Risk Aversion with Nothing to Lose

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Abstract

In a continuous-time game, a risk-neutral decision-maker chooses the volatility of a state variable, and a stopper terminates the game. I provide conditions under which the decision-maker becomes risk averse endogenously and minimizes volatility near termination, even if he faces myopic incentives to gamble for resurrection. The conditions introduce forward-looking incentives to preserve economic rents. I study two applications: a levered corporation and a mutual fund with uncertain productivity. When investors are about to default or withdraw their capital, managers attempt to preserve their rents by minimizing risk. Rents originate from current payoffs, growth opportunities, or managerial overconfidence.

Keywords: Endogenous risk aversion, dynamic programming, distress.

JEL Classification: C73, D81, G20, G30.

1 Introduction

Researchers in finance widely use Jensen and Meckling (1976)'s framework to study risk-taking in levered and distressed firms. In this framework, distressed firms gamble for resurrection: when close to default, a firm has little to lose from a risky investment, but potentially a lot to gain. Therefore, according to this theory, distressed firms should engage in risk-shifting and increase risk exposure. However, evidence from the behavior of financial and non-financial institutions exposes the empirical gaps of this theory: in several cases, distressed firms reduce risk-taking.

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Several empirical studies document an increase in risk aversion in distressed firms. For example, Rauh (2009) shows pension funds reduce risk-taking when poorly funded, Gormley and Matsa (2011) document managers diversify into unrelated businesses when liabilities increase, and Gilje (2016) provides evidence that firms reduce investment risk when financially distressed. These findings cannot be reconciled with the static model of Jensen and Meckling (1976).

In a dynamic model, I provide conditions under which a risk-neutral corporate manager becomes risk averse endogenously when the firm is distressed. The model differs from the static framework of Jensen and Meckling (1976), because I explicitly focus on the dynamic incentives of a manager facing the possibility of corporate default. In fact, Jensen and Meckling (1976) observe that, in a dynamic model, risk-shifting incentives may disappear.

Even when the risk-neutral manager experiences a locally convex flow payoff near default, in a dynamic model, he faces a trade-off. On the one hand, the manager could increase risk and myopically take advantage of the convex flow-payoff structure. On the other hand, the manager could reduce risk and preserve his long-run rents. In a dynamic setting, I provide conditions under which these forward-looking incentives dominate.

To establish a general theory, I develop a parsimonious continuous-time model with a state variable. In the model, a decision-maker controls the volatility of the state variable. By controlling volatility, the decision-maker influences also his flow payoff and the drift of the state variable. If the state variable falls below a threshold, a stopper terminates the game and the decision-maker receives his outside option.

I then provide sufficient conditions for the decision-maker to become risk averse and minimize volatility near the termination threshold. To become risk averse endogenously, the decision-maker needs to enjoy current or future rents from control near termination, as reflected in positive flow payoffs or growth opportunities. Moreover, if the marginal effects of volatility on the flow payoffs and on the drift of the state variable are suitably bounded, the decision-maker minimizes risk near termination.

Starting from this general theory, I consider two applications in finance. In the first application, I study a levered firm similar to Leland (1994) and Leland (1998), in which equity holders choose the default time optimally. Compared with previous models, I introduce a risk-neutral manager who allocates assets between risky and safe investments. Although default is optimal for equity holders, it is not for the manager. Near the default threshold, the risk-neutral manager becomes risk averse and chooses an investment strategy that minimizes risk.

In a variation of the model, I let equity holders choose the risk exposure of the firm, but

I introduce a regulator who takes control of the firm when asset values fall below a threshold. With this variation, I study regulated firms, such as banks and insurance companies, which are subject to regulatory intervention when they violate their capital requirements. When the regulator intervenes too early from the equity holders' perspective, the latter become risk averse and minimize risk near the intervention threshold. Hence, regulated institutions reduce risk exposure before capital requirements are binding.[1]

In the second application, I study a continuous-time version of the Berk and Green (2004) model of a mutual fund with uncertain productivity. Compared with Berk and Green (2004), I allow the fund to choose its allocation to an uncertain investment. Investors learn about the investment's productivity by observing performance. When the manager allocates more capital to the uncertain investment, investors' posterior beliefs become more sensitive to realized returns. If the funds' perceived productivity deteriorates and the fund is near termination, the fund manager becomes risk averse if the investment is productive in the long run. Moreover, the manager chooses to minimize the fund allocation to the uncertain asset, thus minimizing the volatility of investors' posterior beliefs.[2]

I also consider a variation of this model in which the manager is compensated for performance and the investment's long-term productivity is uncertain. The manager could be either unbiased or overconfident. An unbiased manager learns about the investment's productivity together with investors. An overconfident manager is convinced his investment is productive. Because the overconfident manager subjectively believes he possesses long-term rents, he becomes risk averse and minimizes his allocation to the uncertain investment near termination. The unbiased manager always maximizes the allocation to the uncertain investment.

Finally, I extend the general model to allow for slow adjustment in the volatility of the state variable. With this extension, I study situations in which managers cannot modify their investments instantaneously. Here, I show the decision-maker reduces (increases) volatility near termination if, by doing so, he is able to distance himself from the termination threshold.

[1] In their empirical papers, Ben-David et al. (2020) and Kirti (2020) argue regulated institutions, when distressed, reduce risk for reasons beyond their need to meet capital requirements.

[2] The result is different from the model in Kuvalekar and Lipnowski (2020). In Kuvalekar and Lipnowski (2020), an agent reduces the volatility of his principal's beliefs when his perceived productivity is high, not when it is low.

2 RELATED LITERATURE

RELATED THEORETICAL LITERATURE. My paper provides a general theory that explains why decision-makers reduce risk rather than gamble for resurrection. Moreover, it highlights the common factor that drives results in previous literature: the existence of long-term rents. Previous literature showed managers are deterred from taking risks, because of banks' franchise value (Hellmann et al., 2000; Keeley, 1990; Repullo, 2004),[3] funds' fee revenue (Drechsler, 2014; Panageas and Westerfield, 2009), and concerns about reputation (Diamond, 1989; Hirshleifer and Thakor, 1992). Within the framework of my model, franchise value, fee revenues, and reputation represent specific forms of long-term rents for decision-makers.

Among continuous-time models, this paper generalizes a model by the same author showing distressed intermediaries shift to safe portfolios instead of gambling for resurrection (Pegoraro, 2017). Li and Mayer (2022) and Dai et al. (2021) show stablecoin issuers and multi-division firms also prefer safe investments when close to termination or short on cash. However, none of these three papers provides general conditions under which endogenous risk aversion emerges, thus limiting the scope of their application to their own specific framework.

I provide applied models related to the literature on continuous-time corporate finance (DeMarzo and He, 2021; Goldstein et al., 2001; Leland, 1994, 1998; Malenko and Tsoy, 2020) and asset management with learning (Berk and Green, 2004; Pástor and Stambaugh, 2012). Compared with previous literature, I introduce a manager who controls the volatility of the relevant state variable, and I obtain empirical predictions on the risk-taking behavior of managers near termination.

To mathematically characterize the model, I rely heavily on existing results from the theory of stochastic control and differential equations. Strulovici and Szydlowski (2015) provide important and useful results on the properties of value functions. Textbooks by Fleming and Soner (2006) and Pham (2009) also represent crucial references.

EMPIRICAL EVIDENCE. Empirical evidence on risk-shifting is currently mixed. My model sheds some light as to why. In fact, according the the model, decision-makers behave differently based on their forward-looking incentives to maintain their rents.

On the one hand, several researchers document that highly leveraged firms do not engage in risk-shifting, but instead reduce risk exposures. Besides Gilje (2016), Gormley and Matsa (2011), and Rauh (2009), whom I mentioned in the introduction, Andrade and

[3] Empirically, Gan (2004) shows banks take more risk when their franchise value declines.

Kaplan (1998) find no evidence that distressed firms make riskier investments than other firms. Ben-David et al. (2020), Bidder et al. (2021), Di Patti and Kashyap (2017), and Peydró et al. (2020) show banks reduce risk when distressed or after large losses. Kirti (2020) documents similar patterns for insurance companies.

Focusing on corporate decision-makers, researchers have found little to no evidence of risk-shifting in surveys of CFOs (De Jong and Van Dijk, 2007; Graham and Harvey, 2001), in CEOs of firms with high default risk (Milidonis and Stathopoulos, 2014), or in experimental settings with dynamic incentives (Hernández-Lagos et al., 2016).

On the other hand, some studies provide evidence suggesting risk-shifting. Eisdorfer (2008) shows distressed firms invest more when aggregate volatility is higher, whereas Dell'Ariccia et al. (2017) and Drechsler et al. (2016) show less capitalized banks take more risks after monetary policy interventions. Becker and Ivashina (2015) document poorly capitalized insurance companies reach for yield.

In my applied models, managers sacrifice shareholders' profits and firm growth by minimizing risk. My model thus predicts firms take less risk when managers have more discretionary power. The empirical evidence generally agrees with this prediction: firms take fewer risks when managers enjoy stronger protection or control rights (Gormley and Matsa, 2016; John et al., 2008; Laeven and Levine, 2009; Saunders et al., 1990), even at the expense of profitability and growth (Giroud et al., 2012; Gormley and Matsa, 2016; John et al., 2008; Kalcheva and Lins, 2007).

3 General Model

I consider a parsimonious but general model. A risk-neutral decision-maker chooses the volatility of a payoff-relevant state variable, and a stopper decides when to terminate the game. In this section, I do not directly model the stopper's termination decision; rather, I assume the stopper interrupts the game when the state variable falls below a threshold.[4] I then establish sufficient conditions under which the decision-maker develops risk aversion endogenously and minimizes risk near the termination threshold.

Let $(\Omega, \mathcal{F}^*, P)$ be a probability space and let $(Z_t)_{t\geq 0}$ be a Brownian motion on $(\Omega, \mathcal{F}^*)$. Consider a one-dimensional state variable y_t taking values in an interval $\mathcal{Y} \subseteq \mathbb{R}$ and solving the forward stochastic differential equation

$$dy_t = \mu(y_t, \eta_t)dt + \eta_t \sigma(y_t) dZ_t, \quad y_0 = Y_0, \tag{1}$$

[4] In sections 4 and 5, I provide applied models in which investors endogenously stop the game.

where $\eta_t \in [\underline{\eta}, \bar{\eta}]$, with $0 < \underline{\eta} < \bar{\eta} < \infty$. The interval $\mathcal{Y}$ may be unbounded. I say the process $(\eta_t)_{t\geq 0}$ is *admissible* if it is adapted to the filtration $(\mathcal{F}_t)_{t\geq 0}$ and it implies a unique strong solution of (1).

The decision-maker receives a flow payoff $\pi(y_t, \eta_t)$ as long as $y_t > \hat{y}$, where $\hat{y}$ belongs to the interior of $\mathcal{Y}$. When y_t hits the threshold $\hat{y}$, a stopper terminates the game. At that point, the decision-maker receives his outside option, which I normalize to zero.[5] The decision-maker is risk neutral and discounts the future at rate $\rho > 0$.

The decision-maker optimally chooses an admissible process $(\eta_t)_{t\geq 0}$ to maximize his lifetime utility

$$\begin{aligned} u(y_0) = \max_{(\eta_t)_{t\geq 0}} \mathrm{E}\left[\int_0^\tau e^{-\rho t}\pi(y_t, \eta_t)\, dt\right] \\ \text{s.t.} \quad \eta_t \in [\underline{\eta}, \bar{\eta}] \quad \forall t \geq 0, \end{aligned} \tag{2}$$

where $\tau = \inf\{t \geq 0 : y_t \leq \hat{y}\}$ is the stopping time at which the stopper terminates the game.

I impose the following regularity conditions throughout.

ASSUMPTIONS. *(Regularity Conditions)*

(R1) *The functions $\mu(y, \eta)$, $\sigma(y)$, and $\pi(y, \eta)$ are Lipschitz-continuous*[6] *in y for all $\eta \in [\underline{\eta}, \bar{\eta}]$ and differentiable in η for all $y \in \mathcal{Y}$.*

(R2) *The discount rate ρ is large enough that $\rho > C_1^\mu$, where C_1^μ is such that, for all $y \in \mathcal{Y}$, $|\mu(y, \eta)| < C_0^\mu + C_1^\mu |y|$ for some C_0^μ.*[7]

(R3) *$\underline{\sigma} > 0$ exists such that $\sigma(y) \geq \underline{\sigma}$ for all $y \in \mathcal{Y}$ such that $y \geq \hat{y}$.*

Conditions (R1) and (R2) ensure the value function $u(\cdot)$ satisfies linear growth; that is, C_0^u and C_1^u exist such that $u(y) < C_0^u + C_1^u|y|$. For a proof, see Lemma 1 and the subsequent discussion in Strulovici and Szydlowski (2015). Condition (R3) imposes a uniform positive lower bound on the volatility of y, which prevents volatility to be arbitrarily close to zero.

Thanks to these assumptions, I characterize the decision-maker's problem (2) recursively. Because of assumptions (R1) and (R2), classical results (Pham, 2009, Chapter 4)

[5] This normalization is without loss of generality. In fact, one could assume the decision-maker's outside option is $\mathcal{O} \neq 0$ and define $\pi^{\mathcal{O}}(y, \eta) := \pi(y, \eta) - \rho\mathcal{O}$ to obtain an equivalent game with flow payoff equal to $\pi^{\mathcal{O}}$ and zero outside option.

[6] A function $h(x)$ is Lipschitz-continuous in x over $\mathcal{X} \subseteq \mathbb{R}$ if $C > 0$ exists, such that $|h(x_1) - h(x_2)| < C|x_1 - x_2|$ for all $x_1, x_2 \in \mathcal{X}$.

[7] Such C_1^μ exists because of assumption (R1).

establish u is the unique solution of the following Hamilton-Jacobi-Bellman (HJB) equation

$$\rho u(y) = \max_{\eta \in [\underline{\eta}, \bar{\eta}]} \left\{ \pi(y,\eta) + u'(y)\mu(y,\eta) + \frac{1}{2} u''(y)\eta^2 \sigma(y)^2 \right\} \tag{3}$$

on the domain $y \geq \hat{y}$, satisfying the boundary condition $u(\hat{y}) = 0$ and linear growth. Moreover, with assumption (R3), I impose a uniform ellipticity condition, and hence, by Strulovici and Szydlowski (2015), the function u is twice continuously differentiable for $y > \hat{y}$. Let $\eta(y)$ denote the maximizer of (3). With Lemma A.1 and Remark 1 in Appendix A.2, I verify that if the process $(\eta(y_t))_{t\geq 0}$ is admissible, $(\eta(y_t))_{t\geq 0}$ is optimal for the decision-maker.

In this paper, I focus on the decision-maker's behavior near the termination threshold, and I show we cannot directly extend Jensen and Meckling (1976)'s intuition to a dynamic model with Brownian risk. Under certain conditions, the risk-neutral decision-maker minimizes risk near the termination threshold. These conditions allow for the decision-maker to have nothing to lose near termination, that is, $\pi(\hat{y}, \cdot) = 0$. Moreover, these conditions also allow for the decision-maker to have locally convex flow payoffs near termination, that is, $\lim_{y\to\hat{y}^+} \pi_{yy}(y, \cdot) > 0$.

The following assumptions are key to obtaining endogenous risk aversion.

ASSUMPTIONS. *(Economic Assumptions)*

(A1) *For all $y > \hat{y}$, an admissible process $(\eta_t)_{t\geq 0}$ exists with $\eta_t \in [\underline{\eta}, \bar{\eta}]$ for all $t \geq 0$ such that* $\mathrm{E}\left[\int_0^\tau e^{-\rho t}\pi(y_t, \eta_t)\, dt\right] > 0$.[8]

(A2) *$\mu(\hat{y}, \eta) \geq 0$ and $\pi(\hat{y}, \eta) \geq 0$ for at least one $\eta \in [\underline{\eta}, \bar{\eta}]$, with at least one strict inequality.*

(A3) *$\mu(\hat{y}, \eta) \geq \frac{1}{2}\eta\mu_\eta(\hat{y}, \eta)$ and $\pi(\hat{y}, \eta) \geq \frac{1}{2}\eta\pi_\eta(\hat{y}, \eta)$ for any $\eta \in [\underline{\eta}, \bar{\eta}]$, with at least one strict inequality.*

According to Assumption (A1), the decision-maker enjoys a strictly positive lifetime utility as long as the game is not terminated. Hence, the decision-maker dislikes being terminated. According to Assumption (A2), the decision-maker earns current or future rents near termination. Finally, Assumption (A3) prevents the marginal benefit of volatility η from being excessive. Assumption (A3) is satisfied if μ and π are linear or linear-quadratic functions of volatility η. Such functions represent natural modeling choices in a variety of economic contexts, such as those in sections 4 and 5.

I state the main theoretical results of the paper in the next theorem.

[8]A stronger sufficient conditions is that $\pi(y, \eta) > 0$ for all $y \geq \hat{y}$ and all $\eta \in [\underline{\eta}, \bar{\eta}]$.

THEOREM 1. *The following hold.*

(I) *If Assumptions* (A1) *and* (A2) *hold, an* $\varepsilon > 0$ *exists such that, for any* $y \in (\hat{y}, \hat{y} + \varepsilon)$, $u'(y) > 0$ *and* $u''(y) < 0$*;*

(II) *If also* (A3) *holds, an* $\varepsilon > 0$ *exists such that, for any* $y \in (\hat{y}, \hat{y} + \varepsilon)$, $\eta(y) = \underline{\eta}$.

Part (I) of Theorem 1 states that the decision-maker's value function is strictly increasing and concave near termination. This result differs from models of optimal stopping, where the value function is locally flat and convex near the stopping threshold. Here, the decision-maker does not optimally select when to terminate the game. In fact, he strictly prefers not to terminate the game in order to enjoy his rents. As a result, the decision-maker's value function is strictly increasing in a neighborhood of the termination threshold, indicating the decision-maker dislikes termination.

Moreover, near termination, a risk-neutral decision-maker endogenously becomes risk averse, as suggested by the concavity of the value function. Because the decision-maker enjoys current or future rents by assumption (A2), he develops endogenous aversion to the risk of forfeiting those rents.

Finally, according to part (II) of Theorem 1, the decision-maker chooses the lowest risk exposure $\underline{\eta}$ near termination. Because of his endogenous risk aversion, the decision-maker wants to limit risks near termination, as one would expect from Assumption (A3). However, the result in part (II) of Theorem 1 is much stronger. The decision-maker does not want to simply limit risk exposure. He wants to *minimize* risk exposure. That is, no matter how low $\underline{\eta}$ is, the decision-maker will always choose $\eta(y) = \underline{\eta}$ in a neighborhood of $\hat{y}$.

The regularity conditions (R1), (R2), and (R3) are required to apply classical results and establish the value function is the unique solution of equation (3) and that the solution is continuous and twice differentiable. These conditions are not strictly necessary for Theorem 1. Theorem 1 holds for any twice-differentiable solution u to (3) when $\mu(y, \eta)$, $\sigma(y)$, and $\pi(y, \eta)$ are continuous in y and differentiable in η, at least locally in a right neighborhood of $\hat{y}$. Moreover, as long as one finds a twice-differentiable solution u to (3) satisfying $u(\hat{y}) = 0$ and $\lim_{t\to\infty} \mathrm{E}[e^{-\rho t} u(y_t) | \mathcal{F}_0] = 0$, Lemma A.1 shows u is the value function and that the policy function $\eta(y_t)$ is optimal, provided it is admissible.

3.1 DISCUSSION AND EMPIRICAL IMPLICATIONS

According to the model, one could empirically observe a decision-maker minimize risk near termination even if he has no flow payoff to lose, that is, $\pi(\hat{y}, \cdot) = 0$, and his payoff scheme is convex, that is, $\pi_{yy}(y, \cdot) > 0$. To obtain this result, the dynamic structure of the model is crucial. On the one hand, the decision-maker faces a myopic incentive to gamble for resurrection because of the convexity of his flow payoff. On the other hand, Assumptions (A1) and (A2) introduce forward-looking incentives to minimize risk, generating endogenous risk aversion. Because of assumption (A3), these forward-looking incentives dominate the myopic ones and the decision-maker minimizes risk near termination.

Importantly, Theorem 1 holds even if social welfare increases with risk. In fact, in my framework, a decision-maker could reduce social welfare by selecting projects that are too safe. For example, consider a manager of a distressed firm deciding between a safe project of low net present value and a risky project of high net present value. In Jensen and Meckling (1976)'s framework, because the decision-maker prefers to take risks, the socially optimal (risky) project will be implemented. In my model, the decision-maker chooses differently. As long as assumptions (A1), (A2), and (A3) hold, the decision-maker selects the safe project near termination instead of the risky one, even if the risky project is more socially valuable.

Finally, the model highlights sufficient, but not necessary, conditions under which a risk-neutral decision-maker minimizes risk exposure near termination. Conditions (A1), (A2), and (A3) guarantee the results of proposition (1), but one may obtain analogous outcomes even when the conditions partially fail. In this case, the shape of the value function and the optimal choice of the decision-maker depend on parameter. I provide an example in section 4.2. In come situations, however, one can prove the value function is convex and that the decision-maker increases risk exposure near termination. For example, when $\mu(\hat{y}, \eta) \leq 0$ and $\pi(\hat{y}, \eta) \leq 0$, with at least one strict inequality, one can show the decision-maker is risk loving. That is, his value function is convex near termination.

Therefore, my model suggests a series of tests for empirical researchers investigating what drives risk-shifting or risk-avoiding behavior. One could empirically investigate whether risk-shifting (risk-avoiding) decision-makers are experiencing negative (positive) flow payoffs $\pi(y_t, \eta_t)$ near termination, or if they are facing negative (positive) growth opportunities $\mu(y_t, \eta_t)$. Furthermore, one could relate risk-shifting and risk-avoiding behavior to managers' control benefits. In fact, the model predicts managers with larger control benefits are less likely to take risks near termination, because they enjoy larger rents as long as they are not terminated.

4 LEVERED FIRM

In the first example, I consider a model of a levered firm, analogous to that in Leland (1994). Whereas Leland (1994) assumes the firm's risk exposure is constant, I introduce a manager who controls the volatility of the firm's asset growth. Equity holders optimally determine when to default. By defaulting, equity holders force a suboptimal termination for the manager.

As a variation of the model, I then assume equity holders choose the risk exposure of the firm, but they face early termination because of regulatory intervention. For example, banks may be forced into a Prompt Corrective Action (PCA) by US regulators if they are poorly capitalized. PCA could even lead to the institution's liquidation under regulatory supervision.

4.1 BASELINE MODEL

In the baseline model, I introduce a separation between equity holders and management. Equity holders decide when to default, whereas a manager controls the investment strategy of the firm, which can be adjusted instantaneously.[9] Hence, the model best reflects the situation of financial firms investing in liquid assets, including banks, insurance companies, and defined-benefit pension plans. However, the key insight of the model may also be applied to non-financial firms.

The total assets of the firm evolve as

$$dV_t = \mu\eta_t V_t dt + \eta_t \sigma V_t dZ_t, \tag{4}$$

where $\mu > 0$ and $\sigma > 0$ are parameters, and where firm growth depends on the risk exposure of the firm, η_t, representing the fraction of risky investments in the firm's portfolio. The process $(Z_t)_{t\geq 0}$ is a Brownian motion over the probability space $(\Omega, \mathcal{F}^*, P)$. Let $\mathcal{F} = (\mathcal{F}_t)_{t\geq 0}$ be the filtration generated by the process $(V_t)_{t\geq 0}$, possibly augmented by the collection of P-null sets.

The total cash flow of the firm depends on the firm's asset value:

$$dC_t = \delta V_t dt, \tag{5}$$

where $\delta > 0$.

[9] In section 6, I consider a general model in which the firm's investment strategy is subject to slow adjustment.

The manager is risk neutral and chooses the firm's risk exposure η_t, subject to the condition that $\eta_t \in [\underline{\eta}, \bar{\eta}]$, with $0 < \underline{\eta} < \bar{\eta} < \infty$. Hence, the manager and equity holders interact in a framework with incomplete contracting: as long as the manager chooses the fraction of risky assets within the interval $[\underline{\eta}, \bar{\eta}]$, equity holders cannot influence the manager's investment decision.

The manager is compensated with a fraction $\theta > 0$ of the firm's cash flow. One could model more complex compensation schemes, but the results hold as long as the manager has limited liability or, more generally, as long as he does not receive a negative expected compensation.

The firm has a fixed amount of outstanding liabilities and owes payments c to debt holders each period. We can interpret these liabilities as debt in financial and non-financial corporations or as pension liabilities in a pension fund. Therefore, the firm generates free cash flow at rate $\delta V_t - c$.

Equity holders receive the free cash flow of the firm after paying the manager. As a result, equity holders receive a flow payoff of $(1-\theta)\delta V_t - c$ every period. When V_t is small enough, equity holders experience losses. They cover these losses by injecting new equity.

Because equity holders have limited liability, they optimally choose to default when losses become excessive. Risk-neutral equity holders discount future cash flows at rate $r > \mu\bar{\eta}$ and select an optimal default time τ to maximize the value of equity, that is,

$$\begin{aligned} \max_{\tau \geq 0} \quad & \mathrm{E}\left[\int_0^\tau e^{-rt}[(1-\theta)\delta V_t - c]\,dt \middle| \mathcal{F}_0\right] \\ \text{s.t.} \quad & (4). \end{aligned} \tag{6}$$

Because the manager chooses the control process $(\eta_t)_{t\geq 0}$, equity holders take the investment in risky assets as given.

The risk-neutral manager chooses the investment allocation between risky and safe assets, as long as the firm remains solvent. The manager discounts future cash flows at rate $\rho > \mu\bar{\eta}$. The manager therefore solves

$$\begin{aligned} \max_{(\eta_t)_{t\geq 0}} \quad & \mathrm{E}\left[\int_0^\tau e^{-\rho t}\theta\delta V_t\,dt \middle| \mathcal{F}_0\right] \\ \text{s.t.} \quad & (4) \text{ and } \eta_t \in [\underline{\eta}, \bar{\eta}], \quad \forall t \geq 0, \end{aligned} \tag{7}$$

where τ is the time of default. When the firm defaults, the manager is fired and obtains a continuation value that is normalized to zero.

4.1.1 Markov-Perfect Equilibria

As a solution concept of this game between the manager and the equity holder, I consider a Markov-perfect equilibrium, which is defined as follows.

Definition 1 (Markov-Perfect Equilibrium)**.** *A Markov-perfect equilibrium consists of a $\mathcal{F}$-measurable process $(\eta_t^*)_{t\geq 0}$ for the share of risky assets and an $\mathcal{F}$-measurable stopping time $\tau^* \geq 0$ for equity default such that*

(I) *The stopping time τ^* solves* (6) *given $(\eta_t^*)_{t\geq 0}$;*

(II) *$(\eta_t^*)_{t\geq 0}$ solves* (7) *given τ^*;*

(III) *A set $\mathcal{D} \subseteq R^+$ exists such that $\tau^* = \inf\{t \geq 0\colon V_t \notin \mathcal{D}\}$;*

(IV) *A function $\eta\colon \mathcal{D} \to [\underline{\eta}, \bar{\eta}]$ exists such that $\eta(V_t) = \eta_t^*$.*

The first two conditions define an equilibrium in public strategies, in which the manager and the equity holders base their decisions on the history of asset values $(V_t)_{t\geq 0}$. The remaining two conditions restrict equilibria to Markov-perfect ones, in which the manager's and equity holders' choices depend solely on the current value of the firm's asset, V_t, rather than its entire history.[10]

In Lemma A.2 of Appendix A.2, I show that in a Markov-perfect equilibrium, the optimal stopping rule is a *threshold strategy*. That is, equity holders strategically default when the firm's assets drop below a threshold $\hat{V}$.

Because strategies are functions of the firm's asset, in a Markov-perfect equilibrium, the manager's and equity holders' continuation values are also functions of current firm assets. I denote them by $u(V)$ and $E(V)$, respectively, when $V_0 = V$. I then characterize a Markov-perfect equilibrium recursively. The HJB equation associated with the manager's decision problem is

$$\rho u(V) = \max_{\eta \in [\underline{\eta}, \bar{\eta}]} \left\{ \theta \delta V + u'(V) \mu \eta V + \frac{1}{2} u''(V) \eta^2 V^2 \right\}, \tag{8}$$

for $V > \hat{V}$; otherwise $u(V) = 0$. I define the default threshold $\hat{V} \coloneqq \sup\{V \geq 0\colon E(V) = 0\}$, where E solves the variational inequality associated with the shareholders' problem:

$$\min\{rE(V) - H(V, E'(V), E''(V)), E(V)\} = 0, \tag{9}$$

[10] I focus on Markov-perfect equilibria because such equilibria do not require commitment. Other equilibria exist in which, for example, equity holders commit to default as soon as the manager chooses risky investments below the maximum level $\bar{\eta}$. By doing so, equity holders would enforce an equilibrium in which the manager always maximizes risky investments. However, this equilibrium is based on an unrealistic assumption about equity holders' commitment.

with

$$H(V, E'(V), E''(V)) = (1-\theta)\delta V - c + E'(V)\mu\eta(V)V + \frac{1}{2}E''(V)\eta(V)^2V^2,$$

and where $\eta(V)$ is the maximizer in (8).

For any given $\hat{V}$, results in Pham (2009) and Strulovici and Szydlowski (2015) imply the manager's value function is the unique twice-differentiable solution of (8) for $V \geq \hat{V}$. In Lemma A.3 of Appendix A.2, I show the optimal policy $\eta(V)$ is continuous in V for $V \geq \hat{V}$. I impose $\eta(V) = \lim_{V' \to \hat{V}} \eta(V')$ for $V \leq \hat{V}$, so that $\eta(V)$ is continuous for all $V > 0$. Hence, the equity value is the unique continuous solution of (9) satisfying linear growth (Pham, 2009, Theorem 5.2.1 and Remark 5.2.1).

A Markov-perfect equilibrium therefore solves a fixed-point problem. Given the manager's policy function $\eta(\cdot)$, the stopping time $\tau = \inf\{t \geq 0\colon V \leq \hat{V}\}$ must be optimal for the equity holders. At the same time, given the stopping time τ and its associated default threshold $\hat{V}$, the policy function $\eta(\cdot)$ must be optimal for the manager.

In particular, let the functions u and E solve the system given by (8) and (9) with $\hat{V} := \sup\{V \geq 0\colon E(V) = 0\}$ and where $\eta(V)$ is the maximizer in (8). Let $\eta_t^* = \eta(V_t)$ for all $t \geq 0$ and $\tau^* := \inf\{t \geq\colon V_t \leq \hat{V}\}$. If $(\eta_t^*)_{t\geq 0}$ is admissible, $(\eta_t^*)_{t\geq 0}$ and τ^* constitute a Markov-perfect equilibrium. By Lemma A.1 and Remark 1 in Appendix A.2, $(\eta_t^*)_{t\geq 0}$ is optimal for the manager. Moreover, τ^* is optimal for equity holders because

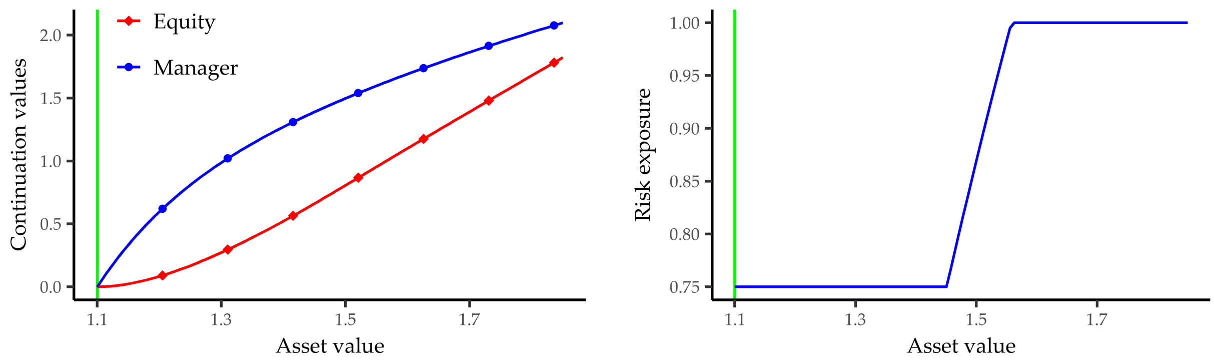
$$\mathrm{E}\left[\int_0^{\tau^*} e^{-\rho t}\{(1-\theta)\delta V_t - c\}\, dt \middle| \mathcal{F}_0\right] = E(V_0) = \max_{\tau} \mathrm{E}\left[\int_0^{\tau} e^{-\rho t}\{(1-\theta)\delta V_t - c\}\, dt \middle| \mathcal{F}_0\right],$$

where the first equality follows from the dynamic programming principle and $E(V_{\tau^*}) = 0$, and the second follows because $E(V_0)$ is the equity holders' value function. Therefore, τ^* is optimal for equity holders.

Important for my argument, the manager's value function is characterized by the HJB equation (8), and the manager is subject to termination when $V_t \leq \hat{V}$. Next, I use the results from 3 to characterize the manager's strategy near default.

4.1.2 Risk Avoidance in Distressed Firms

In its basic structure, the Markov-perfect equilibrium of this model is analogous to the abstract model of section 3. In both cases, a decision-maker (the manager) controls the volatility of a state variable (the firm assets) and faces termination when the state variable falls below a threshold ($\hat{V}$). Moreover, Assumptions (A1), (A2), and (A3) are satisfied in this model. Therefore, I characterize the risk preferences of the manager near the default



(a) Continuation values $E(V)$ and $u(V)$

(b) Risk exposure $\eta(V)$

Figure 1: Continuation values and risk exposure when equity holders optimally default and a corporate manager chooses the firm's risk exposure. The vertical green line marks the termination threshold. The parameter values are $\mu = 2\%$, $\delta = 4\%$, $c = 10\%$, $\sigma = 10\%$, $\bar{\eta} = 100\%$, $\underline{\eta} = 75\%$, $r = \rho = 3\%$, $\theta = 30\%$.

threshold. The next corollary is a direct application of Theorem 1, and its proof is omitted.

COROLLARY 1. *An $\epsilon > 0$ exists such that, for any $V \in (\hat{V}, \hat{V} + \epsilon)$, $u''(V) < 0$ and $\eta(V) = \underline{\eta}$.*

Moreover, in Lemma A.4 of Appendix A.2, I show $E''(V) > 0$ in a right neighborhood of $\hat{V}$. That is, equity holders are risk-loving near the termination threshold and they would like to manager to gamble for resurrection. Intuitively, because equity holders optimally default when assets fall below $\hat{V}$, they have no future rents they are willing to preserve at $\hat{V}$. However, the default threshold $\hat{V}$ is not optimal for the manager. At that asset value, the manager possesses current and future rents, which make him averse to risk. He thus implements an unprofitable investment strategy by making safe investments.

In Figure 1, I numerically illustrate the results of the model. The numerical solutions are obtained using a finite-difference method. Figure 1(a) shows the value functions for the manager and for the equity holders. The value function of the manager is concave near the default threshold, whereas the value function of equity holders is convex. Figure 1(b) shows the manager minimizes risky investments near the default threshold.

5 Active Portfolio Management

I now consider a fund manager who invests in a strategy with uncertain profitability, and investors learn about profitability by observing performance. Investors' posterior beliefs represent the manager's reputation. Investors refuse to provide any capital to the fund when the manager's reputation falls below a threshold. At that threshold, the fund is terminated. The manager controls the activeness of the fund. By choosing a more active strategy, the manager increases the sensitivity of reputation to performance and hence its volatility.

In the baseline model, a manager maximizes the lifetime fee revenues earned by the fund, and the investment is profitable in the long run. Near the termination threshold, the manager chooses to minimize activeness to minimize the informativeness of the return signal and hence the volatility of his reputation.

Then, I consider a model in which the manager is compensated linearly in returns, reflecting the empirical observation that portfolio managers are mainly compensated based on performance (Ibert et al., 2018; Ma et al., 2019). I also assume the long-term profitability of the manager is uncertain in the long run. Here, I show an overconfident manager is more likely than an unbiased manager to reduce activeness near the termination threshold.

5.1 BASELINE MODEL

A population of competitive investors, whose measure is normalized to 1, supply capital $K_t \geq 0$ to a fund manager and pay proportional fees $f_t \geq 0$ on the assets under management. The manager invests in a portfolio of risky assets and decides on the allocation to a strategy with uncertain profitability. I denote with η_t the fraction of the portfolio that is allocated to the uncertain strategy and call this fraction *activeness*.

I denote the strategy's profitability at time t by $h_t \in \{0, 1\}$. When $h_t = 1$, the strategy is profitable; when $h_t = 0$, the strategy is unprofitable. When implementing a profitable (unprofitable) strategy, the manager produces higher (lower) returns by choosing higher activeness. In particular, the gross cash flow generated by a portfolio manager with profitability h_t, assets under management $K_t > 0$, and activeness η_t is

$$dC_t = K_t(\sigma\eta_t(h_t - \hat{\phi}))dt - c(K_t)dt + K_t\sigma dZ_t. \tag{10}$$

Here, $(Z_t)_{t\geq 0}$ is a Brownian motion on the probability space $(\Omega, \mathcal{F}^*, P)$, and $(h_t)_{t\geq 0}$ is a continuous-time, finite-state Markov chain on $(\Omega, \mathcal{F}^*)$, and it is independent of $(Z_t)_{t\geq 0}$. Let $\mathcal{F} = (\mathcal{F}_t)_{t\geq 0}$ be the filtration generated by the process $(C_t)_{t\geq 0}$, possibly augmented by the collection of P-null sets. Then, the processes $(K_t)_{t\geq 0}$ and $(\eta_t)_{t\geq 0}$ are real-valued $\mathcal{F}$-adapted processes.

The parameter $\hat{\phi} \in (0, 1)$ introduces a performance cost for an unprofitable strategy. The function $c(K_t)$ represents an increasing and convex cost of active fund management, as in Berk and Green (2004). I assume $c(x)$ is quadratic; that is, $c(x) = qx^2/2$ for some

$q > 0$. The return volatility $\sigma > 0$ is a known parameter.

Crucially, by increasing activeness, the manager increases the informativeness of returns. In fact, η_t coincides with the signal-to-noise ratio of returns. I impose $\eta_t = 0$ when investors delegate no capital to the manager and $K_t = 0$. That is, a manager cannot be active with no capital. Thus, no new information is acquired by investors when they cease to finance the manager.

The profitability process $(h_t)_{t\geq 0}$ evolves as a continuous-time Markov chain with generator

$$\Lambda(\eta_t) = \begin{pmatrix} -\lambda_0 & \lambda_0 \\ \lambda_1 & -\lambda_1 \end{pmatrix}, \tag{11}$$

where $\lambda_0 > 0$ and $\lambda_1 > 0$, and where the state vector is $(0, 1)'$. I assume

$$p := \frac{\lambda_0}{\lambda_0 + \lambda_1} > \hat{\phi},$$

so that, in the long run, the manager is profitable on average.

Both the investors and the manager observe total cash flows and the activeness η_t and update their assessment of the manager's productivity. In particular, the cash flow (10) represents a public signal of a game of symmetric learning, and players form their assessment of the manager's current productivity, ϕ_t, using Bayes' rule, that is, $\phi_t = \mathrm{E}[h_t|\mathcal{F}_t]$. I refer to ϕ_t as the manager's reputation, because it represents the posterior probability that the manager's strategy is profitable at time t.

Each competitive investor i chooses her supply of capital $k_{it} \geq 0$ to the fund to maximize her lifetime net cash flows:

$$\mathrm{E}\left[\int_s^\infty e^{-r(t-s)} k_{it}(dR_t - f_t dt)\Big|\mathcal{F}_s\right] \quad \forall s \geq 0, \tag{12}$$

where $dR_t := \frac{dC_t}{K_t}$.

When $\phi_t \leq \hat{\phi}$, investors expect non-positive gross returns from the manager and, thus, do not provide any capital to the fund; that is, $k_{it} = 0$ for all i. To streamline the model, I assume the fund is permanently closed when investors withdraw their capital. In section XXX of the online appendix, I show the results of this section survive even if the manger is allowed to open a new fund after the first one is liquidated as long as the manager incurs some positive cost in opening a new fund.

As compensation, the manager receives the fund's profit $f_t K_t$.[11] He chooses fund

[11] Here, I abstract from agency issues between the manager and the fund management company, and I assume the portfolio manager acts in the best interest of the fund management company. In section **??**, I consider an overconfident manager who is paid linearly in performance.

activeness and fund fees to maximize his lifetime utility:

$$\begin{aligned}
\max_{(\eta_t)_{t\geq s},(f_t)_{t\geq s}} \quad & \mathrm{E}\left[\int_s^\tau e^{-\rho(t-s)} f_t K_t \, dt \middle| \mathcal{F}_s\right] \quad \forall s \geq 0, \\
\text{s.t.} \quad & \eta_t \in [\underline{\eta}, \bar{\eta}] \quad \forall t \geq s \\
& f_t \geq 0 \quad \forall t \geq s,
\end{aligned} \tag{13}$$

where $\tau = \inf_{t\geq s}\{t : K_t \leq 0\}$. At $t = \tau$, the fund is terminated and no more signals are generated, so that the manager's reputation will be $\phi_s = \phi_\tau$ for all $s \geq \tau$.

5.1.1 Equilibrium

I solve for a sequential equilibrium of this game. In a sequential equilibrium, individual investors optimally supply capital and take total fund size as given. The manager sets fees and activeness after accounting for investors' willingness to supply capital given the fees and activeness of the fund.

Definition 2 (Public Sequential Equilibrium)**.** *A public sequential equilibrium consists of a set of the following progressively measurable processes with respect to $\mathcal{F}$: a fee process $(f_t^*)_{t\geq 0}$, a process for the assets under management $(K_t^*)_{t\geq 0}$, a process for fund activeness $(\eta_t^*)_{t\geq 0}$, and a reputation process $(\phi_t)_{t\geq 0}$. These processes are such that the following conditions hold:*

(i) *Assets under management coincide with the aggregate supply of capital to the fund, that is,*

$$K_t^* = \int_0^1 k_{it}^* \, di, \quad \forall t \geq 0;$$

(ii) $(k_{it}^*)_{t\geq s}$ *maximizes* (12) *for all investors $i \in [0,1]$ and for all $s \geq 0$;*

(iii) *Given the public strategy profile $(\eta_t^*)_{t\geq 0}$ and an initial prior $\phi_0 \in [0,1]$, managerial reputation $(\phi_t)_{t\geq 0}$ is updated using Bayes' rule and is consistent with the public strategy profile.*

(iv) $(\eta_t^*)_{t\geq s}$ *and* $(f_t^*)_{t\geq s}$ *maximize the manager's lifetime utility* (19) *for any $s \geq 0$, given* (i), (ii) *and* (iii).

In any public sequential equilibrium, investors act myopically because an individual investor cannot influence the aggregate size of the fund or its net performance. Thus, each investor i chooses capital as

$$k_{it}^* \in \arg\max_{k\geq 0} k \left[\sigma \eta_t^* (\phi_t - \hat{\phi}) - \frac{c(K_t^*)}{K_t^*} - f_t^*\right].$$

Therefore, in any equilibrium, we must have $\sigma\eta_t^*(\phi_t - \hat{\phi}) - \frac{c(K_t^*)}{K_t^*} - f_t^* \leq 0$. Because $K_t^* = \int_0^1 k_{it}^* \, di$ and $f_t^* \geq 0$, when $\phi_t > \hat{\phi}$, we must have

$$\sigma\eta_t^*(\phi_t - \hat{\phi}) - \frac{c(K_t^*)}{K_t^*} - f_t^* = 0. \tag{14}$$

When $\phi_t \leq \hat{\phi}$, investors do not provide any capital, and $K_t^* = 0$. Therefore, $\hat{\phi}$ represents the termination threshold in equilibrium.

Moreover, the fund sets its fees myopically to maximize revenues $f_t K_t$, because fees do not affect the evolution of the public signal, which is the gross cash flow. When $\phi_t > \hat{\phi}$, a one-to-one relation exists between fees and size given by (14). In this case, the fund sets fees in order to reach a target size K_t^* that satisfies

$$K_t^* \in \arg\max_{K \geq 0} K\sigma\eta_t^*(\phi_t - \hat{\phi}) - c(K). \tag{15}$$

When $\phi_t \leq \hat{\phi}$, the fund is indifferent between any fee choice, because investors will supply zero capital.

Focusing on the case $\phi_t > \hat{\phi}$, size must satisfy $c'(K_t^*) = \sigma\eta_t(\phi_t - \hat{\phi})$ to solve (15) because $c(\cdot)$ is quadratic. With a quadratic cost function $c(x) = \frac{q}{2}x^2$, optimal profits are a quadratic function of reputation and activeness. In particular, equilibrium profits at time t are $f_t^* K_t^* = \pi(\eta_t^*, \phi_t) := \frac{(\sigma\eta_t^*(\phi_t - \hat{\phi}))^2}{2q}$.

Because managers with different productivity induce different probability distributions over returns, investors exploit the history of returns to learn the type of the manager. According to standard filtering results (Liptser and Shiryaev, 2001, Chapter 9), the belief process $(\phi_t)_{t\geq 0}$ is consistent with the strategy $(\eta_t)_{t\geq 0}$ if, when $K_t > 0$,

$$d\phi_t = (\lambda_0 + \lambda_1)(p - \phi_t)dt + \eta_t\phi_t(1 - \phi_t)\left(d\tilde{R}_t - \eta_t\phi_t dt\right), \tag{16}$$

where

$$d\tilde{R}_t = \frac{dC_t + (\sigma\eta_t\hat{\phi}K_t + c(K_t))dt}{\sigma K_t};$$

whereas when $K_t = 0$, $d\phi_t = 0$.

By changing activeness η_t, the manager also controls the signal-to-noise ratio of returns, which determines the volatility of the manager's reputation and its drift toward the long-run mean p. When activeness is high, cash flow is very informative of the skill of the manager. Investors therefore reevaluate their assessment of the manager's skill by a larger extent for any given performance realization.

When reputation falls below the threshold $\hat{\phi}$, the expected cash flow of the manager is negative for any possible value of K. At that value of reputation, investors do not supply capital to the manager, the fund is terminated, and the manager earns zero profits going forward.

Hence, in equilibrium, fund profits are a function of reputation and activeness only, and the termination threshold is determined by reputation only. Therefore, similar to section 3, in a public sequential equilibrium, the manager's continuation value at time t is equal to $u(\phi_t)$, where u is the unique bounded and continuous solutions of the HJB equation

$$\rho u(\phi) = \max_{\eta \in [\underline{\eta}, \bar{\eta}]} \pi(\eta\phi) + u'(\phi)(\lambda_0 + \lambda_1)(p - \phi) + \frac{1}{2}u''(\phi)\eta^2\phi^2(1-\phi)^2 \tag{17}$$

in the interval $\phi \in [\hat{\phi}, 1]$, with boundary condition $u(\hat{\phi}) = 0$. Moreover, the function u is twice continuously differentiable in $(\hat{\phi}, 1)$. Let $\eta_t^* = \eta(\phi_t)$, where $\eta(\phi)$ is the maximizer of (17). If $(\eta_t^*)_t$ is admissible, the verification result in Lemma A.1 of Appendix A.2 applies because u is bounded and $(\eta_t^*)_t$ is an optimal control.

Hence, I can represent the problem in a recursive form and apply the results of section 3 to show that, near termination, the fund minimizes activeness.

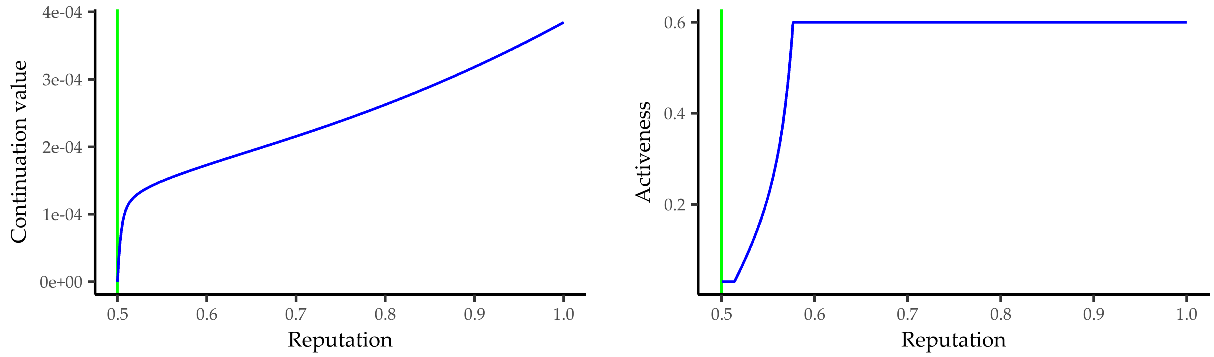
5.1.2 Risk Aversion in Poorly Performing Funds

Given the result in section 5.1.1, this model represents a special case of the general model in section 3. In particular, the model satisfies assumptions (A1), (A2), and (A3). I therefore apply Theorem 1 and obtain the following corollary, whose proof I omit.

Corollary 2. *An $\varepsilon > 0$ exists such that, for all $\phi \in (\hat{\phi}, \hat{\phi} + \varepsilon)$, we have that $u'(\phi) > 0$, $u''(\phi) < 0$ and $\eta(\phi) = \underline{\eta}$.*

Although the manager's payoff is globally convex, a risk-neutral manager acts as a risk-averse decision-maker. When terminated, the manager has to forego future rents. These future rents are reflected in the positive drift of the manager's reputation in the neighborhood of the termination threshold, suggesting the manager is profitable in the long run. When confronted with termination risk, the manager prefers to give up current profits, rather than risk forfeiting his future rents.

Figure 3 plots the value function of the manager and his choice of activeness as functions of the manager's reputation for certain parameters. The manager maximizes activeness for most values of ϕ. However, when ϕ is close to the termination threshold $\hat{\phi}$, the manager minimizes his activeness. Moreover, the value function of the manager is



(a) Manager's continuation value $u(\phi)$ **(b)** Activeness $\eta(\phi)$

Figure 2: Continuation value and activeness as functions of the manager's reputation. The vertical green line marks the threshold at which the manager is terminated. The parameter values are $r = \rho = 5\%$, $\sigma = 10\%$, $\hat{\phi} = 0.5$, $\underline{\eta} = 0.03$, $\bar{\eta} = 0.6$, $q = 1$, $\lambda_0 = 1.5$, $\lambda_1 = 1$

concave in a right neighborhood of the threshold $\hat{\phi}$, but it is convex for higher levels of reputation.

6 Slow Risk Adjustment

In the models I have considered so far, the decision-maker could instantaneously adjust his risk exposure. However, in several situations, a decision-maker may be able to change risk exposure only at a slow rate.

In this section, I consider an extension of the model with slow adjustment in the risk exposure. To preserve tractability, I consider a specific form of slow adjustment, namely, a bounded rate of change in risk exposure. Thanks to this modeling choice, I characterize the solution of the model in terms of a partial differential equation (PDE) whose key properties I establish analytically.

6.1 A General Model with Slow Risk Adjustment

Like in section 3, I consider a state variable evolving according to (1). Now, however, η_t follows the process

$$d\eta_t = i_t dt, \quad \eta_0 = e_0,$$

where $i_t \in [-I, I]$ for all $t \geq 0$ with $0 < I < \infty$. Therefore, the decision-maker can now change η_t only continuously over time, thus preventing discrete changes.

The decision-maker optimally chooses the process $(i_t)_{t\geq 0}$ to maximize his lifetime utility:

$$u(y_0, \eta_0) = \max_{(i_t)_{t \geq 0}} \mathrm{E}\left[\int_0^\theta e^{-\rho t} \pi(y_t, \eta_t)\, dt\right]$$
$$\text{s.t.} \quad i_t \in [-I, I] \quad \forall t \geq 0,$$

where $\theta = \inf\{t \geq 0 : y_t \leq T(\eta_t)\}$ is the stopping time at which the stopper terminates the game. I allow for a general stopping rule $T(\cdot)$, which could depend on the current risk exposure. I impose the restriction that $T(\cdot)$ is continuous and differentiable. By letting the stopping rule depend on the current risk exposure, I allow for cases in which the game stops when y_t falls below a fixed threshold, as in section 3, which correspond to a constant $T(\cdot)$. However, I also allow for cases in which the threshold increases or declines with the current risk exposure. Consider, for example, the model in Section 4.2. In this case, the regulator may take regulatory action earlier if the firm is exposed to more risk and the function $T(\cdot)$ would thus be increasing.

By the results in Pham (2009), the value function u is the unique continuous solution of the following HJB equation:

$$\rho u(y, \eta) = \pi(y, \eta) + u_y(y, \eta)\mu(y, \eta) + \frac{1}{2} u_{yy}(y, \eta)\eta^2\sigma(y)^2 + \max_{i \in [-I, I]} \{i u_\eta(y, \eta)\}, \tag{18}$$

with boundary condition $u(T(\eta), \eta) = 0$ and state constraints $\underline{\eta} \leq \eta \leq \bar{\eta}$. Although one cannot establish the differentiability of u because (18) is no longer elliptic, if a differentiable solution exists, the following theorem characterizes the behavior of the decision-maker.

THEOREM 2. *If $u(y, \eta)$ is twice differentiable in y and once differentiable in η, then:*

(I) *If Assumptions* (A1) *and* (A2) *hold, for any $\eta \in [\underline{\eta}, \bar{\eta}]$ an $\varepsilon > 0$ exists such that $u_y(y, \eta) > 0$ and $u_{yy}(y, \eta) < 0$ for all $y \in (T(\eta), T(\eta) + \varepsilon)$;*

(II) *If $T'(\hat{\eta}) > 0$ for a given $\hat{\eta}$, an $\varepsilon > 0$ exists such that, for any $\eta \in (\hat{\eta} - \varepsilon, \hat{\eta})$, $i(T(\hat{\eta}), \eta) = -I$.*

(III) *If $T'(\hat{\eta}) < 0$ for a given $\hat{\eta}$,an $\varepsilon > 0$ exists such that, for any $\eta \in (\hat{\eta}, \hat{\eta} + \varepsilon)$, $i(T(\hat{\eta}), \eta) = I$.*

Theorem 2 is the counterpart of Theorem 1 when volatility can be modified only slowly. Under identical conditions, the decision-maker displays endogenous risk aversion. However, the decision-maker's choice is primarily driven by the shape of the termination threshold $T(\cdot)$.

When $T'(\hat{\eta}) > 0$, the decision-maker approaches the termination threshold when η marginally increases from $\hat{\eta}$ while keeping y constant. According to Part (II), the decision-maker chooses to reduce risk exposure close to $T(\hat{\eta})$ to distance himself from the termination threshold. Conversely, according to part (III), the decision-maker increases risk exposure if $T'(\hat{\eta}) < 0$ because, by doing so, he moves away from the termination threshold.

When $T'(\hat{\eta}) = 0$, the decision-maker's behavior cannot be unambiguously characterized. Next, I solve an example where T is a constant function, and hence, $T'(\eta) = 0$ for all η. In this case, one can find a set of parameters for which numerical solutions may or may not display risk reduction near the termination threshold.

6.2 FUND MANAGEMENT WITH SLOW PORTFOLIO REALLOCATION

As an illustration of the case of a constant $T(\eta)$, consider the model in section 5.1 and assume the fund's activeness can be adjusted only at a bounded rate. A slow adjustment in activeness represents portfolio adjustment costs that prevent the manager from reallocating money across strategies instantaneously. The objective of the fund manager is

$$\begin{aligned} \max_{(i_t)_{t\geq s},(f_t)_{t\geq s}} \quad & \mathrm{E}\left[\int_s^\tau e^{-\rho(t-s)} f_t K_t \, dt\right] \quad \forall s \geq 0, \\ \text{s.t.} \quad & i_t \in [-I, I] \quad \forall t \geq s \\ & f_t \geq 0 \quad \forall t \geq s, \end{aligned} \tag{19}$$

where $\tau = \inf_{t\geq s}\{t : K_t \leq 0\}$. The definition of a public sequential equilibrium is analogous to Definition 2 but with the portfolio adjustment process $(i_t)_{t\geq 0}$ replacing the activeness process $(\eta_t)_{t\geq 0}$. In particular, fees and fund size are still characterized by (14) and (15), and the evolution of beliefs is still characterized by equation (16) as long as $\phi_t > \hat{\phi}$, with $d\phi_t = 0$ when $\phi_t \leq \hat{\phi}$.

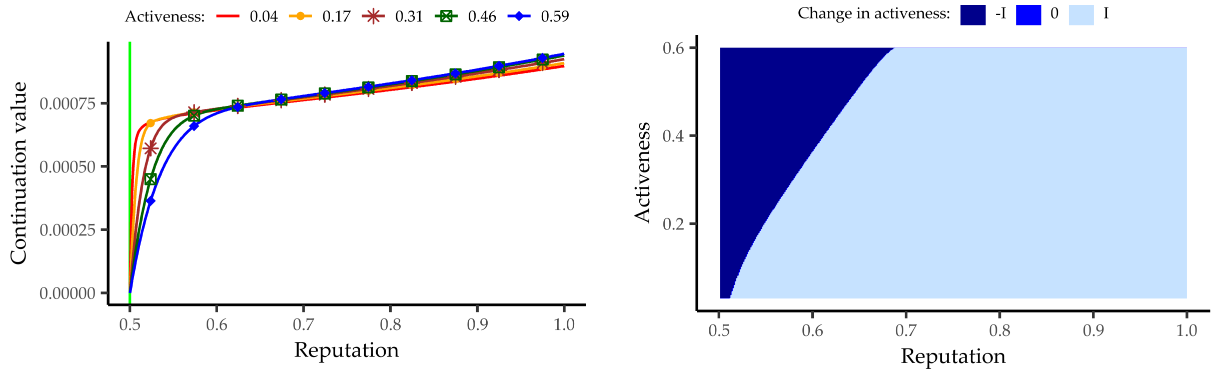
Therefore, the equilibrium value function is still Markovian in beliefs ϕ and activeness η, and characterized by the following HJB equation:

$$\rho u(\phi,\eta) = \pi(\eta\phi) + u_\phi(\phi,\eta)\eta(-\lambda_1\phi + \lambda_0(1-\phi)) + \frac{1}{2}u_{\phi\phi}(\phi,\eta)\eta^2\phi^2(1-\phi)^2 + \max_{i\in[-I,I]}\{iu_\eta(\phi,\eta)\}, \tag{20}$$

with boundary condition $u(T(\eta),\eta) = 0$ and state constraints $\underline{\eta} \leq \eta \leq \bar{\eta}$.

If a solution to this equation ecists that is twice differentiable in y and once differentiable in η, standard arguments similar to those in Lemma A.1 imply a control $(i_t)_{t\geq 0}$ such that $i_t = i(\phi_t,\eta_t) = \arg\max_{i\in[-I,I]}\{iu_\eta(\phi_t,\eta_t)\}$ is an optimal control for the manager.

Instead of plotting the three-dimensional function $u(\phi,\eta)$, in Figure 4(a), I show the manager's continuation value as a function of his reputation for five values of activeness. For this figure, I used the same parameters as in Figure 3 and $I = 1$. Figure 4(b) shows



(a) Manager's continuation value $u(\phi, \eta)$ **(b)** Change in activeness $i(\phi, \eta)$

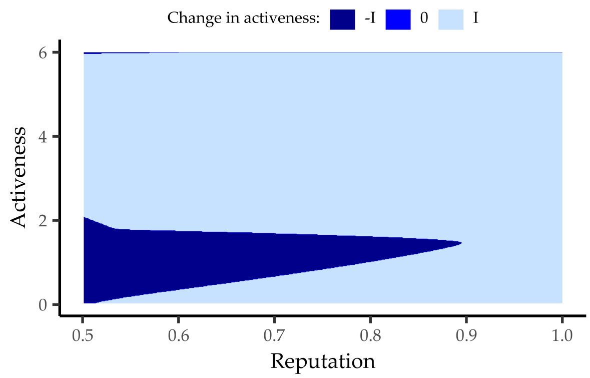
Figure 3: Continuation value and change in activeness as functions of reputation and current activeness. The continuation value is provided as a function of reputation for five distinct levels of activeness. The change in activeness is represented by different colors over the entire state space. The parameter values are $r = \rho = 5\%$, $\sigma = 10\%$, $\hat{\phi} = 0.5$, $\underline{\eta} = 0.03$, $\bar{\eta} = 0.6$, $q = 1$, $\lambda_0 = 1.5$, $\lambda_1 = 1$, and $I = 1$.

the associated optimal control over the entire state space.

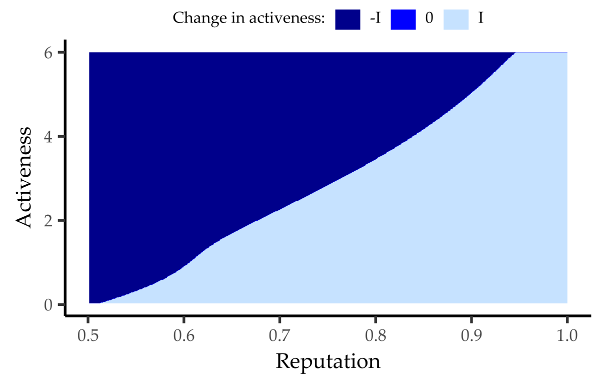
As expected from part (I) of Proposition 2, the value function is increasing and concave in reputation near the termination threshold. Moreover, the numerical results also show the manager reduces risk exposure in the vicinity of the termination threshold. Interestingly, the manager begins reducing risk exposure at larger values of reputation ϕ when the current risk exposure η is higher.

However, the result in Figure 4(b) is not always guaranteed. Figures 5(a) and 5(b) show optimal controls when $\bar{\eta} = 6$, instead of 0.6. In Figure 5(a), the parameter I is set equal to 1, like in Figure 4. All other parameters are unchanged. Although the value function continues to be increasing and concave near termination by Theorem 2(I), for large enough values of η, the manager now does not reduce risk exposure near the termination threshold.

In Figure 5(a), I increase the parameter I to 100 and keep $\bar{\eta} = 6$. In this case, we observe again a reduction in activeness as reputation approaches the termination threshold even if $\bar{\eta}$ is large. Intuitively, if η is very large and adjustment is very slow, the decision may fail to reduce a large risk exposure quickly enough near the termination threshold. As a result, the decision-maker may prefer to enjoy large current payoffs and gamble for resurrection.



(a) Change in activeness $i(\phi, \eta)$ with $I = 1$



(b) Change in activeness $i(\phi, \eta)$ with $I = 10$

Figure 4: Change in activeness as function of reputation and current activeness when $\bar{\eta}$ is large. The change in activeness is represented by different colors over the entire state space. The parameter values are $r = \rho = 5\%$, $\sigma = 10\%$, $\hat{\phi} = 0.5$, $\underline{\eta} = 0.03$, $\bar{\eta} = 6$, $q = 1$, $\lambda_0 = 1.5$, $\lambda_1 = 1$. The parameter I changes between the two figures.

7 Conclusion

In this paper, I provided general conditions under which decision-makers reduce risk exposure when close to termination. I then developed two applied models. In the first model, I considered a leveraged corporation whose equity investors may strategically default. In the second model, I considered a mutual fund manager with unknown productivity and who experiences outflows of funds when his reputation deteriorates.

Because of the separation between managers and investors, investors' termination decisions are not optimal for the manager. To preserve his long-term rents, the manager minimizes risk-taking near the termination threshold. My theory highlights that, in a dynamic model, forward-looking incentives to avoid termination may offset the myopic risk-shifting incentives identified by Jensen and Meckling (1976), thus explaining the mixed empirical evidence on risk-shifting in distressed firms.

A.1 Proofs of the Main Theorems

A.1.1 Proof of Theorem 1

Proof of Theorem 1(i) To begin, note

$$u(y) = \max_{\eta_t \in [\underline{\eta}, \bar{\eta}], \forall t \geq 0} \mathrm{E}\left[\int_0^{\tau} e^{-\rho t} \pi(y_t, \eta_t)\, dt\right] > 0 = u(\hat{y}),$$

where the first inequality follows from Assumption (A1).

Next, I want to show $\liminf_{y\to\hat{y}} u'(y) \geq 0$. Suppose, toward a contradiction, $\liminf_{y\to\hat{y}} u'(y) < 0$. Therefore, in any interval $(\hat{y}, \hat{y}+\varepsilon)$, there exists a sequence $(y_n)_{n=0}^{\infty}$ converging to $\hat{y}$ such that $\lim_{n\to\infty} u'(y_n) < 0$. For each n, define $\varepsilon_n := y_n - \hat{y} > 0$. By a first-order Taylor expansion,

$$u(y_n) = u(\hat{y}) + u'(y_n)\varepsilon_n + o(\varepsilon_n) = \left(u'(y_n) + \frac{o(\varepsilon_n)}{\varepsilon_n}\right)\varepsilon_n.$$

Then, there exists an $\bar{n} > 0$ large enough that $\left(u'(y_{\bar{n}}) + \frac{o(\varepsilon_{\bar{n}})}{\varepsilon_{\bar{n}}}\right)\varepsilon_{\bar{n}} < 0$, which contradicts the fact that $u(y_{\bar{n}})$ must be non-negative.

I then show $\limsup_{y\to\hat{y}} u''(y) \leq 0$. Using the HJB equation (3),

$$(\rho u(y) - \pi(y,\eta) - u'(y)\mu(y,\eta) \geq \frac{1}{2}u''(y)\sigma(y)^2\eta^2, \quad \forall\eta \in [\underline{\eta}, \bar{\eta}]. \tag{A.1}$$

Because $\liminf_{y\to\hat{y}} u'(y) \geq 0$, $\lim_{y\to\hat{y}} u(y) = 0$ by continuity, and $\lim_{y\to\hat{y}} \pi(y,\eta) \geq 0$ by assumption (A2), equation (A.1) implies $\limsup_{y\to\hat{y}} u''(y) \leq 0$.

Next, I show limits exist for the first and second derivatives of u. That is, I show $\liminf_{y\to\hat{y}} u'(y) = \limsup_{y\to\hat{y}} u'(y)$ and $\limsup_{y\to\hat{y}} u''(y) = \liminf_{y\to\hat{y}} u''(y)$. Suppose, by way of contradiction, $\limsup_{y\to\hat{y}} u'(y) > \liminf_{y\to\hat{y}} u'(y)$. Because of the continuity of u', there exist H' and H, with $H' > H$, such that $u'(y) = H$ for infinitely many points y in any neighborhood of $\hat{y}$, and such that $u'(y') = H'$ for infinitely many points y' in any neighborhood of $\hat{y}$. In particular, a sequence $(y_n, y_n')_{n=0}^{\infty}$ exists with $y_n < y_n'$, $u'(y_n) = H$, $u'(y_n') = H'$, $y_n' - y_n \to 0$, and $y_n \to \hat{y}$. Hence, for any M, $n_M > 0$ exists such that

$$\frac{u'(y_n') - u'(y_n)}{y_n' - y_n} > M$$

for all $n > n_M$. By the mean-value theorem, for every $n > n_M$, an $y_n^M \in (y_n, y_n')$ exists such that $u''(y_n^M) > M > 0$. Because $y_n^M \to \hat{y}$ as $n \to \infty$, this conclusion contradicts $\limsup_{y\to\hat{y}} u''(y) \leq 0$. We therefore conclude $\liminf_{y\to\hat{y}} u'(y) = \limsup_{y\to\hat{y}} u'(y)$. This finding, together with equation (A.1), implies also that $\limsup_{y\to\hat{y}} u''(y) = \liminf_{y\to\hat{y}} u''(y)$.

I now prove $\lim u'(y) > 0$ and $\lim u''(y) > 0$. With some abuse of notation, I define $u'(\hat{y}) := \lim_{y\to\hat{y}^+} u'(y)$ and $u''(\hat{y}) := \lim_{y\to\hat{y}^+} u''(y)$.

Case 1. I first consider the case in which $\pi(\hat{y},\eta) > 0$ and $\mu(\hat{y},\eta) \geq 0$ for some η.

By (A.1), we immediately have $\lim_{y\to\hat{y}} u''(y) < 0$. Hence, an $\varepsilon > 0$ exists such that $u''(y) < 0$ for all $y \in (\hat{y}, \hat{y}+\varepsilon)$. Then,

$$u(\hat{y}+\varepsilon) = \int_{\hat{y}}^{\hat{y}+\varepsilon} u'(x)dx = \int_{\hat{y}}^{\hat{y}+\varepsilon} \left(u'(\hat{y}) + \int_{\hat{y}}^{x} u''(z)\,dz\right) dx.$$

If $u'(\hat{y}) = 0$, $u(\hat{y} + \varepsilon) < 0$, which is a contradiction. Hence, $u'(\hat{y}) > 0$.

Case 2. I now consider the case in which $\pi(\hat{y}, \eta) \geq 0$ and $\mu(\hat{y}, \eta) > 0$. By (A.1), $\lim_{y\to\hat{y}} u'(y) = 0 \iff \lim_{y\to\hat{y}} u''(y) = 0$. I proceed by contradiction and assume $\lim_{y\to\hat{y}} u'(y) = 0$.

Let $\omega > 0$ be such that $\rho(y - \hat{y}) < \mu(y, \eta)$ for all $y \in (\hat{y}, \hat{y} + \omega)$. Such ω exists because $\lim_{y\to\hat{y}} \mu(y, \eta) = \mu(\hat{y}, \eta) > 0$. Define the set $C = \{y \in (\hat{y}, \hat{y} + \omega) \colon u'(y) > 0, u''(y) \geq 0\}$. Because $u(y) > 0$ for all $y > \hat{y}$, $u(\hat{y}) = 0$, and $\lim_{y\to\hat{y}} u'(y) = 0$, C is not empty and, for any $\varepsilon > 0$, an $y \in (\hat{y}, \hat{y} + \varepsilon)$ exists such that $y \in C$.

First, assume a $y^C \in C$ exists such that $u'(y^C) \geq \frac{u(y^C)}{y^C - \hat{y}}$. Then,

$$\rho u(y^C) \geq \mu(y^C, \eta) u'(y^C) \implies \rho(y^C - \hat{y}) \geq \mu(y^C, \eta),$$

which contradicts $\rho(y - \hat{y}) < \mu(y, \eta)$ for all $y \in (\hat{y}, y_\omega)$.

Otherwise, assume that for all $y^C \in C$, $u'(y^C) < \frac{u(y^C)}{y^C - \hat{y}}$. Consider $y_0 \in C$. By the mean-value theorem, there exists $y_0' \in [\hat{y}, y_0]$ such that $u'(y_0) < u'(y_0')$. Because $\lim_{y\to\hat{y}} u'(y) = \lim_{y\to\hat{y}} u''(y) = 0$, by the continuity of $u'(y)$ and $u''(y)$, there exists $y_0'' \in C$ such that $u'(y_0'') \geq u'(y_0)$, and $y_0'' < y_0$. Let

$$y_1 \coloneqq \inf\{y'' \leq y_0 \colon y'' \in C, u'(y'') \geq u'(y_0)\}. \tag{A.2}$$

I now show $y_1 = \hat{y}$. I proceed by contradiction and assume $y_1 > \hat{y}$. I want to show $y_1 \in C$. By definition of y_1, either $y_1 \in C$ and $u'(y_1) \geq y'(y_0) > 0$, or there exists a sequence $(y_n'')_{n=0}^\infty$ with $y_n'' \in C$ and $u'(y_n'') \geq u'(y_0)$ for all n with $y_n'' \to y_1$ as $n \to \infty$. In the latter case, because u is twice continuously differentiable, $u'(y_1) = \lim_{n\to\infty} u'(y_n)$, $u''(y_1) = \lim_{n\to\infty} u''(y_n)$. Because $u'(y_n) \geq u'(y_0) > 0$ and $u''(y_n) \geq 0$ for all n, then $u'(y_1) > 0$ and $u''(y_1) \geq 0$; hence, $y_1 \in C$ necessarily. However, because $y_1 \in C$ and $u'(y_1) < \frac{u(y_1)}{y_1 - \hat{y}}$ by assumption, by the previous argument there exists $y_1' < y_1$ with $y_1' \in C$ such that $u'(y_1') > u'(y_1) \geq u'(y_0)$, contraddicting the definition of y_1 in (A.2). Hence, we must have $y_1 = \hat{y}$.

Because $y_1 = \hat{y}$, by definition of y_1, a sequence $(y_n)_{n=0}^\infty \to \hat{y}$ exists with $\lim_{n\to\infty} u'(y_n) \geq u'(y_0) > 0$. This result, however, contradicts $\lim_{y\to\hat{y}} u'(y) = 0$. Therefore, $\lim_{y\to\hat{y}} u'(y) > 0$ and $\lim_{y\to\hat{y}} u''(y) < 0$. □

Proof of Theorem 1(ii) Consider the first derivative of the right-hand side of (3) with respect to η evaluated at an arbitrary $\eta \in [\underline{\eta}, \bar{\eta}]$:

$$D(y, \eta) \equiv \pi_\eta(y, \eta) + u'(y)\mu_\eta(y, \eta) + u''(y)\sigma(y)^2\eta. \tag{A.3}$$

Suppose, by way of contradiction, that $D(y,\eta) \geq 0$ in a neighborhood of $\hat{y}$. Because of (A.3), we have

$$u''(y)\sigma(y)^2\eta \geq -\pi_\eta(y,\eta) - u'(y)\mu_\eta(y,\eta).$$

Substituting this inequality in (3), we obtain

$$\rho u(y) \geq \left(\pi(y,\eta) - \frac{1}{2}\eta\pi_\eta(y,\eta)\right) + u'(y)\left(\mu(y,\eta) - \frac{1}{2}\eta\mu_\eta(y,\eta)\right).$$

Because of Assumption (A3), and because $u(y) \to 0$ as $y \to \hat{y}$, we would conclude $\lim_{y\to\hat{y}} u'(y) \leq 0$, contradicting part (I) of this theorem. Therefore, for y close enough to $\hat{y}$, $D(y,\eta) < 0$ for all $\eta \in [\underline{\eta}, \bar{\eta}]$, and $\eta = \underline{\eta}$ maximizes (3). □

A.1.2 Proof of Theorem 2

Proof of Theorem 2(i). Let $i(y,\eta) := \arg\max_{i\in[-I,I]} iu_\eta(y,\eta)$. Then, $i(y,\eta)u_\eta(y,\eta) \geq 0$ and the proof of part (I) is identical to the proof Theorem 1(I) with $\pi(y,\eta) + i(y,\eta)u_\eta(y,\eta)$ replacing $\pi(y,\eta)$. □

Proof of Theorem 2(ii). First, note $i(y,\eta) = I$ if $u_\eta(y,\eta) > 0$, and $i(y,\eta) = -I$ if $u_\eta(y,\eta) < 0$. To prove part (II) of the Theorem, it is therefore sufficient to prove that if $T'(\hat{\eta}) > 0$ for a given $\hat{\eta}$, an $\varepsilon > 0$ exists such that, for any $\eta \in (\hat{\eta} - \varepsilon, \hat{\eta})$, $u_\eta(T(\hat{\eta}), \eta) < 0$.

Consider $\hat{\eta} \in [\underline{\eta}, \bar{\eta}]$ with $T'(\hat{\eta}) > 0$, and a sequence $(\eta_n)_{n=0}^\infty$ with $\eta_n < \hat{\eta}$ for all n and $\eta_n \to \hat{\eta}$ as $n \to \infty$. Note

$$\frac{u(T(\hat{\eta}),\hat{\eta}) - u(T(\hat{\eta}),\eta_n)}{\hat{\eta} - \eta_n} = -\frac{u(T(\hat{\eta}),\eta_n) - u(T(\eta_n),\eta_n)}{T(\hat{\eta}) - T(\eta_n)}\frac{T(\hat{\eta}) - T(\eta_n)}{\hat{\eta} - \eta_n}.$$

Using a first-order Taylor expansion, we have

$$\frac{u(T(\hat{\eta}),\hat{\eta}) - u(T(\hat{\eta}),\eta_n)}{\hat{\eta} - \eta_n} = u_\eta(T(\hat{\eta}),\eta_n) + \frac{o(|\hat{\eta} - \eta_n|)}{\hat{\eta} - \eta_n}$$

and

$$\frac{u(T(\hat{\eta}),\eta_n) - u(T(\eta_n),\eta_n)}{T(\hat{\eta}) - T(\eta_n)} = u_y(T(\eta_n),\eta_n) + \frac{o(|T(\hat{\eta}) - T(\eta_n)|)}{T(\hat{\eta}) - T(\eta_n)}.$$

Substituting the latter two expressions and taking the limit, I obtain

$$\limsup_{n\to\infty} u_\eta(T(\hat{\eta}),\eta_n) = -T'(\hat{\eta}) \limsup_{n\to\infty} u_y(T(\eta_n),\eta_n).$$

I then show $\limsup_{n\to\infty} u_y(T(\eta_n),\eta_n) = u_y(T(\hat\eta),\hat\eta)$. Suppose, by way of contradiction, that

$$\limsup_{n\to\infty} |u_y(T(\eta_n),\eta_n) - u_y(T(\hat\eta),\hat\eta)| > D$$

for some $D > 0$. Then, there exists a $\delta > 0$ such that, for all $\delta' \in (0,\delta)$,

$$\limsup_{n\to\infty} \left| \frac{u(T(\eta_n)+\delta',\eta_n) - u(T(\hat\eta)+\delta',\hat\eta)}{\delta'} \right| > D,$$

which implies

$$\limsup_{n\to\infty} |u(T(\eta_n)+\delta',\eta_n) - u(T(\hat\eta)+\delta',\hat\eta)| > D\delta',$$

contradicting that $u(y,\eta)$ is a continuous function.

Hence, we have

$$\limsup_{n\to\infty} u_\eta(T(\hat\eta),\eta_n) = -T'(\hat\eta)u_y(T(\hat\eta),\hat\eta) < 0,$$

where the inequality follows from the assumption that $T'(\eta) > 0$ and from part (I) of the theorem.

Because the sequence $(\eta_n)_{n=0}^\infty$ is arbitrary, we conclude an $\varepsilon > 0$ exists such that, for all $\eta \in (\hat\eta - \varepsilon, \hat\eta)$, $u_\eta(T(\hat\eta),\eta) < 0$. □

PROOF OF THEOREM 2(III). The proof of this part of the Theorem is analogous to part (II). In this case, one must show $\liminf_{n\to\infty} u_\eta(T(\hat\eta),\eta_n) > 0$. To do so, consider a sequence $(\eta_n)_{n=0}^\infty$ with $\eta_n > \hat\eta$ for all n and $\eta_n \to \hat\eta$ as $n \to \infty$ and note $T'(\eta) < 0$. The remaining steps of the proof are identical. □

A.2 AUXILIARY RESULTS

LEMMA A.1. *Let $(\eta(V_t))_{t\geq 0}$ be admissible and let $u(\cdot)$ be a twice-differentiable solution of* (3). *If $\lim_{t\to\infty} \mathrm{E}[u(y_t)e^{-\rho t}|\mathcal{F}_0] = 0$, $u(\cdot)$ is the decision-maker's value function and $(\eta(y_t))_{t\geq 0}$ is optimal for the decision-maker.*

Proof. Consider a localizing sequence of stopping times $(\tau_n)_{n=0}^\infty$ such that $\tau_n \to \infty$ as $n \to \infty$. Then, for any arbitrary admissible strategy $(\eta_t)_{t\geq 0}$ such that $\eta_t \in [\underline{\eta},\bar\eta]$, by the Dynkin's formula (Øksendal, 2003, Chapter 7.4),

$$E[e^{-\rho\tau_n}u(y_{\tau_n})|\mathcal{F}_0] - u(y_0) = \mathrm{E}\left[\int_0^{\tau_n} e^{-\rho t}\left\{u'(y_t)\mu(y_t,\eta_t) + \frac{1}{2}u''(y_t)\eta_t^2\sigma(y_t)^2 - \rho u(y_t)\right\} dt \Big| \mathcal{F}_0\right]$$

$$\leq -\mathrm{E}\left[\int_0^{\tau_n} e^{-\rho t}\pi(y_t,\eta_t)\,dt\Big|\mathcal{F}_0\right],$$

with equality if $\eta_t = \eta(y_t)$.

By assumption $E[e^{-\rho\tau_n}u(y_{\tau_n})|\mathcal{F}_0] \to 0$ as $n \to 0$. Taking the limit and using the dominated convergence theorem, I obtain $u(y_0) \geq \mathrm{E}\left[\int_0^\infty e^{-\rho t}\pi(y_t,\eta_t)\,dt\big|\mathcal{F}_0\right]$, with equality if $\eta_t = \eta(y_t)$. Hence, $(\eta(y_t))_{t\geq 0}$ must be an optimal control and

$$u(y_0) = \mathrm{E}\left[\int_0^\infty e^{-\rho t}\pi(y_t,\eta(y_t))\,dt\Big|\mathcal{F}_0\right].$$

□

REMARK 1. If a function $u(\cdot)$ satisfies linear growth, it also satisfies $\lim_{t\to\infty}\mathrm{E}[u(y_t)e^{-\rho t}|\mathcal{F}_0] = 0$. In fact,

$$0 \leq \left|\lim_{t\to\infty}\mathrm{E}[u(y_t)e^{-\rho t}|\mathcal{F}_0]\right| \leq \lim_{t\to\infty}\mathrm{E}[|u(y_t)|e^{-\rho t}|\mathcal{F}_0] \leq \lim_{t\to\infty}\mathrm{E}[e^{-\rho t}(C_0\mu + C_1^\mu|y_t|)|\mathcal{F}_0] = 0,$$

where the last equality follows from C_0^μ and C_1^μ being constants and from Lemma 1 in Strulovici and Szydlowski (2015), which applies because of Assumption (R2).

LEMMA A.2. *Consider the model of section 4.1. In any Markov-perfect equilibrium, a $\hat{V} > 0$ exists such that $\mathcal{D} = \{V \in \mathbb{R}^+ : V > \hat{V}\}$, and hence, $\tau^* = \inf\{t \geq 0 : V_t \leq \hat{V}\}$.*

Proof. It is sufficient to prove the complement of $\mathcal{D}$, which I denote as $\mathcal{D}^C$, is a non-empty, bounded, and closed interval in the form $[0, \hat{V}]$.

First, I prove $\mathcal{D}^C$ is non-empty. I proceed by contradiction. Suppose $\mathcal{D}^C$ is empty; then, E coincides with the equity value when $\tau = \infty$, which I denote by $\tilde{E}(V)$. Note

$$\tilde{E}(V) \leq \max_{(\eta_t)_{t\geq 0}} E\left[\int_0^\infty e^{-rt}[(1-\theta)\delta V_t - c]\,dt\Big|\mathcal{F}_0\right] = \frac{(1-\theta)\delta}{r-\bar{\eta}\mu}V - \frac{c}{r},$$

and therefore, there exists a $\tilde{V} > 0$ such that $\tilde{E}(V) < 0$ for all $V \in [0, \tilde{V})$. But this finding contradicts that $\tau = \infty$ is optimal for the equity holders. Therefore, $\mathcal{D}^C$ is not empty.

Next, I show $\mathcal{D}^C$ is bounded. In particular, I show $\mathcal{D}^C$ is a subset of the bounded interval $\mathcal{N} := \{V \in \mathbb{R}^+ : (1-\theta)\delta V - c \leq 0\} = \left[0, \frac{c}{(1-\theta)\delta}\right]$. To show $\mathcal{D}^C \subseteq \mathcal{N}$, I proceed by contradiction. In particular, suppose there exists $V \in \mathcal{D}^C$ such that $(1-\theta)\delta V - c > 0$. Because $V \in \mathcal{D}^C$, $E(V) = 0$. Given $V_0 = V$, consider a stopping time $\bar{\tau} = \inf\{t \geq 0 : (1-\theta)\delta V_t - c \leq 0\}$. Then, $\mathrm{E}\left[\int_s^{\bar{\tau}} e^{-rt-s}[(1-\theta)\delta V_t - c]\,dt\right] > 0 = E(V)$. But this result contradicts $\tau = 0$ being optimal when $V_0 = V$.[12] Therefore, $\mathcal{D}^C$ must be a subset of $\mathcal{N}$.

[12] Note that, by definition of a Markov equilibrium, $\tau = 0$ is optimal for equity holders when $V_0 \in \mathcal{D}^C$.

Then, I show $\mathcal{D}^C$ is an interval. By way of contradiction, assume there exist two sets D_1 and D_2, subsets of $\mathcal{D}^C$, with $D_1 \cap D_2 = \emptyset$, $V^1 = \sup D_1 < \inf D_2 = V^2$ and such that $[V^1, V^2] \cap \mathcal{D} \neq \emptyset$. Then, for any $V \in [V^1, V^2] \cap \mathcal{D}$, we must have $E(V) > 0$. With $V_0 = V$, define $\tau' := \inf\{t \geq 0 \colon V_t \in D_1 \cup D_2\}$. By the dynamic programming principle,

$$E(V) = \mathrm{E}\left[\int_0^{\tau'} e^{-rt}((1-\theta)\delta V_t - c)\,dt + e^{-r\tilde{\tau}}E(V_{\tilde{\tau}})\middle|\mathcal{F}_0\right].$$

By the definition of τ', we have that $E(V_{\tau'}) = 0$. Moreover, because $\mathcal{D}^C \subseteq \mathcal{N}$, $(1-\theta)\delta V - c \leq 0$ for all $V \leq V^2$, and the integral in the previous expression is (weakly) negative. This result would imply $E(V) \leq 0$, thus contradicting that $V \in \mathcal{D}$. Therefore, we conclude $\mathcal{D}^C$ is an interval.

Finally, I show $\mathcal{D}^C$ is closed. First, notice $0 \in \mathcal{D}^C$. I then need to show $\hat{V} := \sup \mathcal{D}^C \in \mathcal{D}^C$. Let $\tau_V^0 := \inf\{t \geq 0 : V_t \in \mathcal{D}^C, V_0 = V\}$. By the Blumenthal zero-one law (Karatzas and Shreve, 1998, Chapter 2.7.C), either $P(\tau_{\tilde{V}}^0 = 0|V_0 = \tilde{V}) = 0$ or $P(\tau_{\tilde{V}}^0 = 0|V_0 = \tilde{V}) = 1$. By symmetry, the first case is impossible. Therefore, $P(\tau_{\tilde{V}}^0 = 0|V_0 = \tilde{V}) = 1$ and $E(\tilde{V}) = 0$.

□

LEMMA A.3. *Consider the model of section 4.1. Let $\eta(V)$ be the maximizer in* (8). *Then, $\eta(\cdot)$ is a continuous function for $V > \hat{V}$.*

Proof. By the maximum theorem, the set of maximizers of (8) is an upper-hemicontinuous correspondence in V. I therefore need to show it is single-valued. For $V > \hat{V}$, there are two cases in which (8) has multiple maximizers: (i) $u''(V) = 0$ and $u'(V)\mu = 0$, or (ii) $u''(V) > 0$ and $u'(V)\mu(\bar{\eta} - \underline{\eta}) + \frac{1}{2}u''(V)s^2V(\bar{\eta}^2 - \underline{\eta}^2) = 0$.

I rule out these two cases by showing u is strictly increasing for $V \geq \hat{V}$ and that, if $\mu = 0$, u is strictly concave for $V \geq \hat{V}$. If u is strictly increasing, the we can rule out both cases when $\mu > 0$. When $\mu = 0$, the strict concavity of u rules out both cases as well.

To show that u is strictly increasing, consider $V_0^1 > V_0^0 \geq \hat{V}$. For $i \in \{0, 1\}$, let

$$\ln V_t^{i,\eta} = \ln V_0^i + \int_0^t \left(\mu\eta_t - \frac{1}{2}\eta_t^2\right)\,dt + \int_0^t \eta_t\,dZ_t, \tag{A.4}$$

where I set $\eta_t = \eta\left(e^{\ln V_t^{0,\eta}}\right) = \eta\left(e^{\ln V_t^{1,\eta} + \ln V_0^0 - \ln V_0^1}\right) = \eta\left(V_t^{1,\eta}\frac{V_0^0}{V_0^1}\right)$, which is a Markovian control also for $V_0 = V_0^1$. In particular, it coincides with the manager's optimal control when $V_0 = V_0^0$ for any given realized path $(Z_u)_{0 \leq u \leq t}$ of the Brownian motion.

Let $\tau^0 = \inf\{t \geq 0 : V_t^{0,\eta} \leq \hat{V}\}$. Note $V_{\tau^0}^{1,\eta} > \hat{V}$. Then,

$$u(V_0^1) \geq \mathrm{E}\left[\int_0^{\tau_0} e^{-\rho t}\theta\delta V_t^{1,\eta}\,dt + e^{-\rho\tau^0}u(V_{\tau^0}^{1,\eta})\right] \geq u(V_0^0) + \mathrm{E}\left[e^{-\rho\tau^0}u(V_{\tau^0}^1)\right] > u(V_0^0). \quad \text{(A.5)}$$

Hence, u is strictly increasing for $V \geq \hat{V}$.

Next, I prove strict concavity when $\mu = 0$. First, note that, from (8), we have $\rho u(V) - \theta\delta V \geq \frac{1}{2}u''(V)V^2\eta^2$ for any $\eta \in [\underline{\eta}, \bar{\eta}]$. Next, note

$$u(V) < \tilde{u}(V) = \max_{(\eta_t)_{t\geq s}} \mathrm{E}\left[\int_s^\infty e^{-\rho(t-s)}\theta\delta V_t\,dt\right] = \frac{\theta\delta V}{\rho},$$

where the first inequality follows because $\theta\delta V_t > 0$ and $\tau \coloneqq \inf\{t \geq 0\colon V_t \leq \hat{V}\} < \infty$. The equality follows because $\theta\delta V/\rho$ is a solution of (8) with $\hat{V} = 0$ and, by Pham (2009) and Strulovici and Szydlowski (2015), it is the unique solution of (8). We therefore conclude $\rho u(V) - \theta\delta V < 0$, and hence, $u''(V) < 0$ for any $V \geq \hat{V}$.

In conclusion, the maximizer of (8) is a continuous function of V. □

Lemma A.4. *Consider the model of section 4.1. An $\varepsilon > 0$ exists such that E is twice differentiable in $(\hat{V}, \hat{V} + \varepsilon)$ with $E''(V) < 0$.*

Proof. I begin by showing E is twice differentiable in a right neighborhood of $\hat{V}$. By corollary 1 and by the restriction that $\eta(V') = \lim_{V\to V^+}\eta(V)$ for $V' \leq \hat{V}$, there exists $\varepsilon' > 0$ such that $\eta(V) = \underline{\eta}$ for $V < \hat{V} + \varepsilon'$. For a $0 < \varepsilon_0 < \varepsilon_1 < \varepsilon'$, consider the following variational inequality for $V \in (\hat{V} + \varepsilon_0, \hat{V} + \varepsilon_1)$:

$$re(V) - H(V, e'(V), e''(V)) = 0,$$

with boundary conditions $e(\hat{V} + \varepsilon_0) = E(\hat{V} + \varepsilon_0)$ and $e(\hat{V} + \varepsilon_1) = E(\hat{V} + \varepsilon_1)$. It can be immediately verified that $e = E$ is a solution of this variational inequality and that, by the usual arguments in Pham (2009), it is the unique continuous solution.

Because $\eta(V) = \underline{\eta}$ for $V \leq V_1$, the functions $(1-\theta)\delta V - c$, $\mu\eta(V)V$, and $\eta(V)\sigma V$ are all twice differentiable in V and bounded in the interval $(\hat{V} + \varepsilon_0, \hat{V} + \varepsilon_1)$. Classical results (Fleming and Soner, 2006, Ch. IV.4) imply e is twice differentiable in $(\hat{V} + \varepsilon_0, \hat{V} + \varepsilon_1)$. Because $\varepsilon_0 > 0$ can be arbitrarily small, we conclude e, and hence E, is twice differentiable in $(\hat{V}, \hat{V} + \varepsilon_1)$.

Next, I show $E''(V) > 0$ in a right neighborhood of $\hat{V}$. By the smooth-fit principle

(Pham, 2009), $E(V) = E'(\tilde{V}) = 0$ and $E(V) \to 0$ and $E'(V) \to 0$ as $V \to \tilde{V}^{+}$. By (9),

$$rE(V) - [(1-\theta)\delta V - c] - E'(V)\mu\eta(V)V = +\frac{1}{2}E''(V)\eta(V)^2V^2,$$

and hence,

$$\lim_{V \to \hat{V}^{+}} \frac{1}{2}E''(V)\eta(V)^2V^2 = \lim_{V \to \hat{V}^{+}} -[(1-\theta)\delta V - c].$$

It therefore suffices to show $[(1-\theta)\delta\hat{V} - c] < 0$. To show this, note

$$E(V) > \min_{(\eta_t)_{t\geq 0}} \mathrm{E}\left[\int_0^{\infty} e^{-rt}[(1-\theta)\delta V_t - c]\,dt\right] \quad \text{s.t. } \eta_t \in [\underline{\eta}, \bar{\eta}],\ \forall t \geq 0,$$

where the strict inequality follows because $\mathcal{D}^C$ is non-empty, and hence, $\tau^* < \infty$, and where the right-hand side of this expression is equal to $\frac{(1-\theta)\delta}{r-\mu\underline{\eta}}V - \frac{c}{r}$. For $V = \hat{V}$, it follows that $0 > \frac{(1-\theta)\delta}{r-\mu\underline{\eta}}\hat{V} - \frac{c}{r}$, and hence, $(1-\theta)\delta\tilde{V} - c < -c\mu\underline{\eta}/r \leq 0$. We therefore conclude $E''(V) > 0$ in a right neighborhood of $\hat{V}$. □

LEMMA A.5. *Consider the model of section* **??**. *The value function of an unbiased manager is* $u_u(\phi) = \frac{\beta\sigma\bar{\eta}(\phi-\hat{\phi})}{2\rho}$. *Moreover,* $\eta_t = \bar{\eta}$ *for all* t *maximizes* (**??**) *for an unbiased manager.*

Proof. By Pham (2009) and Strulovici and Szydlowski (2015), u_u is the unique bounded (classical) solution of the following HJB equation:

$$\rho u_u(\phi) = \max_{\eta \in [\underline{\eta}, \bar{\eta}]} \beta\frac{\sigma\eta(\phi-\hat{\phi})}{2} + \frac{1}{2}u_u''(\phi)\eta^2\phi^2(1-\phi)^2. \tag{A.6}$$

One can verify $\frac{\beta\sigma\bar{\eta}(\phi-\hat{\phi})}{2\rho}$ is a solution of the HJB equation. By uniqueness, $u_u(\phi) = \frac{\beta\sigma\bar{\eta}(\phi-\hat{\phi})}{2\rho}$.

Therefore, the maximizer in (A.6) is constant in ϕ and $\eta_u(\phi) = \bar{\eta}$. Because the process $(\eta_u(\phi_t))_{t\geq 0}$ is constant, $(\eta_u(\phi_t))_{t\geq 0}$ is admissible. By Lemma A.1, $\eta_t = \bar{\eta}$ for all t maximizes (**??**) for the unbiased manager. □

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