} \\
& p' = \left[(1-\lambda_{\mathrm{D}}) R_{\mathrm{P}} p + \lambda_{\mathrm{N}} n + \lambda_{\mathrm{K}} k \right] / \left[(1-c) R_{\mathrm{W}} \right] \\
& k' = \left[k + n - \lambda_{\mathrm{N}} n - \lambda_{\mathrm{K}} k \right] / \left[(1-c) R_{\mathrm{W}} \right] \\
& 0 \leq c \leq \max\{1-p, 0\} \\
& n \geq 0
\end{aligned}
\tag{A.1}
$$

The value function of this problem is not smooth, so we use robust numerical methods, such as linear interpolation and discretization of choice variables, which are described below. We also exploit the inequality in Proposition 1, normalized by post-consumption wealth:

$$
1 - \omega_{\mathrm{S}} - \omega_{\mathrm{P}} \geq a_{\mathrm{K}} k/(1-c) + a_{\mathrm{N}} n/(1-c)
$$

The coefficients a_{K} and a_{N} are given in Proposition 1. Numerically, at each point in the state space, we ignore any combination of choice variables that does not satisfy this constraint.

In our value function iteration algorithm, we represent the state space with a grid of (p, k) values with 51×51 points, which are evenly distributed over the unit square $[0, 1]^2$. In each dimension, the distance between one point and the next is 2%.

We start the algorithm with a guess for the value function that is constant across all grid points, and which equals the constant value v_{L} of the case with perfect market liquidity (see section B of this appendix).

In each iteration, we solve the constrained maximization problem to obtain the optimal policies and the resulting value at every point of the grid. This generates an updated guess for the value function that replaces the initial one. With the new guess, a new iteration begins, and the procedure is repeated until convergence. Specifically, we stop the algorithm when the absolute difference between the current and the new value function, at each point of the grid, becomes lower than $v_{\mathrm{L}} \times 10^{-4}$.

To calculate expectations, we need to integrate with respect to the risky returns, $r_{\mathrm{P}} = \ln(R_{\mathrm{P}})$ and $r_{\mathrm{S}} = \ln(R_{\mathrm{S}})$, and interpolate the value function $v(k', p')$. We use linear interpolation to evaluate this function with arguments outside the (k, p) grid. We use Gauss-Hermite quadrature to integrate with respect to the risky returns. We employ 3 quadrature points for each risky asset, resulting into a total of 9 quadrature points.

From iteration 1 to 9, we represent the choice set with a grid made of $8 \times 201 \times 101$ points in the $(c, \omega_{\mathrm{S}}, n)$ space.[15] At each (k, p) in the discretized state space, we select from that grid the point that maximizes the objective function while satisfying proposition 1. Then, at the beginning of iteration 10, we build new grids for the choice set that allow us to refine the solution. Specifically, for each (k, p) in the discretized state space, we build a new grid in the $(c, \omega_{\mathrm{S}}, n)$ space, which covers a smaller area while being finer than the starting grid. Importantly, the center of these new grids are the optimal choices obtained from iteration 9. We maintain these new grids until iteration 20, when we refine them further using the same idea. After this second refinement, we let the algorithm run unchanged for 10 more simulations. Finally, starting from iteration 30, we apply a standard policy iteration procedure in order to speed up convergence.

[15] In particular, we use the vector $(0.0001, 0.01, 0.02, \ldots, 0.07)$ for c, $(0, 0.005, 0.01, 0.015, \ldots, 1)$ for ω_{S}, and $(0, 0.005, 0.01, 0.015, \ldots, 0.5)$ for n.

The algorithm is implemented in R. It takes about 5 minutes to solve the model on a standard computer running Linux.

A.1 Monte Carlo Simulation

Every time we solve the model, the solution is then used in a Monte Carlo simulation. In the simulation, we consider a LP starting without any PE interests ($k_1 = 0$ and $p_1 = 0$). The LP employs optimal strategies as it transitions randomly in the state space depending on risky returns which are drawn from their lognormal distribution. The simulation lasts for 11000 periods, and we discard the first 1000 periods to ensure that our assumption about initial PE interests, k_1 and p_1, does not affect the results. For the remaining 10000 periods, we save the realization of state variables, choice variables, and returns. These data are then used to compute properties of the marginal and joint distribution of those variables. We use those data, for example, to obtain average portfolio allocations, and also to compute how likely is the investor to visit a certain area of the state space.

B Liquid Model

We consider a simple liquid model where the LP can buy and sell private assets directly, instead of having to make initial commitments that are gradually called and eventually distributed. In this liquid model, PE is effectively another liquid stock. The model with two stocks is standard, and it has been studied since, at least, Samuelson (1969). Nevertheless, this liquid model provides a useful baseline.

Total wealth, W, is the only state variable of this model, and the Bellman equation is as follows:

$$
\begin{aligned}
& V_{\mathrm{L}}(W) = \max_{(c,\omega_{\mathrm{P}},\omega_{\mathrm{S}})} \left\{ (cW)^{1-\gamma} + \delta \mathrm{E}\left[V_{\mathrm{L}}(W')^{1-\gamma} \right] \right\}^{\frac{1}{1-\gamma}} \\
& \text{subject to} \\
& R_{\mathrm{W}} = \omega_{\mathrm{P}} R_{\mathrm{P}} + \omega_{\mathrm{S}} R_{\mathrm{S}} + \left(1 - \omega_{\mathrm{P}} - \omega_{\mathrm{S}}\right) R_{\mathrm{F}}, \\
& W' = W(1-c) R_{\mathrm{W}} \\
& 0 \leq c \leq 1
\end{aligned}
\tag{B.1}
$$

In this expression, c is the consumption-to-wealth ratio, while ω_{P} and ω_{S} are portfolio weights in PE and stocks, respectively. It is convenient to state explicitly the return to LP's total wealth, R_{W}, even though this return is i.i.d and not a state variable.

Since the LP cannot consume more resources than what it owns, it is unable to consume in states with zero wealth. Those states, however, are infinitely painful for LPs with $\gamma \geq 1$, who will then make sure to never run out of wealth. To that end, the LP maintains a positive allocation to risk-free bonds, and does not short sell any risky asset. The resulting constraints (i.e. $\omega_{\mathrm{P}} + \omega_{\mathrm{S}} \leq 1$, $\omega_{\mathrm{P}} \geq 0$, and $\omega_{\mathrm{S}} \geq 0$) are introduced explicitly as we normalize and solve the liquid model.

The liquid model is homogeneous in wealth, and the value function is $V_{\mathrm{L}}(W) = W v_{\mathrm{L}}$ where v_{L}

is a positive constant that solves the normalized problem:

$$v_{\mathrm{L}} = \max_{(c,\omega_{\mathrm{P}},\omega_{\mathrm{S}})} \left\{ c^{1-\gamma} + \delta \mathrm{E}\Big[[v_{\mathrm{L}}(1-c)R_{\mathrm{W}}]^{1-\gamma} \Big] \right\}^{\frac{1}{1-\gamma}}$$

subject to

$$R_{\mathrm{W}} = \omega_{\mathrm{P}} R_{\mathrm{P}} + \omega_{\mathrm{S}} R_{\mathrm{S}} + (1 - \omega_{\mathrm{P}} - \omega_{\mathrm{S}})\, R_{\mathrm{F}} \tag{B.2}$$

$$0 \leq c \leq 1$$

$$\omega_{\mathrm{P}} + \omega_{\mathrm{S}} \leq 1$$

We solve this normalized problem numerically for 21 evenly distributed values of $\gamma \in [1, 3]$. Figure 7 in the main text shows the resulting optimal portfolio allocations for different values of the LP's relative risk aversion, γ. Naturally, when the LP's relative risk aversion increases, the optimal allocations to PE and stocks decline, and the bond allocation increases. Conversely, as relative risk aversion decreases, the LP first reduces its bond allocation to zero. Due to the no-leverage constraint in the model, the LP cannot reduce its bond allocation below zero, and to keep increasing its PE allocation, the LP reduces its stock allocation. The LP's choice between PE and stocks depends on their relative risks and returns. In the current parametrization, PE is more attractive than stocks, and a less risk averse LP prefers to increase its PE allocation at the cost of a lower stock allocation.

C Other Specifications of PE Performance

The baseline specification of the reduced model is reported in Table 1 of the main text. In that specification, PE performance is characterized by $\alpha = 3\%, \beta = 1.6$, $\rho = 0.8$, and $\sigma_{\mathrm{P}} = 40\%$. In this section, we compare the solution of the reduced model under different assumptions about PE performance. We consider the following parameter values:

$$\alpha \in (1\%, 3\%, 5\%)$$

$$\beta \in (1.2, 1.6, 1.8)$$

$$\rho = 0.8 \quad \text{or} \quad \sigma_{\mathrm{P}} = 40\%$$

All other parameters remain constant at their baseline level. We obtain 18 ($= 3 \times 3 \times 2$) specifications of PE performance, which are reduced to 15 after eliminating duplicates. In particular, there are only 5 unique combinations of β, ρ, and σ_{P}:

β	ρ	σ_{P}
1.2	0.8	30%
1.2	0.6	40%
1.6	0.8	40%
1.8	0.8	45%
1.8	0.9	40%

These 5 combinations and the 3 possible values of α result into 15 unique cases.

We report output in the following order:

1. Average portfolio weights resulting from optimal policies (plots, p. 54)
2. Optimal portfolio weights in the Liquid Model (plots, p. 55)
3. Value function (plots, p. 56-57)
4. Optimal new commitment (plots at p. 58-59 and tables at p. 60)

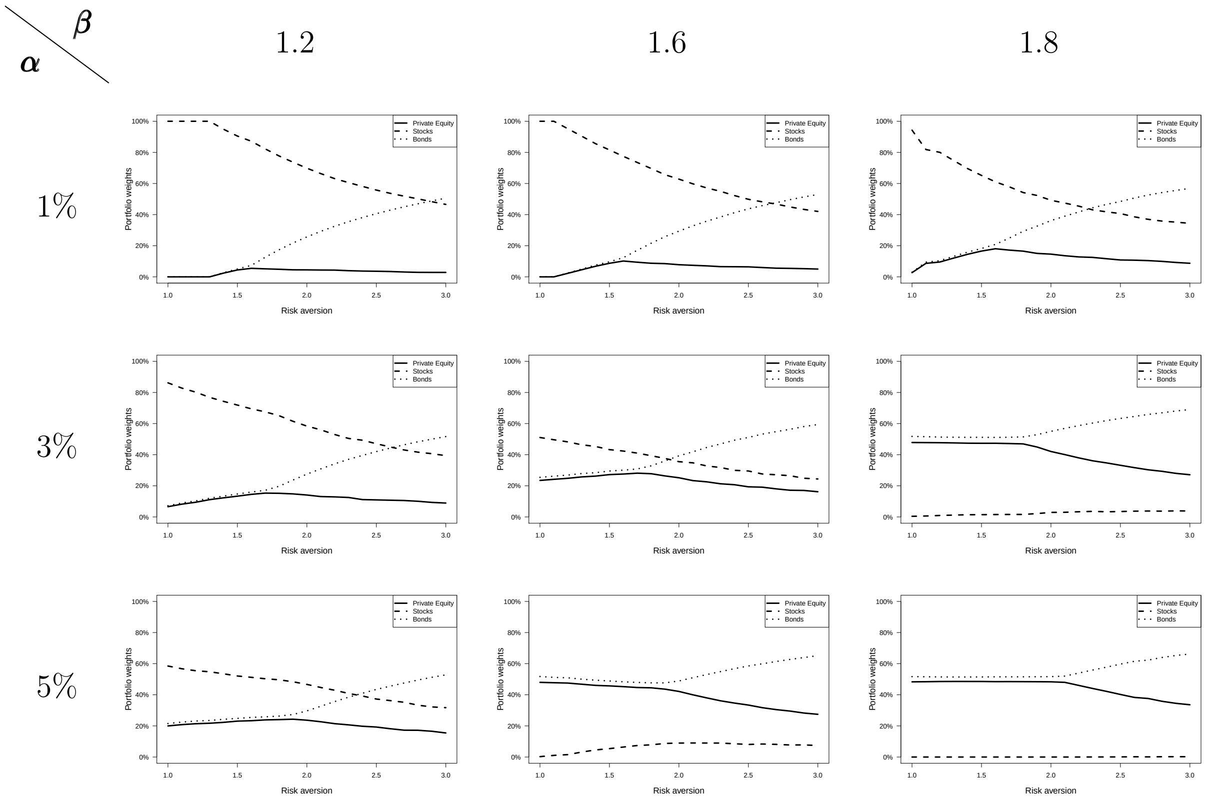
5. Optimal stock allocation (plots at p. 61-62 and tables at p. 63)

6. Optimal consumption-to-wealth ratio (plots, p. 64-65)

In the figures of this section, the middle column of Panel A is identical to that of Panel B, and the center of each panel corresponds to the baseline specification in Table 1 of the main text. In the tables of Figure C.7 (p. 60) and Figure C.10 (p. 63), empty columns are shown when average PE uncalled commitments are less than 3% of wealth. In Figure C.10, empty columns are shown also when average portfolio weight in stocks is 3% or less.

Figure C.1: Average portfolio weights resulting from optimal policies

Panel A: $\sigma_{\mathrm{P}} = 40\%$



Panel B: $\rho = 0.8$

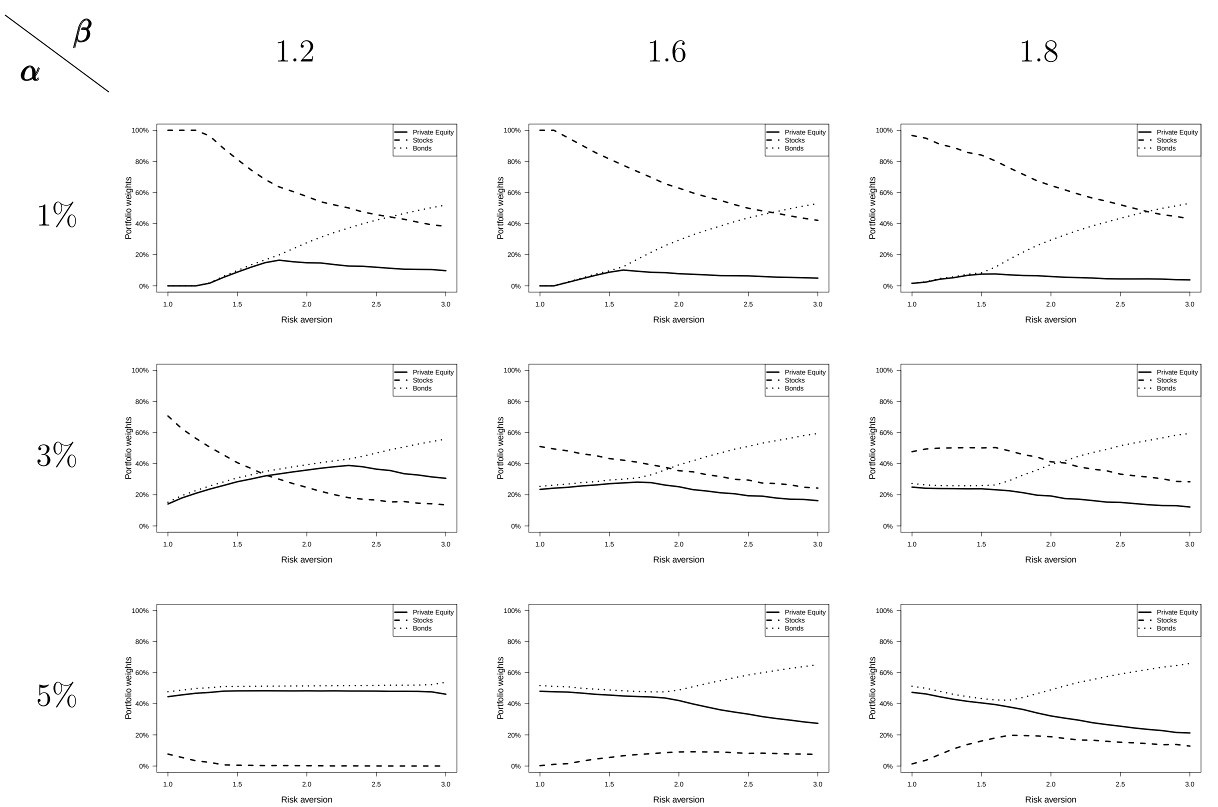
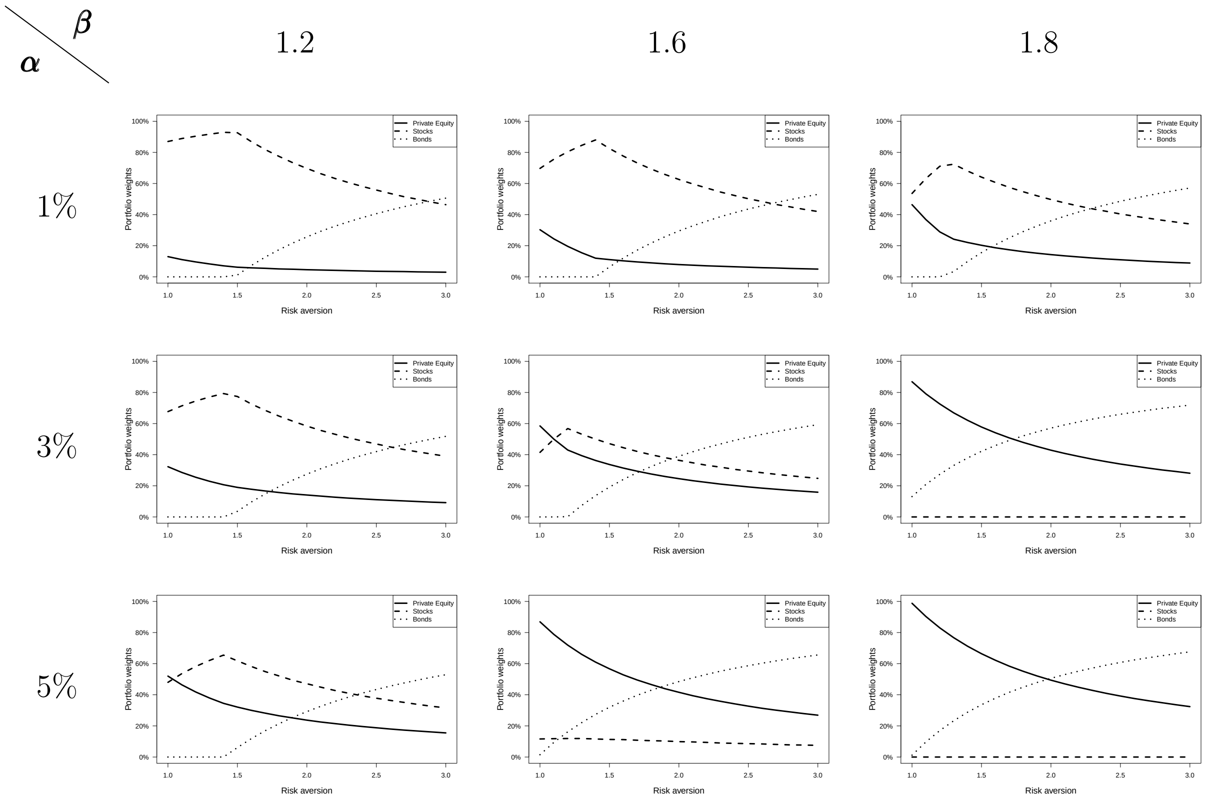


Figure C.2: Optimal portfolio weights in the Liquid Model

Panel A: $\sigma_{\mathrm{P}} = 40\%$



Panel B: $\rho = 0.8$

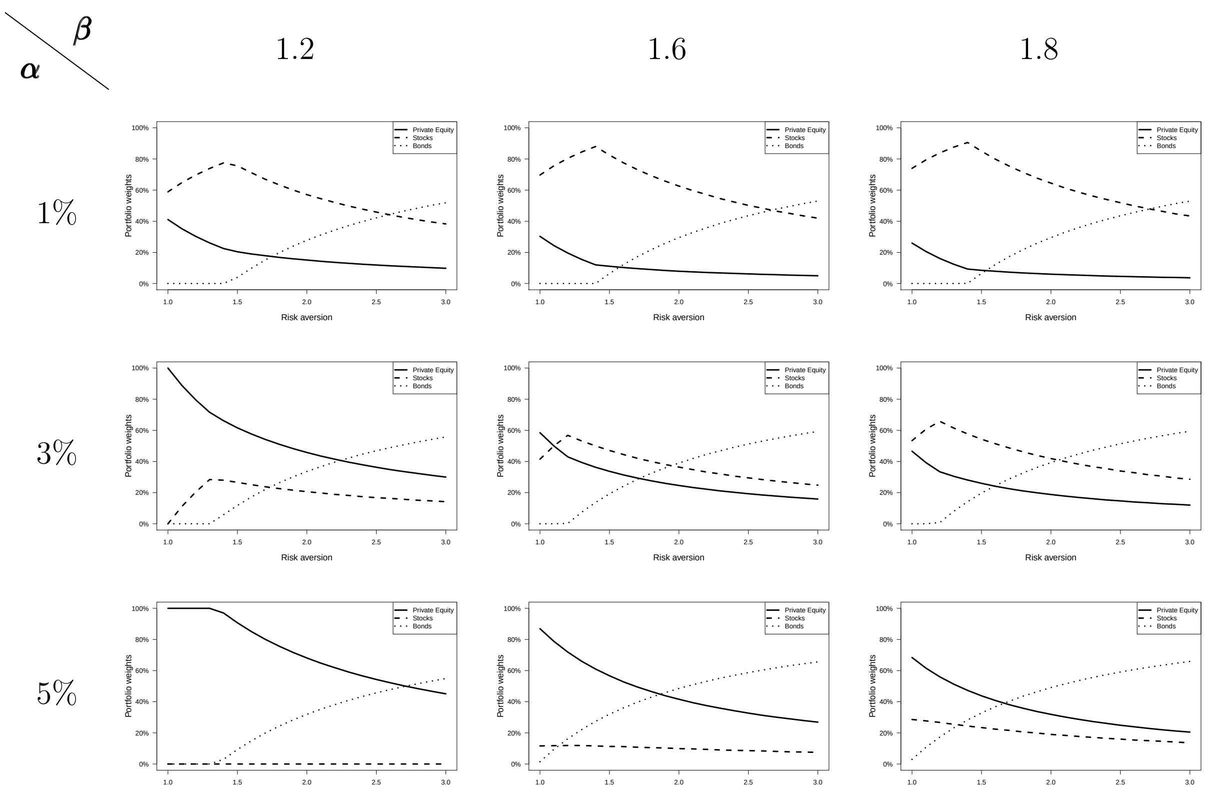
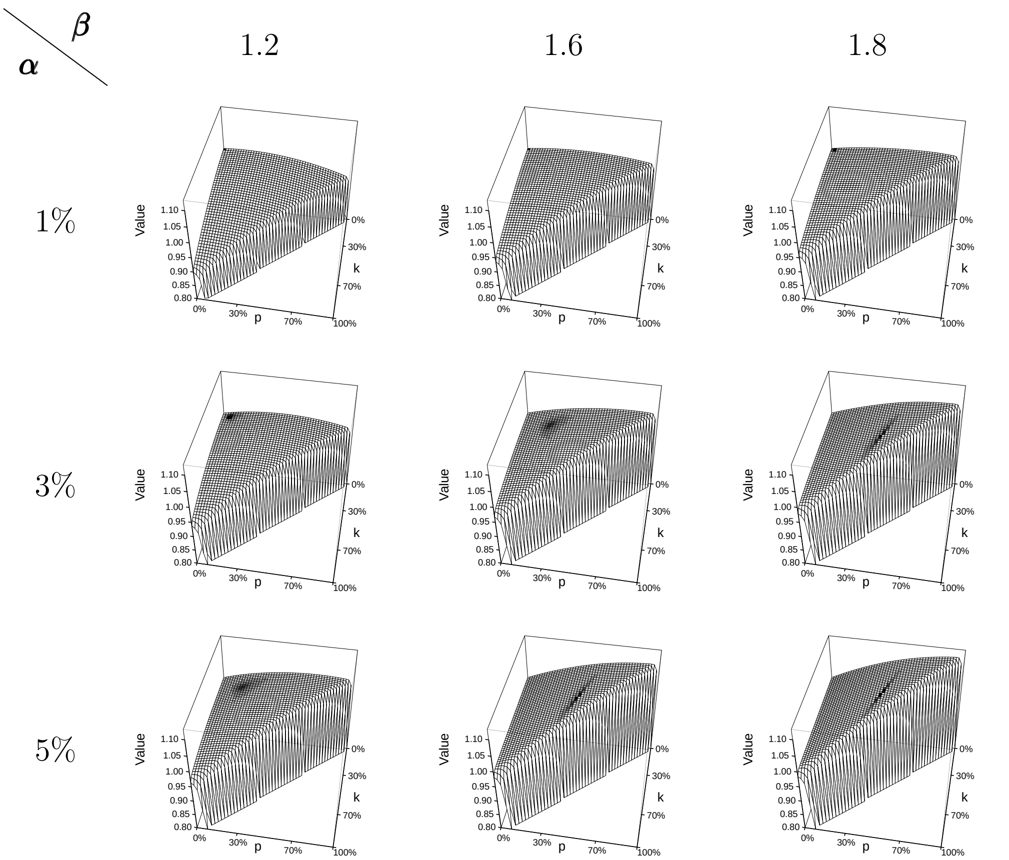


Figure C.3: Value function of aggressive LP ($\gamma = 1$)

Panel A: $\sigma_{\mathrm{P}} = 40\%$



Panel B: $\rho = 0.8$

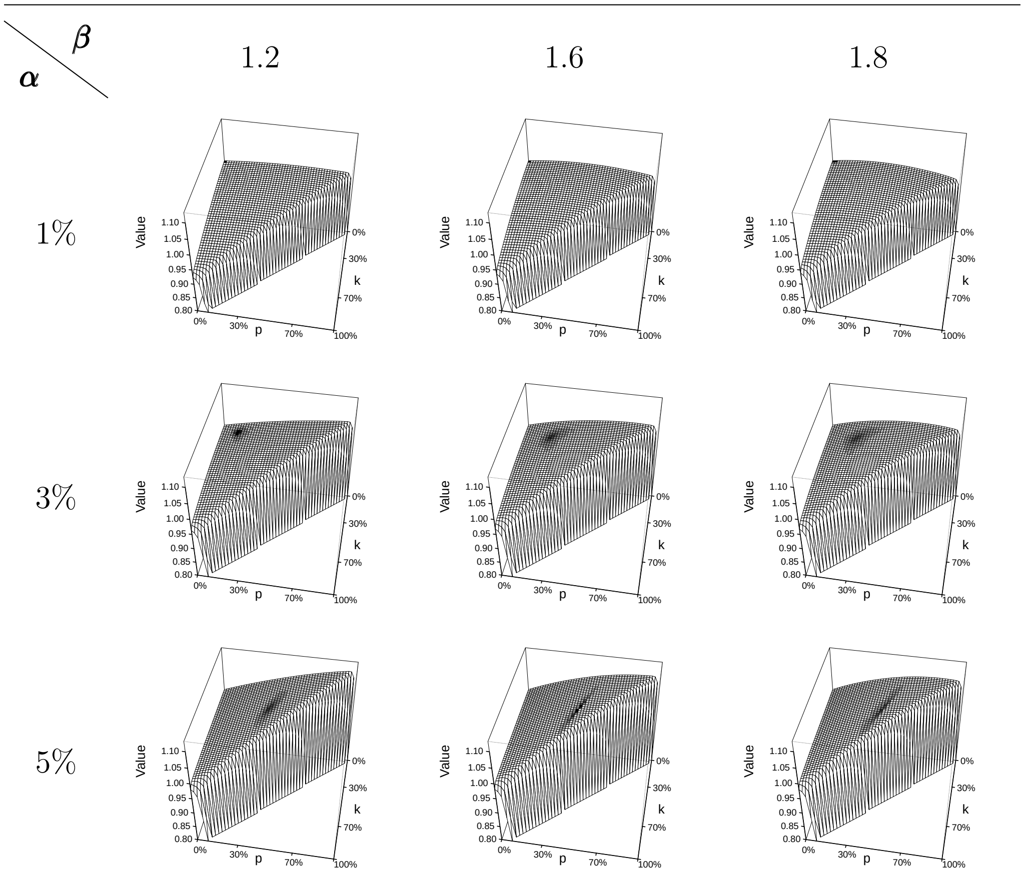
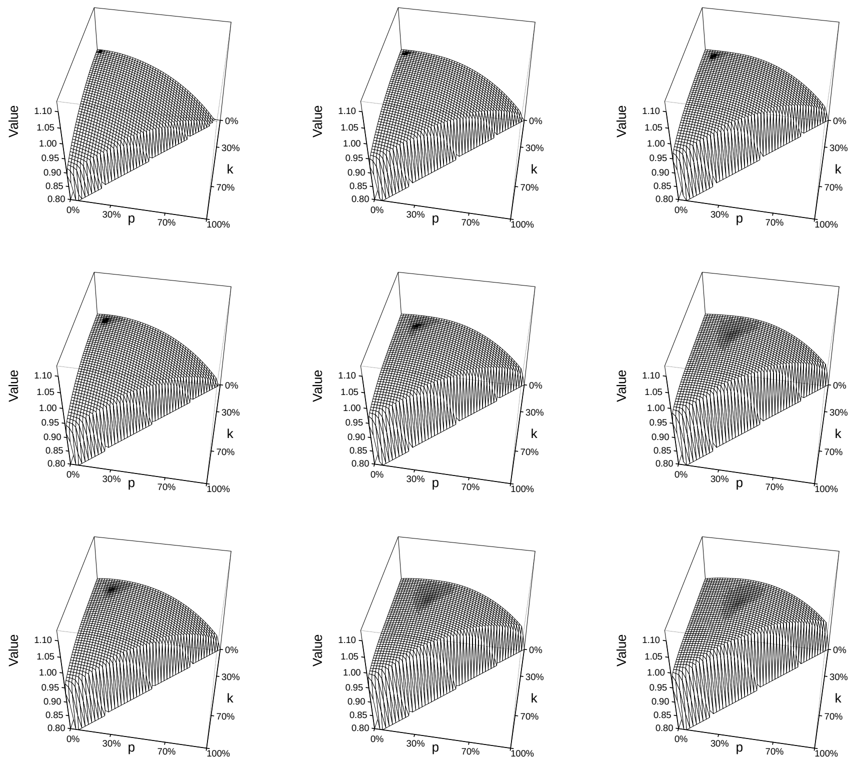


Figure C.4: Value function of conservative LP ($\gamma = 3$)

Panel A: $\sigma_{\mathrm{P}} = 40\%$

α \ β	1.2	1.6	1.8
1%			
3%			
5%			



Panel B: $\rho = 0.8$

α \ β	1.2	1.6	1.8
1%			
3%			
5%			

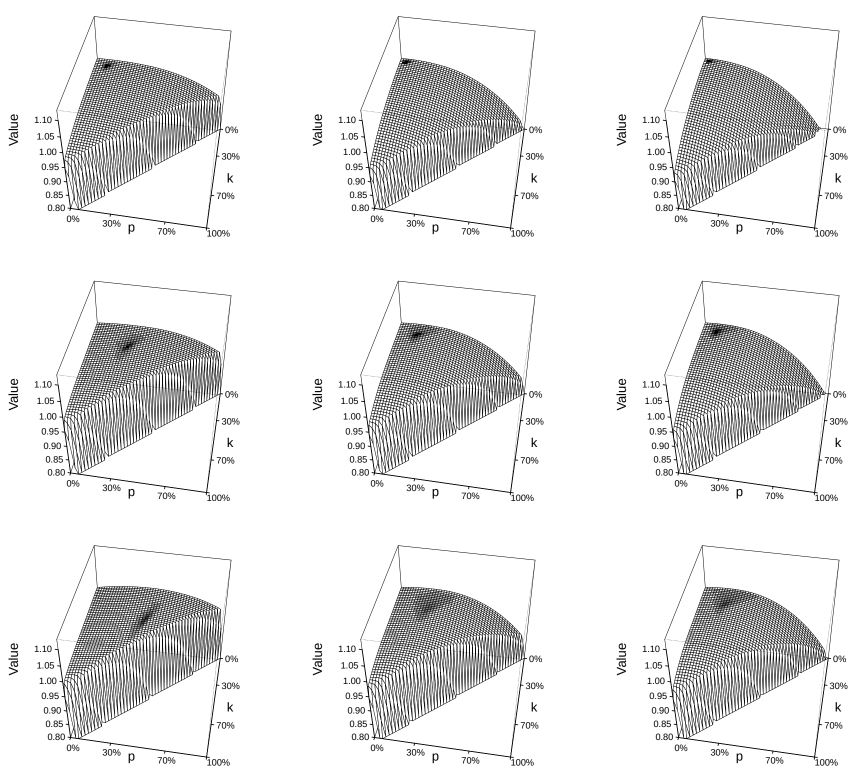
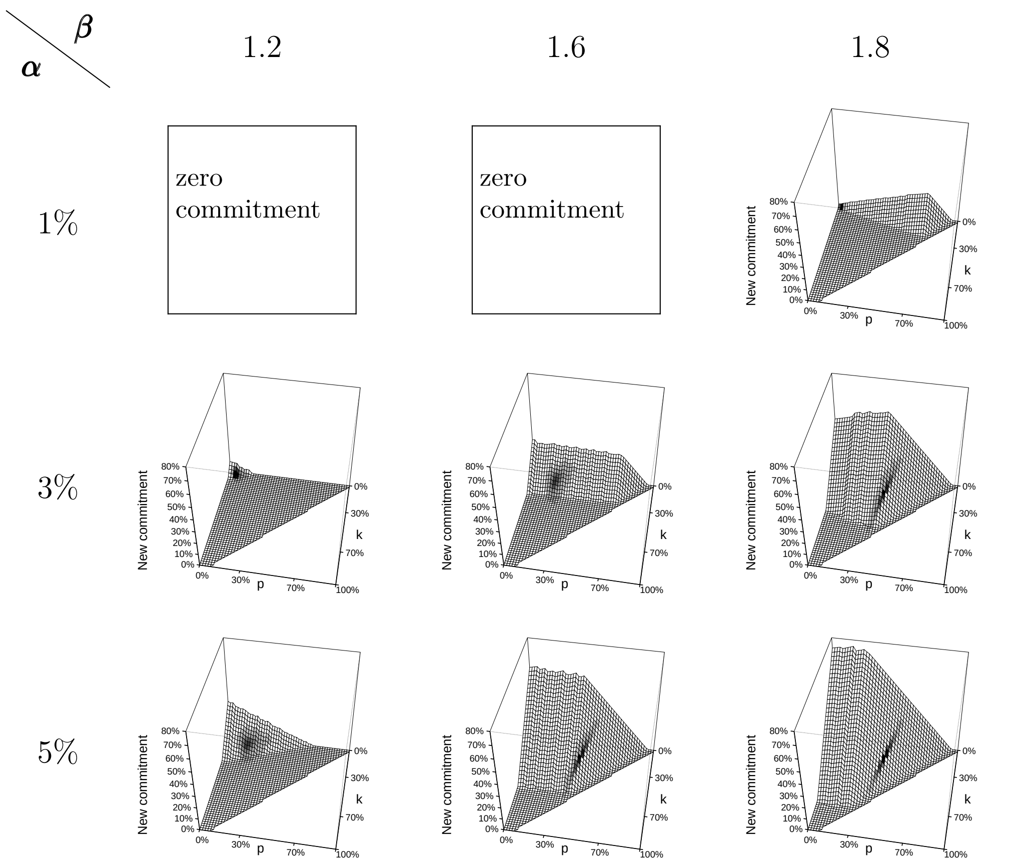


Figure C.5: New commitment of aggressive LP ($\gamma = 1$)

Panel A: $\sigma_P = 40\%$



Panel B: $\rho = 0.8$

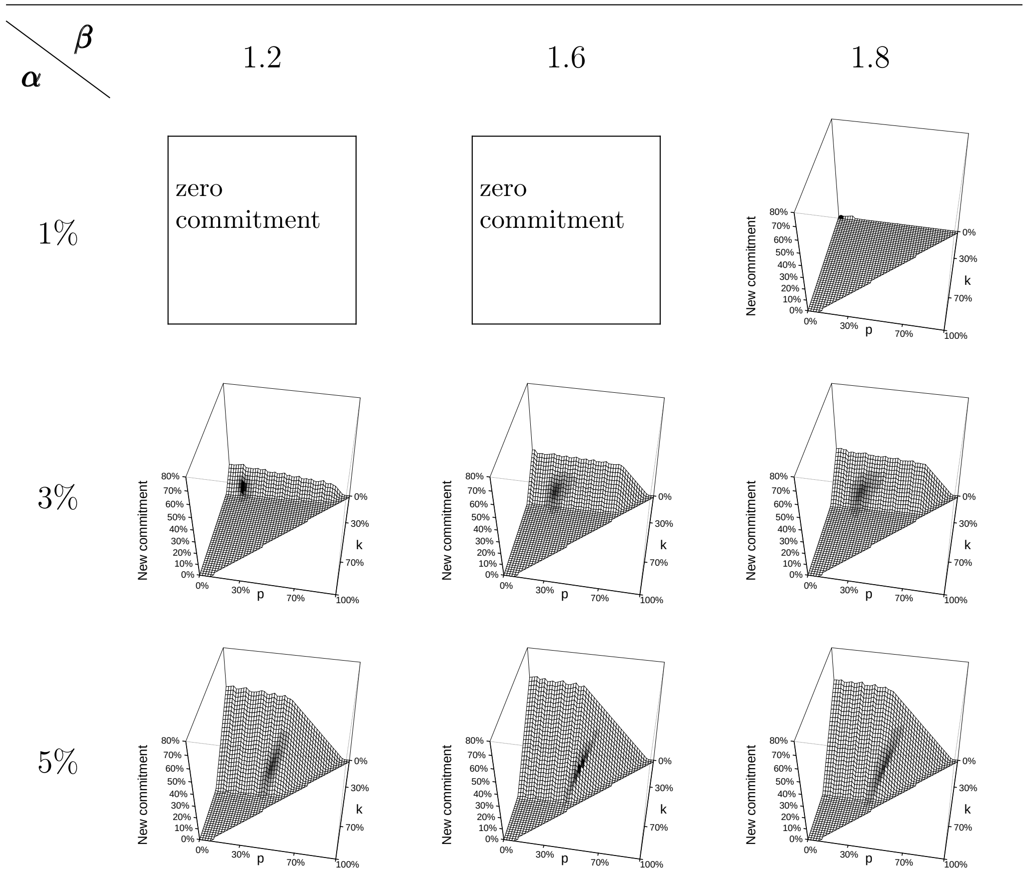
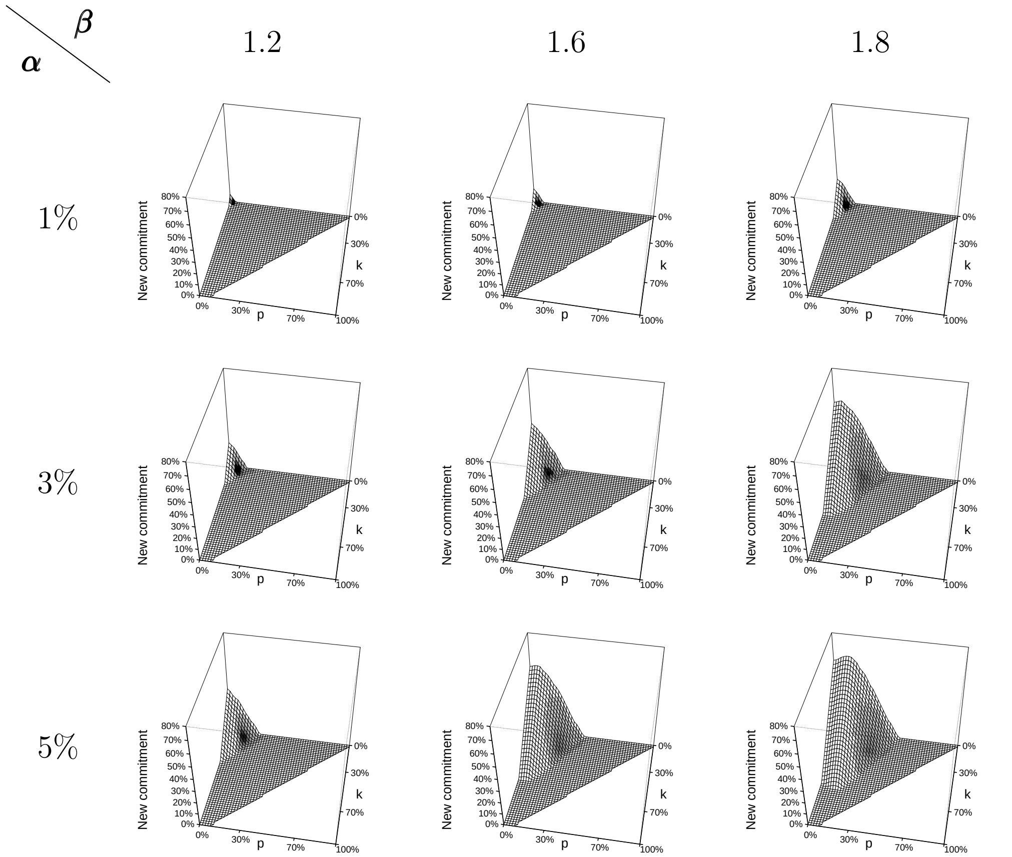


Figure C.6: New commitment of conservative LP ($\gamma = 3$)

Panel A: $\sigma_{\mathrm{P}} = 40\%$



Panel B: $\rho = 0.8$

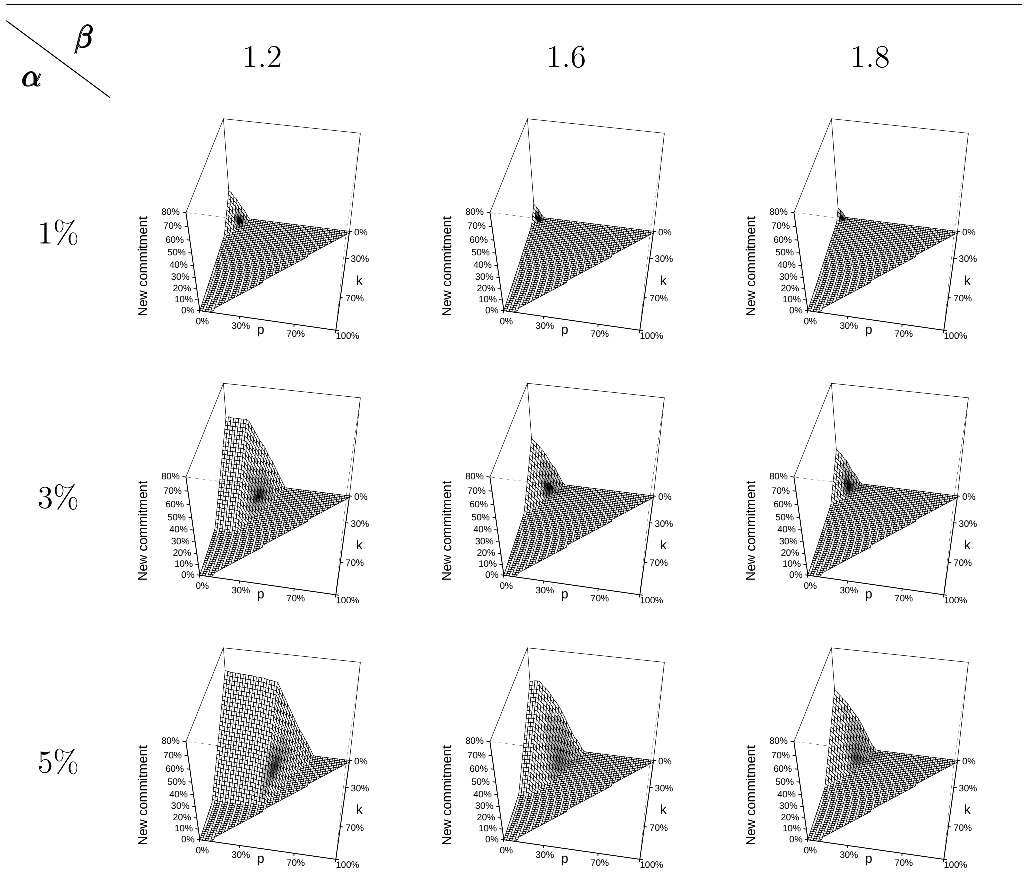


Figure C.7: New commitment of aggressive vs. conservative LP

Panel A: $\sigma_{\mathrm{P}} = 40\%$

α \ β	1.2	1.6	1.8

1%

	Aggressive ($\gamma = 1$)		Conservative ($\gamma = 3$)	
	(1)	(2)	(3)	(4)
Illiquid wealth (p)				
Commitments (k)				
Interaction ($p \times k$)				
Constant				
R^2				

	Aggressive ($\gamma = 1$)		Conservative ($\gamma = 3$)	
	(1)	(2)	(3)	(4)
Illiquid wealth (p)			−1.22	−1.63
Commitments (k)			−0.75	−1.32
Interaction ($p \times k$)				12.05
Constant			0.10	0.12
R^2			0.94	0.95

	Aggressive ($\gamma = 1$)		Conservative ($\gamma = 3$)	
	(1)	(2)	(3)	(4)
Illiquid wealth (p)	0.39	0.57	−1.53	−1.40
Commitments (k)	−0.63	−0.41	−0.91	−0.73
Interaction ($p \times k$)		−8.46		−2.19
Constant	0.01	0.01	0.22	0.21
R^2	0.95	0.95	0.96	0.96

3%

	Aggressive ($\gamma = 1$)		Conservative ($\gamma = 3$)	
	(1)	(2)	(3)	(4)
Illiquid wealth (p)	−0.63	−1.07	−1.43	−1.43
Commitments (k)	−0.76	−1.39	−0.89	−0.89
Interaction ($p \times k$)		9.67		0.06
Constant	0.10	0.13	0.21	0.21
R^2	0.89	0.90	0.96	0.96

	Aggressive ($\gamma = 1$)		Conservative ($\gamma = 3$)	
	(1)	(2)	(3)	(4)
Illiquid wealth (p)	−0.08	0.16	−1.46	−1.55
Commitments (k)	−0.98	−0.66	−0.98	−1.10
Interaction ($p \times k$)		−1.43		0.77
Constant	0.27	0.22	0.40	0.41
R^2	0.98	0.99	0.98	0.98

	Aggressive ($\gamma = 1$)		Conservative ($\gamma = 3$)	
	(1)	(2)	(3)	(4)
Illiquid wealth (p)	−0.64	−1.46	−1.85	−1.86
Commitments (k)	−0.92	−1.92	−0.98	−1.00
Interaction ($p \times k$)		2.19		0.05
Constant	0.79	1.16	0.77	0.77
R^2	0.98	0.99	0.99	0.99

5%

	Aggressive ($\gamma = 1$)		Conservative ($\gamma = 3$)	
	(1)	(2)	(3)	(4)
Illiquid wealth (p)	−0.40	−0.33	−1.55	−1.45
Commitments (k)	−0.99	−0.90	−0.96	−0.82
Interaction ($p \times k$)		−0.46		−0.97
Constant	0.29	0.28	0.39	0.38
R^2	0.97	0.98	0.98	0.98

	Aggressive ($\gamma = 1$)		Conservative ($\gamma = 3$)	
	(1)	(2)	(3)	(4)
Illiquid wealth (p)	−0.73	−1.40	−1.93	−1.99
Commitments (k)	−0.93	−1.75	−0.96	−1.04
Interaction ($p \times k$)		1.79		0.33
Constant	0.83	1.14	0.79	0.80
R^2	0.98	0.99	0.99	0.99

	Aggressive ($\gamma = 1$)		Conservative ($\gamma = 3$)	
	(1)	(2)	(3)	(4)
Illiquid wealth (p)	−1.02	−0.99	−1.93	−2.09
Commitments (k)	−0.98	−0.98	−0.98	−1.20
Interaction ($p \times k$)		0.03		0.71
Constant	0.98	0.97	0.97	1.02
R^2	1.00	1.00	1.00	1.00

Panel B: $\rho = 0.8$

α \ β	1.2	1.6	1.8

1%

	Aggressive ($\gamma = 1$)		Conservative ($\gamma = 3$)	
	(1)	(2)	(3)	(4)
Illiquid wealth (p)			−1.36	−1.26
Commitments (k)			−0.97	−0.84
Interaction ($p \times k$)				−1.37
Constant			0.24	0.23
R^2			0.99	0.99

	Aggressive ($\gamma = 1$)		Conservative ($\gamma = 3$)	
	(1)	(2)	(3)	(4)
Illiquid wealth (p)			−1.23	−1.66
Commitments (k)			−0.77	−1.35
Interaction ($p \times k$)				12.59
Constant			0.11	0.13
R^2			0.94	0.96

	Aggressive ($\gamma = 1$)		Conservative ($\gamma = 3$)	
	(1)	(2)	(3)	(4)
Illiquid wealth (p)			−0.94	−1.06
Commitments (k)			−0.78	−0.94
Interaction ($p \times k$)				4.52
Constant			0.07	0.07
R^2			0.96	0.96

3%

	Aggressive ($\gamma = 1$)		Conservative ($\gamma = 3$)	
	(1)	(2)	(3)	(4)
Illiquid wealth (p)	−0.60	−0.56	−1.71	−2.29
Commitments (k)	−0.92	−0.87	−0.99	−1.76
Interaction ($p \times k$)		−0.38		2.66
Constant	0.23	0.22	0.84	1.01
R^2	0.94	0.94	0.98	0.99

	Aggressive ($\gamma = 1$)		Conservative ($\gamma = 3$)	
	(1)	(2)	(3)	(4)
Illiquid wealth (p)	−0.08	0.16	−1.47	−1.57
Commitments (k)	−0.98	−0.66	−0.98	−1.12
Interaction ($p \times k$)		−1.41		0.92
Constant	0.27	0.22	0.40	0.42
R^2	0.98	0.98	0.98	0.98

	Aggressive ($\gamma = 1$)		Conservative ($\gamma = 3$)	
	(1)	(2)	(3)	(4)
Illiquid wealth (p)	−0.07	−0.21	−1.55	−1.37
Commitments (k)	−0.96	−1.15	−0.93	−0.67
Interaction ($p \times k$)		0.81		−2.33
Constant	0.28	0.32	0.30	0.28
R^2	0.98	0.98	0.96	0.96

5%

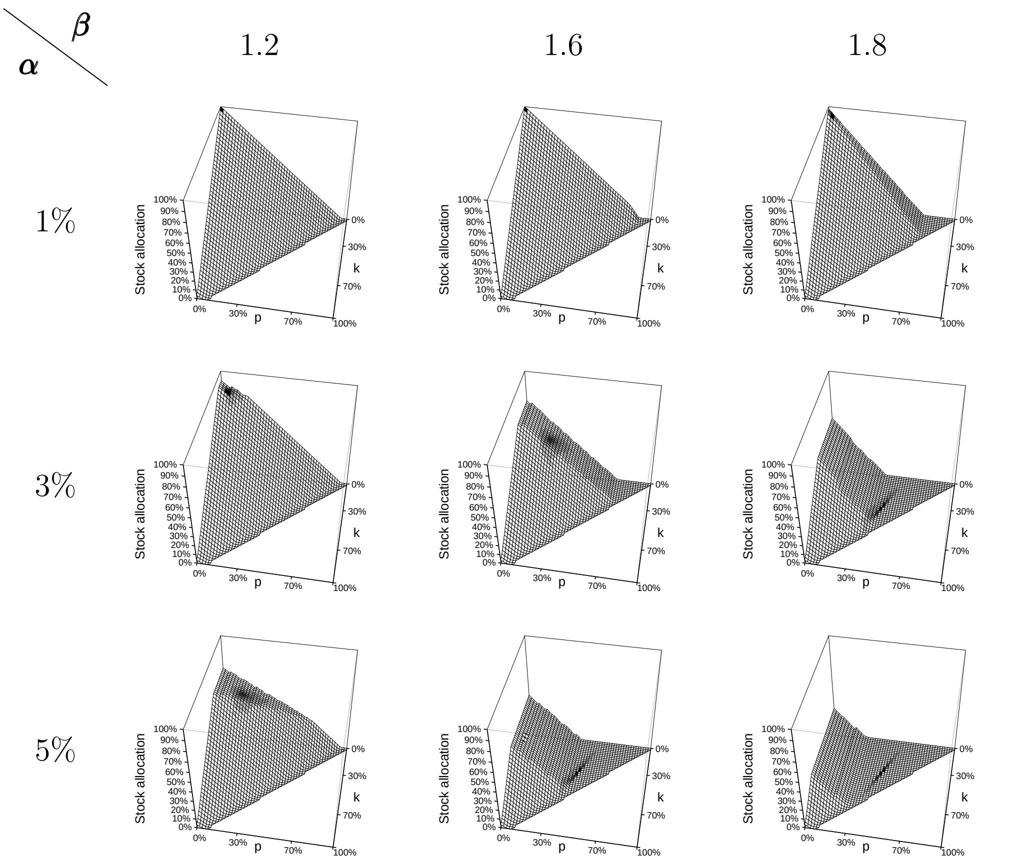
	Aggressive ($\gamma = 1$)		Conservative ($\gamma = 3$)	
	(1)	(2)	(3)	(4)
Illiquid wealth (p)	−0.04	0.40	−1.97	−2.44
Commitments (k)	−0.99	−0.43	−0.99	−1.59
Interaction ($p \times k$)		−1.33		1.37
Constant	0.49	0.31	1.37	1.58
R^2	0.99	0.99	0.98	0.99

	Aggressive ($\gamma = 1$)		Conservative ($\gamma = 3$)	
	(1)	(2)	(3)	(4)
Illiquid wealth (p)	−0.77	−1.40	−1.93	−2.01
Commitments (k)	−0.94	−1.73	−0.96	−1.06
Interaction ($p \times k$)		1.72		0.41
Constant	0.85	1.14	0.79	0.81
R^2	0.98	0.99	0.99	0.99

	Aggressive ($\gamma = 1$)		Conservative ($\gamma = 3$)	
	(1)	(2)	(3)	(4)
Illiquid wealth (p)	−0.51	−1.19	−1.73	−1.98
Commitments (k)	−0.94	−1.77	−0.94	−1.30
Interaction ($p \times k$)		1.82		1.88
Constant	0.73	1.04	0.56	0.61
R^2	0.99	1.00	0.99	0.99

Figure C.8: Stock allocation of aggressive LP ($\gamma = 1$)

Panel A: $\sigma_{\mathrm{P}} = 40\%$



Panel B: $\rho = 0.8$

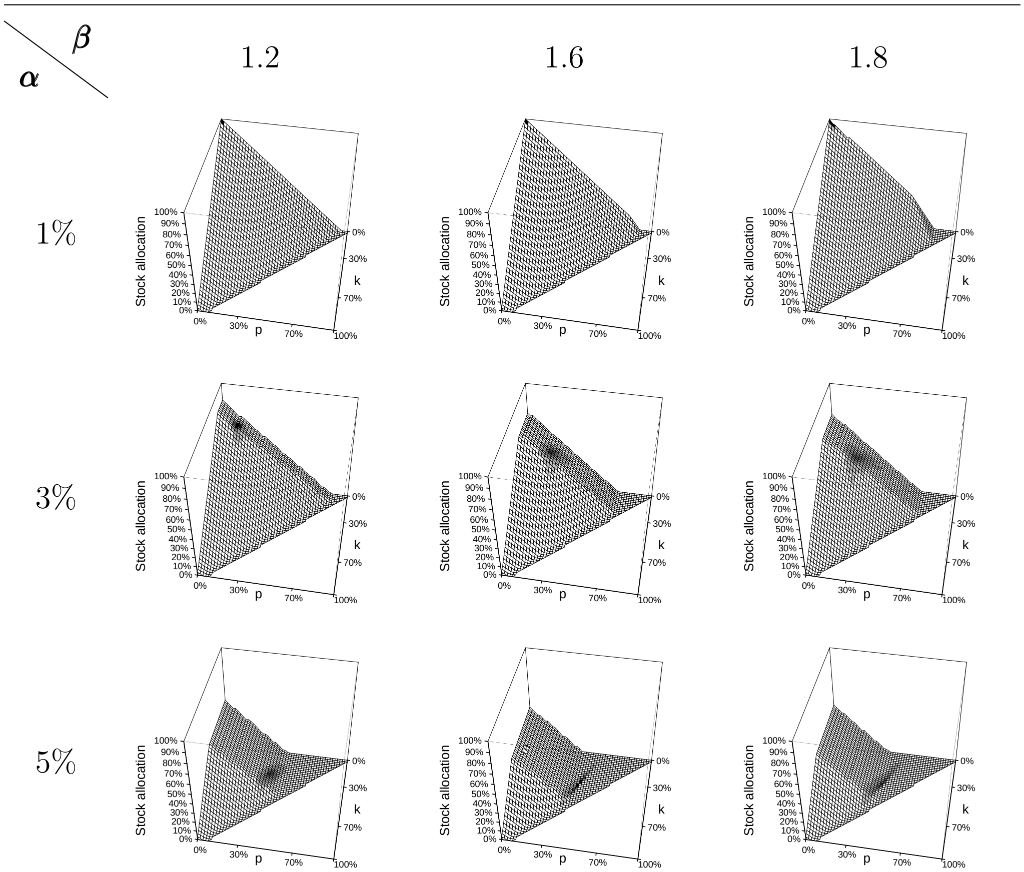
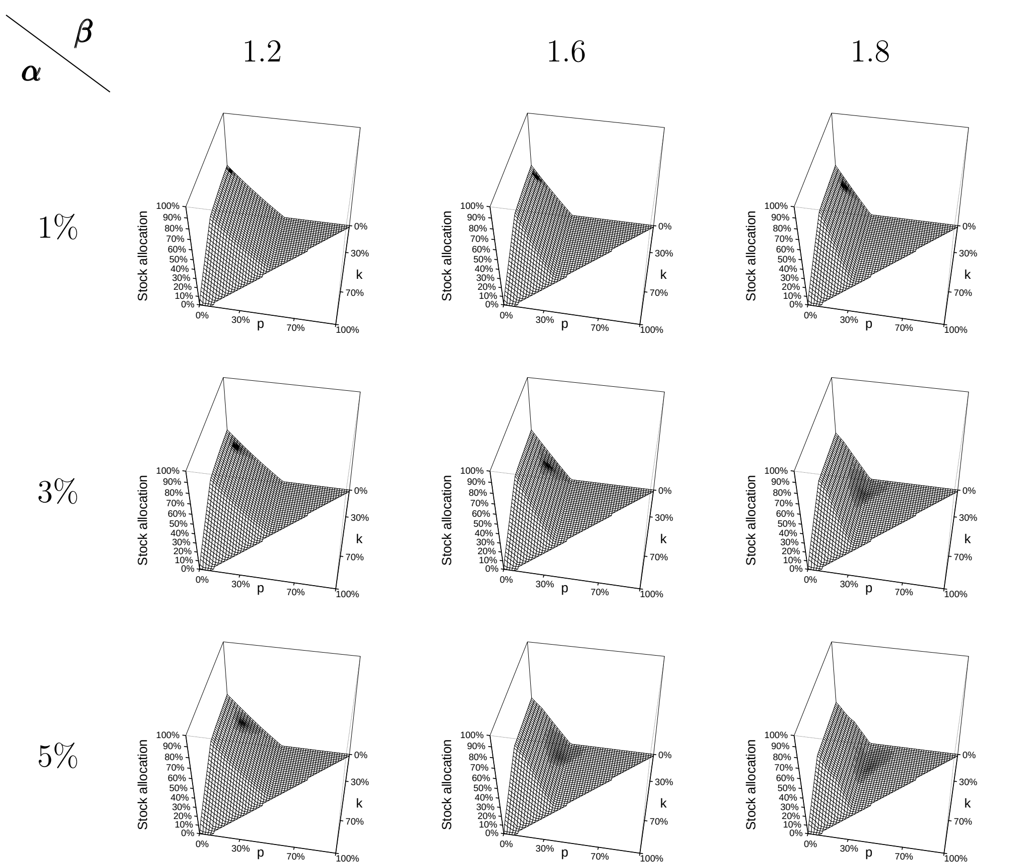


Figure C.9: Stock allocation of conservative LP ($\gamma = 3$)

Panel A: $\sigma_{\mathrm{P}} = 40\%$



Panel B: $\rho = 0.8$

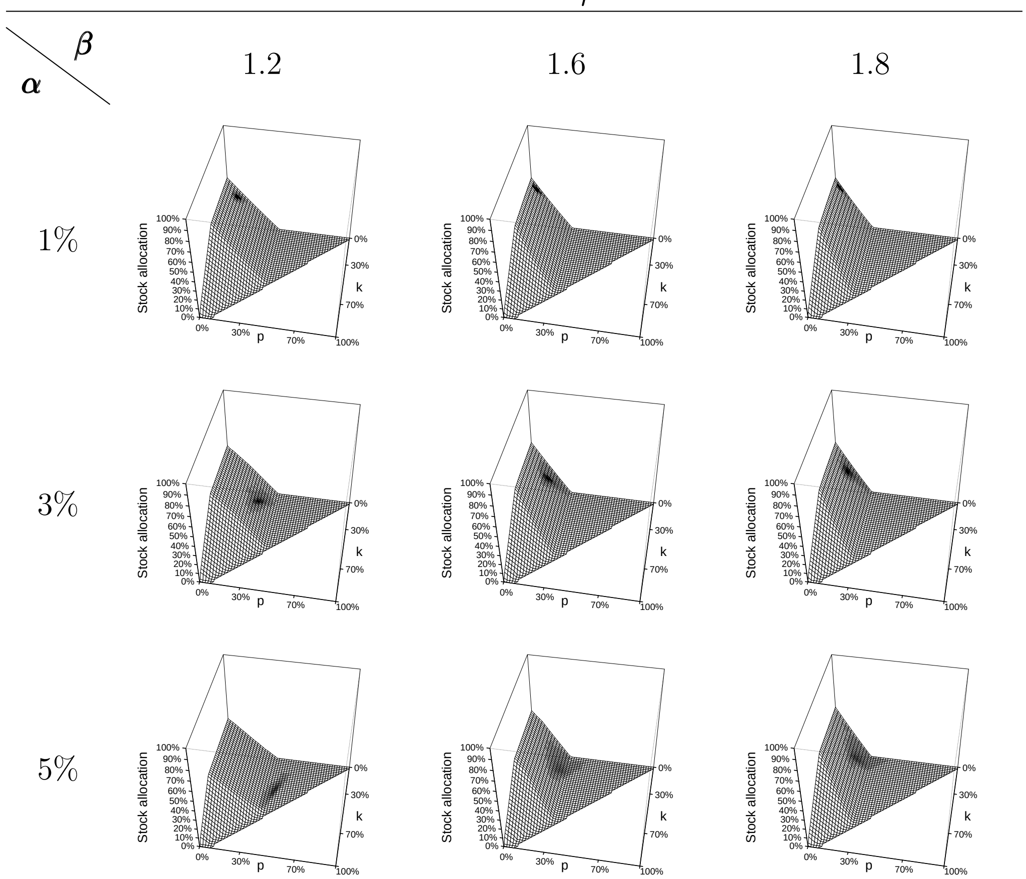


Figure C.10: Stock allocation of aggressive vs. conservative LP

Panel A: $\sigma_{\mathrm{P}} = 40\%$

α \ β	1.2	1.6	1.8

1%

	Aggressive ($\gamma = 1$)		Conservative ($\gamma = 3$)	
	(1)	(2)	(3)	(4)
Illiquid wealth (p)			−1.24	−1.22
Commitments (k)			−0.00	0.04
Interaction ($p \times k$)				−1.31
Constant			0.50	0.50
R^2			1.00	1.00

	Aggressive ($\gamma = 1$)		Conservative ($\gamma = 3$)	
	(1)	(2)	(3)	(4)
Illiquid wealth (p)			−1.65	−1.64
Commitments (k)			0.00	0.01
Interaction ($p \times k$)				−0.21
Constant			0.50	0.50
R^2			1.00	1.00

	Aggressive ($\gamma = 1$)		Conservative ($\gamma = 3$)	
	(1)	(2)	(3)	(4)
Illiquid wealth (p)	−1.44	−1.60	−1.85	−1.84
Commitments (k)	−0.37	−0.57	0.00	0.01
Interaction ($p \times k$)		7.46		−0.12
Constant	0.99	0.99	0.50	0.50
R^2	0.99	0.99	1.00	1.00

3%

	Aggressive ($\gamma = 1$)		Conservative ($\gamma = 3$)	
	(1)	(2)	(3)	(4)
Illiquid wealth (p)	−0.46	0.04	−1.21	−1.19
Commitments (k)	−0.25	0.45	−0.00	0.02
Interaction ($p \times k$)		−10.69		−0.32
Constant	0.90	0.87	0.50	0.50
R^2	0.72	0.77	1.00	1.00

	Aggressive ($\gamma = 1$)		Conservative ($\gamma = 3$)	
	(1)	(2)	(3)	(4)
Illiquid wealth (p)	−0.98	−1.24	−1.59	−1.57
Commitments (k)	−0.03	−0.36	−0.00	0.02
Interaction ($p \times k$)		1.51		−0.16
Constant	0.73	0.79	0.49	0.49
R^2	0.98	0.98	1.00	1.00

	Aggressive ($\gamma = 1$)		Conservative ($\gamma = 3$)	
	(1)	(2)	(3)	(4)
Illiquid wealth (p)			−1.68	−1.58
Commitments (k)			−0.01	0.11
Interaction ($p \times k$)				−0.51
Constant			0.46	0.44
R^2			0.99	0.99

5%

	Aggressive ($\gamma = 1$)		Conservative ($\gamma = 3$)	
	(1)	(2)	(3)	(4)
Illiquid wealth (p)	−0.67	−0.72	−1.19	−1.16
Commitments (k)	−0.03	−0.10	−0.00	0.04
Interaction ($p \times k$)		0.40		−0.27
Constant	0.72	0.73	0.49	0.49
R^2	0.96	0.96	1.00	1.00

	Aggressive ($\gamma = 1$)		Conservative ($\gamma = 3$)	
	(1)	(2)	(3)	(4)
Illiquid wealth (p)			−1.52	−1.40
Commitments (k)			−0.01	0.15
Interaction ($p \times k$)				−0.61
Constant			0.47	0.44
R^2			1.00	1.00

	Aggressive ($\gamma = 1$)		Conservative ($\gamma = 3$)	
	(1)	(2)	(3)	(4)
Illiquid wealth (p)				
Commitments (k)				
Interaction ($p \times k$)				
Constant				
R^2				

Panel B: $\rho = 0.8$

α \ β	1.2	1.6	1.8

1%

	Aggressive ($\gamma = 1$)		Conservative ($\gamma = 3$)	
	(1)	(2)	(3)	(4)
Illiquid wealth (p)			−1.23	−1.20
Commitments (k)			−0.00	0.04
Interaction ($p \times k$)				−0.47
Constant			0.50	0.50
R^2			1.00	1.00

	Aggressive ($\gamma = 1$)		Conservative ($\gamma = 3$)	
	(1)	(2)	(3)	(4)
Illiquid wealth (p)			−1.65	−1.64
Commitments (k)			0.00	0.01
Interaction ($p \times k$)				−0.18
Constant			0.50	0.50
R^2			1.00	1.00

	Aggressive ($\gamma = 1$)		Conservative ($\gamma = 3$)	
	(1)	(2)	(3)	(4)
Illiquid wealth (p)	−1.19	−1.24	−1.82	−1.78
Commitments (k)	−0.34	−0.42	0.00	0.06
Interaction ($p \times k$)		5.00		−1.53
Constant	0.99	0.99	0.50	0.50
R^2	0.99	0.99	1.00	1.00

3%

	Aggressive ($\gamma = 1$)		Conservative ($\gamma = 3$)	
	(1)	(2)	(3)	(4)
Illiquid wealth (p)	−0.48	−0.41	−1.22	−1.20
Commitments (k)	−0.09	0.01	0.00	0.02
Interaction ($p \times k$)		−0.69		−0.08
Constant	0.78	0.77	0.49	0.49
R^2	0.64	0.64	1.00	1.00

	Aggressive ($\gamma = 1$)		Conservative ($\gamma = 3$)	
	(1)	(2)	(3)	(4)
Illiquid wealth (p)	−0.98	−1.24	−1.59	−1.57
Commitments (k)	−0.03	−0.36	−0.00	0.03
Interaction ($p \times k$)		1.47		−0.18
Constant	0.74	0.79	0.49	0.49
R^2	0.97	0.98	1.00	1.00

	Aggressive ($\gamma = 1$)		Conservative ($\gamma = 3$)	
	(1)	(2)	(3)	(4)
Illiquid wealth (p)	−1.00	−0.91	−1.78	−1.74
Commitments (k)	−0.06	0.06	−0.00	0.06
Interaction ($p \times k$)		−0.52		−0.53
Constant	0.73	0.70	0.49	0.49
R^2	0.97	0.97	1.00	1.00

5%

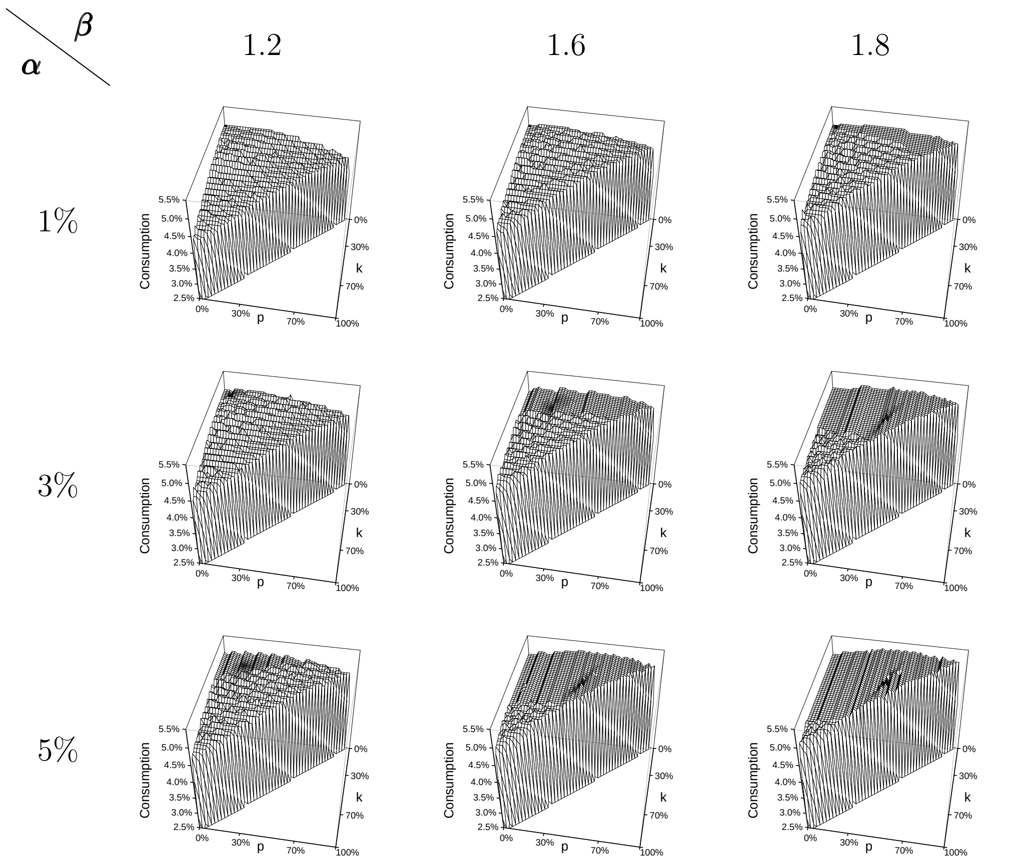
	Aggressive ($\gamma = 1$)		Conservative ($\gamma = 3$)	
	(1)	(2)	(3)	(4)
Illiquid wealth (p)	−1.02	−1.45		
Commitments (k)	−0.01	−0.56		
Interaction ($p \times k$)		1.30		
Constant	0.51	0.69		
R^2	0.96	0.97		

	Aggressive ($\gamma = 1$)		Conservative ($\gamma = 3$)	
	(1)	(2)	(3)	(4)
Illiquid wealth (p)			−1.52	−1.40
Commitments (k)			−0.01	0.15
Interaction ($p \times k$)				−0.64
Constant			0.47	0.44
R^2			1.00	1.00

	Aggressive ($\gamma = 1$)		Conservative ($\gamma = 3$)	
	(1)	(2)	(3)	(4)
Illiquid wealth (p)			−1.74	−1.71
Commitments (k)			−0.00	0.03
Interaction ($p \times k$)				−0.18
Constant			0.48	0.48
R^2			1.00	1.00

Figure C.11: Consumption of aggressive LP ($\gamma = 1$)

Panel A: $\sigma_{\mathrm{P}} = 40\%$



Panel B: $\rho = 0.8$

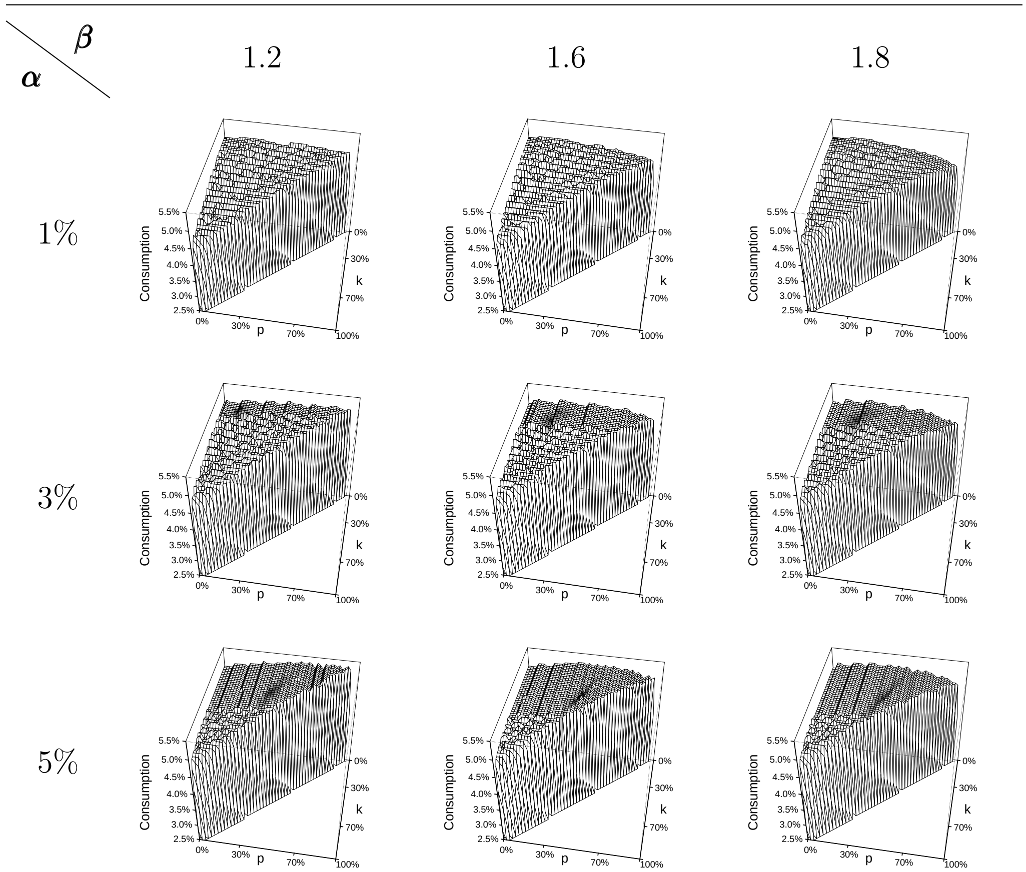
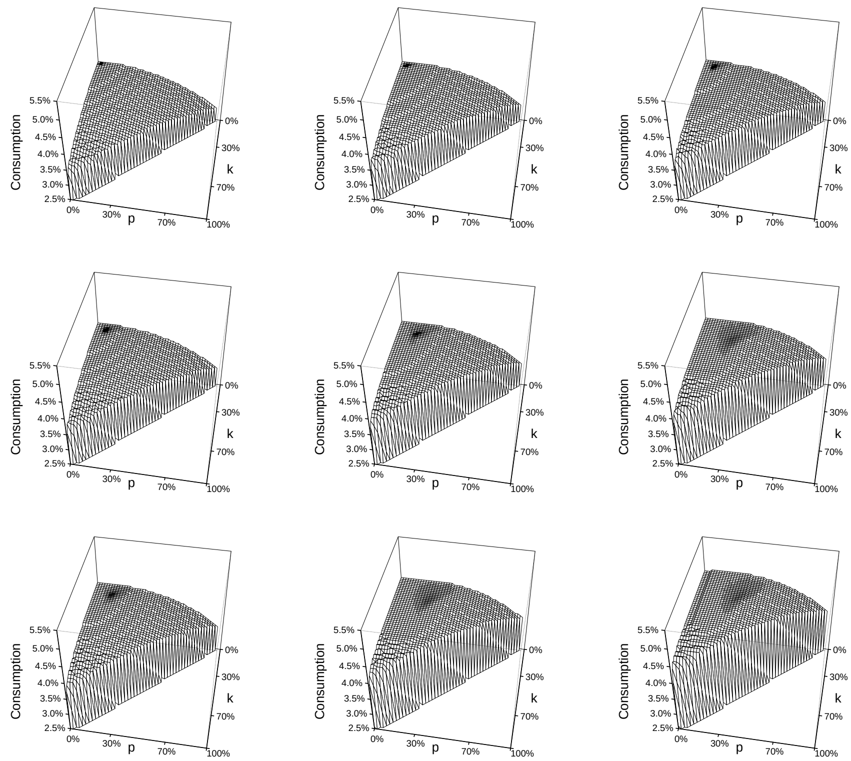


Figure C.12: CONSUMPTION OF CONSERVATIVE LP ($\gamma = 3$)

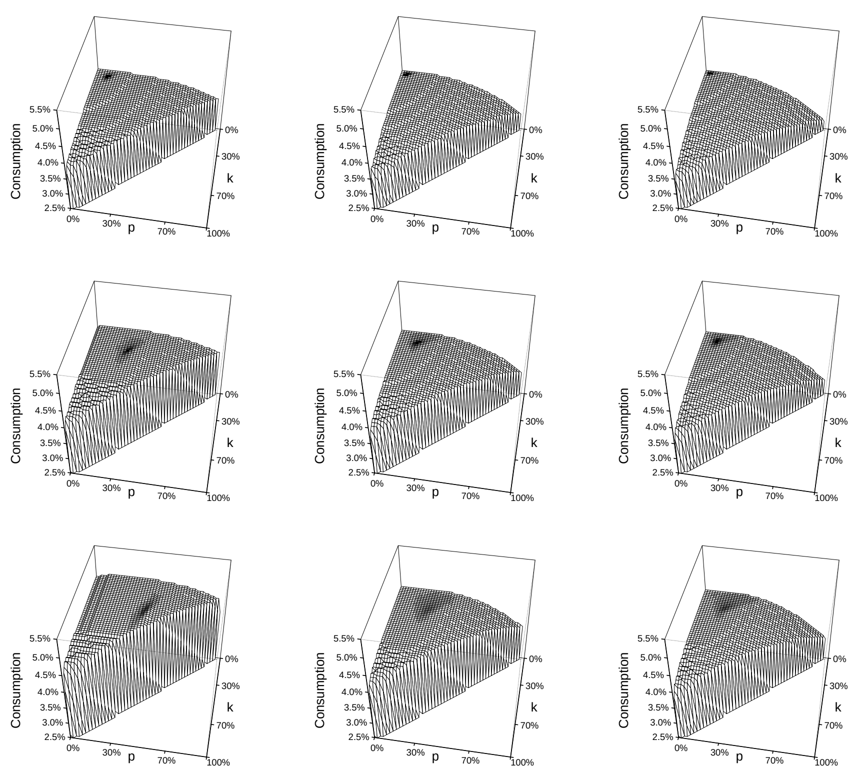
Panel A: $\sigma_{\mathrm{P}} = 40\%$

α \ β	1.2	1.6	1.8
1%			
3%			
5%			



Panel B: $\rho = 0.8$

α \ β	1.2	1.6	1.8
1%			
3%			
5%			



D Model with Secondary Market

The Bellman equation with a secondary market is given by the following expression:

$$
\begin{aligned}
&V(W,P,K) = \max_{(C,N,S,f)} \left\{C^{1-\gamma} + \delta E\left[V(W',P',K')^{1-\gamma}\right]\right\}^{\frac{1}{1-\gamma}} \\
&\text{subject to} \\
&W' = R_{\mathrm{P}}P(1-f) + R_{\mathrm{S}}S + R_{\mathrm{F}}\left(W - C - P(1-f) - S - f(\psi_{\mathrm{P}}P + \psi_{\mathrm{K}}K)\right) \\
&P' = \left(1-\lambda_{\mathrm{D}}\right) R_{\mathrm{P}}(1-f)P + \lambda_{\mathrm{N}}N + \lambda_{\mathrm{K}}(1-f)K \\
&K' = (1-\lambda_{\mathrm{K}})(1-f)K + (1-\lambda_{\mathrm{N}})N \\
&0 \le C \le W - P(1-f) - f(\psi_{\mathrm{P}}P + \psi_{\mathrm{K}}K) \\
&N \ge 0
\end{aligned}
\tag{D.1}
$$

Variables and parameters in this expression are defined in the main text. Normalized by wealth, the Bellman equation with a secondary market becomes:

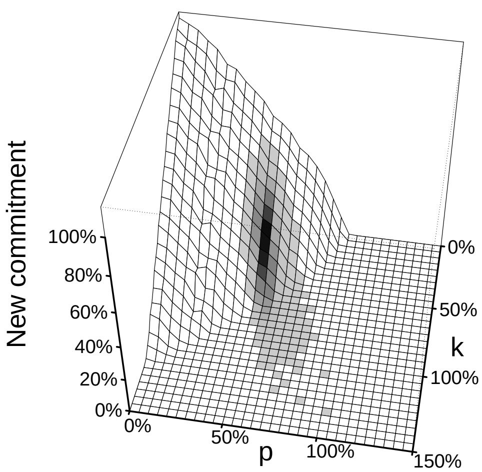
$$
\begin{aligned}
&v(p,k) = \max_{(c,n,\omega_{\mathrm{S}},f)} \left\{c^{1-\gamma} + \delta E\left[\left[\Big(1 - c - f(\psi_{\mathrm{P}}p + \psi_{\mathrm{K}}k)\Big) R_{\mathrm{W}} v(p',k')\right]^{1-\gamma}\right]\right\}^{\frac{1}{1-\gamma}} \\
&\text{subject to} \\
&R_{\mathrm{W}} = \omega_{\mathrm{P}}R_{\mathrm{P}} + \omega_{\mathrm{S}}R_{\mathrm{S}} + \left(1 - \omega_{\mathrm{P}} - \omega_{\mathrm{S}}\right) R_{\mathrm{F}} \\
&p' = \left[\left(1-\lambda_{\mathrm{D}}\right) R_{\mathrm{P}}(1-f)p + \lambda_{\mathrm{N}}n + \lambda_{\mathrm{K}}(1-f)k\right] \Big/ \left[(1 - c - f(\psi_{\mathrm{P}}p + \psi_{\mathrm{K}}k))R_{\mathrm{W}}\right] \\
&k' = \left[(1-\lambda_{\mathrm{K}})(1-f)k + (1-\lambda_{\mathrm{N}})n\right] \Big/ \left[(1 - c - f(\psi_{\mathrm{P}}p + \psi_{\mathrm{K}}k))R_{\mathrm{W}}\right] \\
&0 \le c \le 1 - p(1-f) - f(\psi_{\mathrm{P}}p + \psi_{\mathrm{K}}k) \\
&n \ge 0
\end{aligned}
\tag{D.2}
$$

In this expression, the portfolio weights in PE and stocks are defined respectively as $\omega_{\mathrm{P}} = P(1-f)/[W - C - f(\psi_{\mathrm{P}}P + \psi_{\mathrm{K}}K)]$ and $\omega_{\mathrm{S}} = S/[W - C - f(\psi_{\mathrm{P}}P + \psi_{\mathrm{K}}K)]$. The remaining variables are defined in the main text.

We solve the normalized problem numerically using an algorithm similar to the one described in Appendix A, and the rest of this section reports output from that solution. We plot optimal policies for new commitment, stock allocation, and consumption. We also include regression tables measuring the local sensitivity of optimal policies to the two state variables, k and p. Figures and tables are constructed in the same way as their counterparts from the solution of the reduced model.

D.1 Optimal Commitment with Secondary Market

a. Aggressive ($\gamma = 1$)



b. Conservative ($\gamma = 3$)

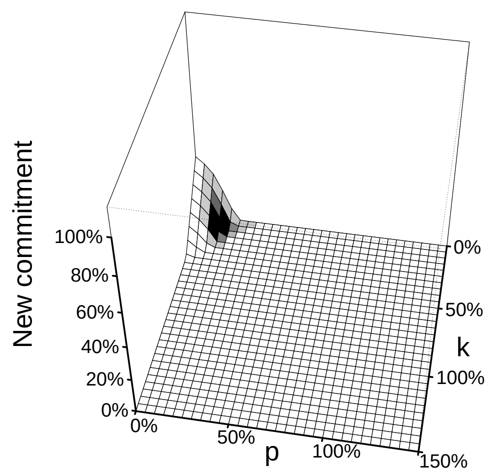


Figure D.13: Optimal commitment strategy with secondary market.

Table A1: Optimal Commitment with secondary market.

	Aggressive ($\gamma = 1$)		Conservative ($\gamma = 3$)	
	(1)	(2)	(3)	(4)
Illiquid Wealth	-0.86	-1.35	-1.49	-1.55
Commitment	-0.97	-1.57	-0.84	-0.94
Interaction		1.12		0.66
Constant	1.08	1.34	0.39	0.40
R^2	0.99	0.99	0.96	0.96

D.2 Optimal Stock Allocation with Secondary Market

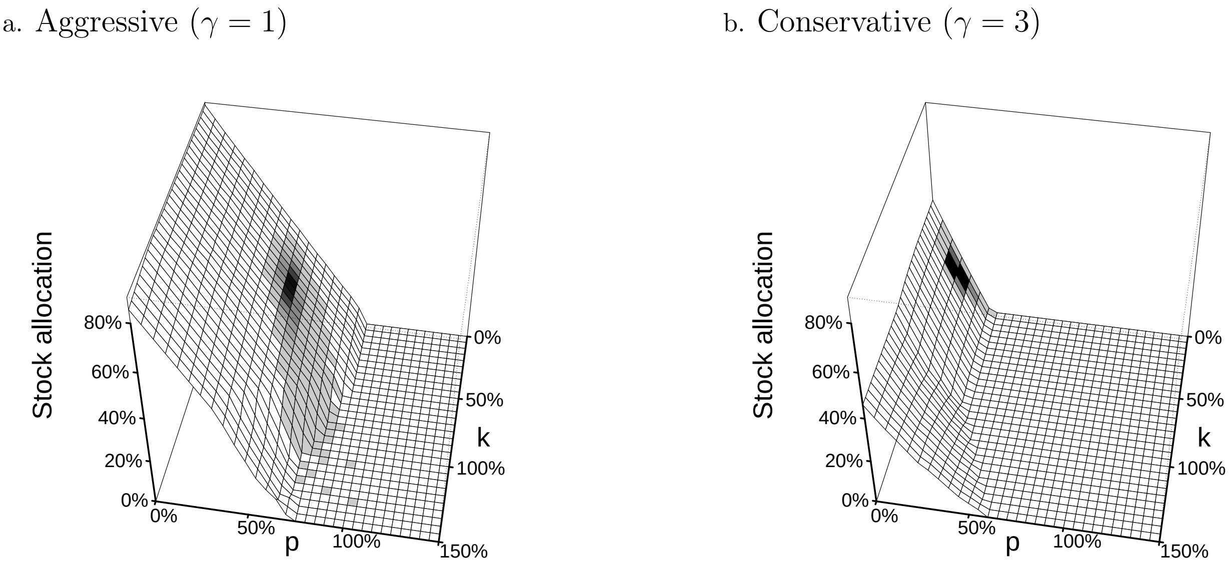


Figure D.14: Optimal stock allocation with secondary market.

Table A2: Optimal stock allocation with secondary market.

	Aggressive ($\gamma = 1$)		Conservative ($\gamma = 3$)	
	(1)	(2)	(3)	(4)
Illiquid Wealth	−1.04	−0.60	−1.58	−1.54
Commitment	−0.02	0.42	0.00	0.06
Interaction		−0.80		−0.40
Constant	0.94	0.70	0.49	0.48
R^2	0.97	0.99	1.00	1.00

D.3 Optimal Consumption with Secondary Market

a. Aggressive ($\gamma = 1$) b. Conservative ($\gamma = 3$)

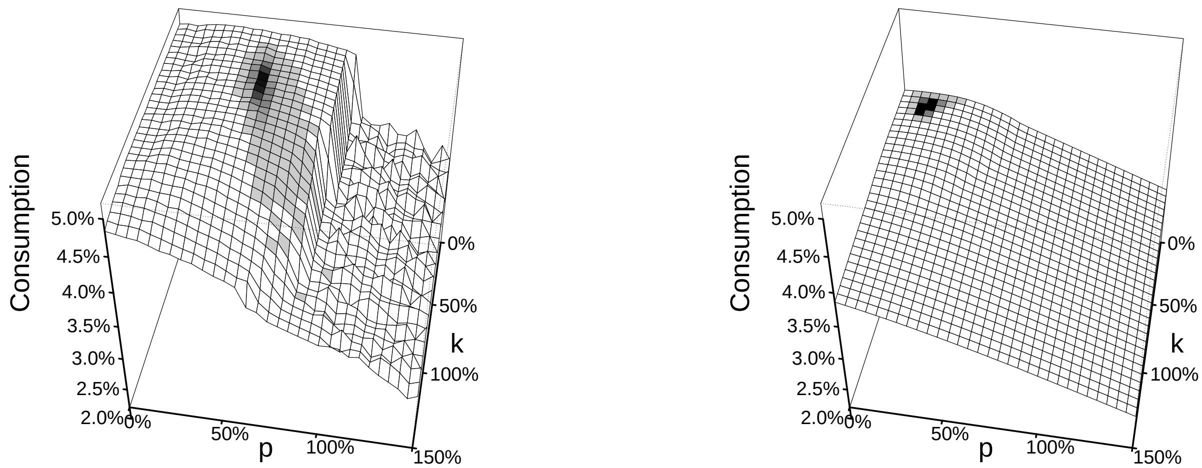


Figure D.15: Optimal consumption-to-wealth ratio with secondary market.