Wealth Inequality, Aggregate Risk, and the Equity Term Structure

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Abstract

This paper studies the feedback between stock market fluctuations and wealth inequality dynamics. We do so by means of a dynamic consumption-based general equilibrium model with endogenous asset returns and a non-degenerate wealth distribution for a continuum of households. Households are heterogeneous in risk aversion and thus choose different expected portfolio returns and portfolio return volatilities, generating time-varying wealth inequality. We show how to solve the model analytically in terms of a cumulant generating function, which encodes information about all the moments of the distribution of risk aversion. With this result, we recover the unobservable distribution of risk aversion using time variation in the slope of the observable equity term structure. We also confront the model with US data on the wealth distribution to recover a second estimate of the distribution of risk aversion. By comparing the two estimates, we show quantitatively that there is significant feedback between stock price dynamics and wealth distribution dynamics.

Keywords: wealth inequality, aggregate risk, equity term structure.

Contents

1 Introduction

The wealth distribution changes over time and impacts asset price dynamics through the stochastic discount factor. Moreover, the dynamic consumption-portfolio choices made by households based on asset prices impact the dynamics of the wealth distribution. In short, there is a natural feedback between wealth inequality and asset prices. Recent work in macroeconomics (see Benhabib and Bisin, 2018, Fagereng, Holm, et al., 2019, Gabaix et al., 2016, Hubmer et al., 2021, Xavier, 2020) emphasizes the necessity of heterogeneity in rates of return for explaining wealth inequality dynamics. However, none of these papers endogenizes rates of return, making it impossible to study the two way feedback between wealth inequality and asset prices. The asset pricing literature (see e.g. Dumas, 1989, Chan and Kogan, 2002, Bhamra and Uppal, 2009, Bhamra and Uppal, 2014, Gârleanu and Panageas, 2015) has studied theoretically the importance of heterogeneity in individual's risky portfolio returns for aggregate stock return dynamics, but rarely considers the implications for wealth inequality.

In this paper, we develop a theoretical framework to understand the feedback between endogenous stock market fluctuations and the endogenous dynamics of the wealth distribution. Central to our model is heterogeneity in exposure to aggregate consumption risk driven by heterogeneity in risk aversion for a continuum of households in a dynamic setting.[1] Consequently, households have different expected portfolio returns and portfolio return volatilities, leading to time-varying wealth inequality. Empirical work by Bach et al. (2020) stresses the relevance of this channel for generating dispersion in wealth across households. The distribution of risk aversion across agents drives their distribution of exposure to aggregate consumption risk and hence both the dynamics of asset prices and wealth inequality. We solve the model analytically for an arbitrary distribution of risk aversion by obtaining the distribution of consumption across households in terms of a cumulant generating function.

[1]We differ from Gollier (2001), who uses a static two-date framework, where individuals have different wealth levels, but identical risk preferences. Gomez (2016) studies wealth inequality in a dynamic model, but with two types of agent who differ in risk aversion; in our model risk aversion varies smoothly across a continuum of agents, giving rise to a continuous distribution.

Previous research in finance documents significant time variation in the slope of the dividend-yield term structure.[2] We show that the correlation of this slope with the growth rate of the stock holdings of the top 1% of individual investors in the US is 34% (see Figure 1). Using dividend yield data from Giglio et al. (2021) and the closed-form expression for the dividend yield slope from our model, we recover the distribution of risk aversion and the wealth distribution.[3] In contrast with expected risk premia, which are estimated with considerable error, dividend yields can be observed. We also use the individual US wealth series from Saez and Zucman (2016) to build empirical estimates of the distribution of financial wealth and the share of wealth invested in risky assets to recover the risk aversion distribution from wealth dynamics. The estimates of the risk aversion distribution based on wealth dynamics overlap considerably with the estimate from asset price dynamics. Our results thus provide evidence for significant feedback between asset price dynamics and wealth distribution dynamics.

The model we use to confront the data is a consumption-based asset pricing model with a continuum of households and dynamically complete financial markets. Individual households have power utility which loads multiplicatively on the standard of living and depends exogenously on the time-weighted history of aggregate consumption.[4] Households are heterogeneous with respect to their risk aversion and their sensitivity to the standard of living index, but have identical rates of time preference. The key determinants of both the dynamics of stock prices and the wealth distribution are the risk aversion distribution summarized by the social planner's Pareto weights together with the model's state variable: the difference between log aggregate consumption and its history, which we refer to as the consumption surplus. By assuming that aggregate consumption is log-normal, we use consumption data to extract the history of consumption shocks, which determines the path of the consumption surplus. We lack data on individual household consumption,

[2]Van Binsbergen, Brandt, et al. (2012) are the first to use dividend strips to study the term structure of equity and Van Binsbergen, Hueskes, et al. (2013) use dividend strips to study the term structure of expected dividend growth and the dividend yield slope. Gormsen (2021) and Bansal et al. (2021) have recently shown that this slope varies across time.

[3]The prior literature has not shown how to recover the wealth distribution solely from asset prices. There is a literature focusing on recovering the physical distribution of market returns from derivative prices without using a specific model, initiated by Ross (2015) and developed further by Borovička et al. (2016), Schneider and Trojani (2019). Our approach uses a specific model to recover the wealth distribution from time variation in the dividend-yield term structure.

[4]The individual household utility specification is a generalization of Chan and Kogan (2002) developed in Muraviev (2013).

so we cannot use it to determine the density for the social planner's Pareto weights. Instead, we recover the density in two distinct ways: (i) by using times series data for the dividend-yield term structure, and (ii) using household wealth shares.

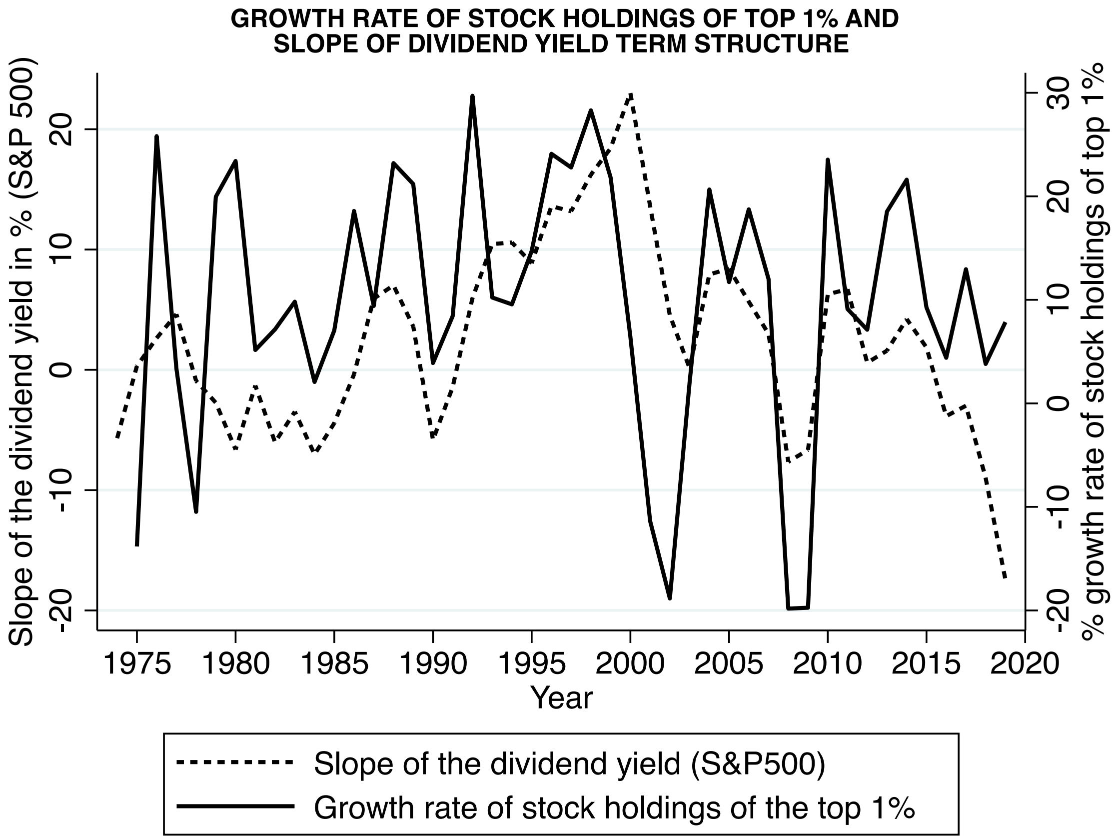

Figure 1: Growth rate of stock holdings of top 1% and slope of dividend yield term structure

Notes: This figure shows the relationship between the growth rate of stock holdings of the top 1% in the US and the slope of the dividend-yield term structure of the S&P 500 over the period 1974-2019. The annual growth rate of stock holdings of the top 1% has been calculated using the microfiles of Saez and Zucman (2016) (`https://gabriel-zucman.eu/usdina/`) and ranking the US population by their level of stock holdings. The slope of the dividend yield term structure data has been computed taking the annual average of the monthly series of dividend yields (`https://www.serhiykozak.com/data`) estimated by Giglio et al. (2021). The slope is the difference between the 15 year dividend strip's annualized yield and the 1 year dividend strip's annualized yield for the "sizeS" cross-sectional portfolio, which corresponds closely to the S&P 500. Both series are shown in units of percentages per annum. The correlation coefficient between these two series is 34%.

We can obtain an estimate for the social planner's Pareto-weights using dividend yield data by using two theoretical results. First, we show how the functional dependence of aggregate risk aversion on the consumption surplus pins down the cumulant generating function of a transformation of the density function of the Pareto weights. Second, we exploit techniques from operator theory (see e.g. Hansen and Scheinkman, 2008 and Hansen, 2012) to show that the slope of the dividend yield is directly proportional to the consumption-share weighted aggregate risk aversion in the economy. By using time series data on the dividend-yield term structure, we can estimate the functional dependence of aggregate risk aversion on the consumption surplus. We can therefore identify the cumulant generating function of a transformation of the density function of the Pareto weights. We turn the empirically estimated cumulant generating function into a moment generating function and hence use an inverse Laplace transform to recover the density function for the social planner's Pareto-weights in closed form.[5]

A second set of estimates for the density function for the social planner's Pareto-weights come from matching the model-implied wealth distribution to the empirical one. When we compare the dividend-yield implied density function with the wealth-share implied density functions, we find considerable overlap and quantify its extent via their relative entropy.

This paper is related to a growing theoretical and empirical literature examining the impact of wealth inequality on asset prices. Gollier (2001) examines how wealth inequality affects the equilibrium level of the equity premium and the risk-free rate. Gomez (2016) studies the interplay between wealth inequality and asset prices both empirically and theoretically using the series of top wealth shares from Kopczuk and Saez (2004). Eisfeldt et al. (2013) examine the joint relation between the wealth distribution and asset prices across markets with different expertise and Kacperczyk et al. (2019) also study the role of investor sophistication in explaining the recent rise in capital income inequality. Recent theoretical studies emphasize the need of heterogeneity in rates of return for explaining wealth inequality dynamics (see Benhabib and Bisin, 2018, Fagereng, Holm, et al., 2019, Gabaix et al., 2016, Hubmer et al., 2021, Xavier, 2020) in light of the recent empirical work showing that these are indeed features of the data (see Bach et al., 2020, Fagereng, Guiso, et al., 2020, Kuhn et al., 2020, Martínez-Toledano, 2020). We build upon these partial equilibrium models

[5]A similar technique is used in astrophysics to infer the density of dark matter (matter which cannot be directly observed), see e.g. Bernardeau and Kofman (1995).

and empirical findings and develop for the first time a general equilibrium framework to study the two-way feedback between endogenous wealth inequality and endogenous asset price dynamics.

Our work is also closely related to the asset pricing literature studying the importance of heterogeneity in individual's risky portfolio returns for aggregate stock return dynamics (Dumas, 1989, Chan and Kogan, 2002, Bhamra and Uppal, 2009, Bhamra and Uppal, 2014, Gârleanu and Panageas, 2015) and the literature relying on dividend yield data to better understand asset prices (Van Binsbergen, Brandt, et al., 2012, Van Binsbergen, Brandt, et al., 2012, Giglio et al., 2021, Gormsen, 2021, Bansal et al., 2021). A novel feature of our approach is the use of the dividend yield to extract the dynamics of the wealth distribution. Taken together, our theoretical and quantitative results reveal that the current macroeconomics and inequality literatures could benefit by incorporating insights from asset pricing.

2 Model

We work in an infinite horizon, continuous-time, dynamic general equilibrium economy with aggregate risk and heterogeneous agents. We now describe how we model the dynamics of aggregate consumption risk and the heterogeneous exposure of agents to this risk.

2.1 The Dynamics of Aggregate Consumption Risk

We assume aggregate consumption flow is exogenous and stochastic. The aggregate consumption flow at time-t is denoted by Y_t and we assume that

$$\frac{dY_t}{Y_t} = \mu_Y dt + \sigma dZ_t, \tag{1}$$

where $Z = (Z_t)_{t\in[0,\infty)}$ is a standard Brownian motion under the physical probability measure $\mathbb{P}$. The time-t expected consumption growth rate is denoted by μ_Y and is constant. The volatility of consumption growth is denoted by σ and is constant.

The standard of living index at time-t is denoted by X_t, where $x_t = \ln X_t$ is the mean of past realizations of the logarithm of the aggregate consumption process, weighted exponentially, so that more recent realizations have a greater impact on the current standard of living, i.e.

$$x_t = x_0 e^{-\lambda t} + \lambda \int_0^t e^{-\lambda(t-u)} y_u \, du, \tag{2}$$

where $\lambda > 0$.

We define the logarithm of the ratio of current aggregate consumption to the standard of living index as the consumption surplus

$$\omega_t \equiv y_t - x_t.$$

A positive consumption surplus reflects an improvement in current economic conditions relative to the past and increases the standard of living index – observe that (2) implies $dx_t = \lambda(y_t - x_t)dt$, so that $dx_t > 0$ when $\omega_t > 0$. The consumption surplus is mean-reverting and its dynamics are given by

$$d\omega_t = \lambda \left(\lambda^{-1} \mu_y - \omega_t \right) dt + \sigma dZ_t, \tag{3}$$

where $\mu_y = \mu_Y - \frac{1}{2}\sigma_Y^2$ and the long-run mean of the surplus is $\overline{\omega} = \mu_y / \lambda$. To understand the role played by the mean-reversion parameter λ, we observe that

$$\omega_{t+u} = e^{-\lambda u} \omega_t + (1 - e^{-\lambda u}) \frac{\mu_y}{\lambda} + \sigma \int_t^{t+u} e^{-\lambda(s-t)} dZ_s,$$

which makes is clear that $e^{-\lambda u}$ is the persistence of ω over a time period of length u (measured in units of years) and $\lim_{u \to \infty} E_t[\omega_{t+u}] = \frac{\mu_Y - \frac{1}{2}\sigma^2}{\lambda}$ is the long-run mean of ω. The half-life of ω is given by $t_{1/2} = \ln 2 / \lambda$ Increasing λ decreases both the persistence of ω and its long-run mean. For example, with a value of λ equal to 1.29 (per annum), the annualized persistence of ω is 0.28, the half life is 0.54 years, and with $\mu_y = 3.11\%$ per annum the long-run mean is $\mu_y / 1.29 = .024$. Figure 2.1 shows the time series for US log aggregate consumption, the implied times series for the log standard of living index and the consumption surplus relative to its long-run mean. From Panel B, we can see that the consumption surplus tends to fall below its long-run mean around and during NBER recessions.

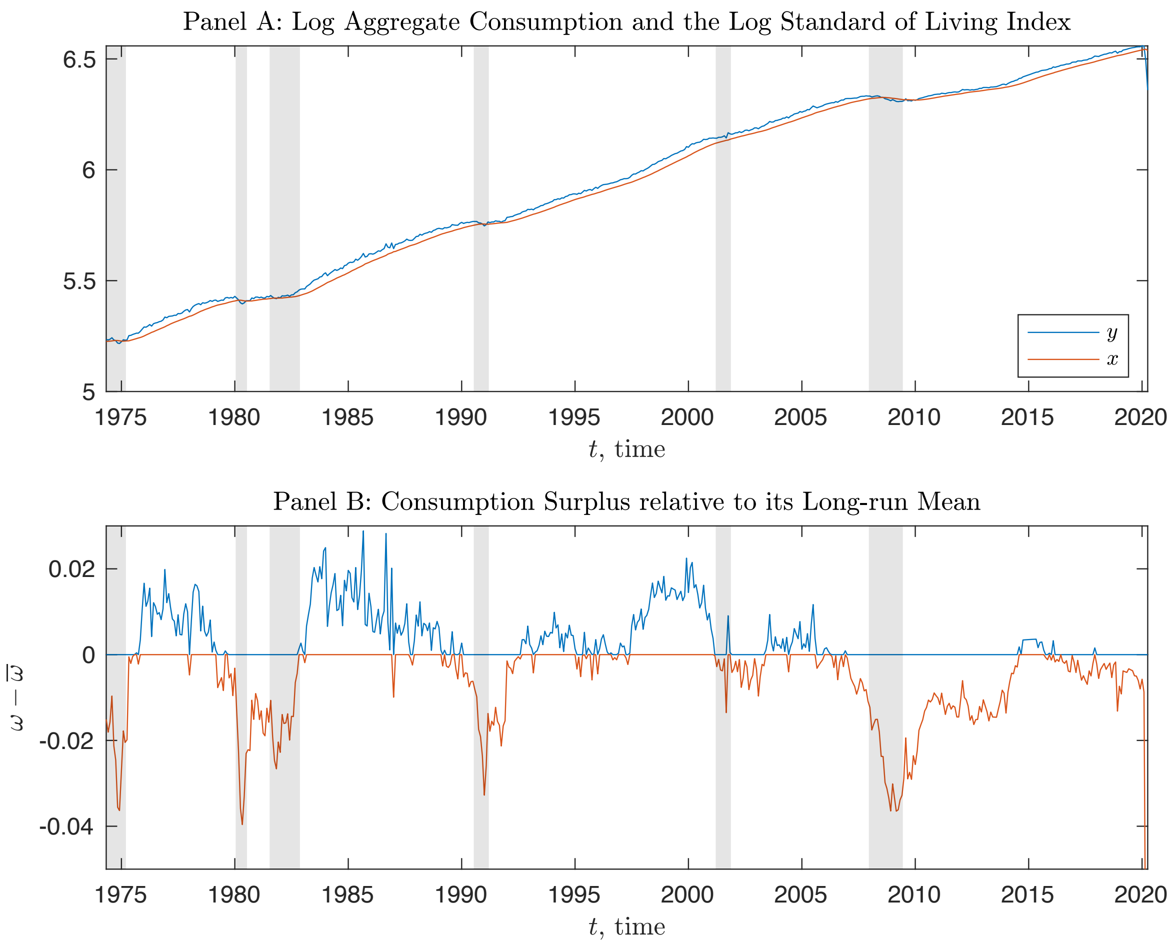

Figure 2: Aggregate Consumption, Standard of Living Index, and Consumption Surplus.

Notes: This figure shows the key aggregate macroeconomic time series used in the model from 1st May 1974 to 1st March 2020 with NBER recessions as grey bars. Panel A shows log aggregate consumption, y, and the implied log standard of living index, x. Panel B displays the implied time series of the consumption surplus relative to its long-run mean, $\omega - \overline{\omega}$. The times series is shown in blue when $\omega - \overline{\omega} > 0$ and red when $\omega - \overline{\omega} < 0$. We construct aggregate consumption from observations of $(Y_{t+1/12} - Y_t)/Y_t$, which is the seasonally adjusted monthly percentage change in US Real Personal Consumption Expenditures available at `https://fred.stlouisfed.org/series/DPCERAM1M225NBEA`. Our estimates $\mu_Y = 3.15\%$ per annum and $\sigma = 2.92\%$ per annum are based on the full times series which runs from 1st Jan 1959 to 1st Feb 2021 and we set $\lambda = 1.29$ per annum.

2.2 Heterogeneous Agents and Aggregate Consumption Risk

There is a continuum of heterogeneous agents, who are consumers whose utility is influenced by the standard of living index. They are all infinitely lived, have identical information, the same rate of time preference, δ, but differ with respect to a single parameter, γ, which determines both relative risk aversion and an agent's sensitivity to the standard of living index.

The consumption rate of an agent with relative risk aversion γ at instant u is denoted by $C_{\gamma,u}$ and the instantaneous utility from consumption is given by the following power function that depends on consumption relative to an agent-specific tracker for the importance of the standard of living index, $H_{\gamma,u}$:

$$U_\gamma(C_{\gamma,u}, H_{\gamma,u}) \equiv e^{-\delta u} \frac{1}{1-\gamma} \left(\frac{C_{\gamma,u}}{H_{\gamma,u}} \right)^{1-\gamma}, \tag{4}$$

where δ is the constant subjective discount rate (that is, the rate of time preference), and γ is the degree of relative-risk aversion.[6]

The quantity $H_{\gamma,u}$ in (4) is defined by

$$H_{\gamma,u} = X_u^{h_\gamma} = e^{h_\gamma x_u},$$

where h_γ is the agent's sensitivity to the standard of living index, as modeled in Muraviev (2013), which generalizes the specification in Chan and Kogan (2002) to allow for heterogeneity in the sensitivity parameter h_γ. If $h_\gamma = 1$, this reduces to the specification in Chan and Kogan (2002), and if $h_\gamma = 0$, one gets the standard isoelastic utility function without any dependence on the standard of living index.

We parameterize h_γ as follows

$$h_\gamma \equiv \frac{\gamma - \frac{1}{\psi}}{\gamma - 1},$$

[6] See Chan and Kogan (2002) for a discussion of this specification for the utility function, and why it is still appropriate to interpret γ as the coefficient of relative-risk aversion. For papers in the literature that study the effect of habit on asset prices in representative-agent models, see Abel (1990) and Abel (1999) and Constantinides (1990).

where $\gamma \in [0, \infty)$ differs across agents, but ψ does not. We can interpret $\frac{1}{\psi}$ as the sensitivity of the risk-free rate to the growth rate of aggregate consumption in the steady state when all agents are identical in the economy without risk.[7]

We assume markets are dynamically complete. The competitive equilibrium is therefore Pareto optimal and we can characterize the consumption-sharing rule via three standard steps. First, we analyze the social planner's problem. Then we construct an Arrow-Debreu economy to support the optimal allocation found in the planner's problem. Finally, we implement the Arrow- Debreu equilibrium as a sequential-trade economy. We can then determine equilibrium asset prices and the wealth distribution. In this section, we restrict ourselves to stating the social planner's problem, which is to maximize

$$\int_0^\infty f(\gamma) u_\gamma \left(\frac{C_{\gamma,t}}{H_{\gamma,t}} \right) d\gamma,$$

subject to the constraint

$$\int_0^\infty C_{\gamma,t} d\gamma = Y_t, \tag{5}$$

where $f(\gamma)$ defined in $\gamma \in [0, \infty)$ is the density function for the social planner's Pareto-Negishi weights, which we refer to as the Pareto-Negishi density.

The Pareto-Negishi density $f(\gamma)$ is a key determinant of both aggregate asset prices and the wealth distribution. The existing literature on heterogeneous agents assumes a specific functional form for the Pareto-Negishi density and then proceeds to solve for equilibrium. We show how to solve for equilibrium without making parametric assumptions about the Pareto-Negishi density.

[7] We start by considering the marginal utility (MU) of consumption at date t for agent, which is given by

$$\mathrm{MU}_{\gamma,t} = e^{-\delta\, t} {H_{\gamma,t}}^{\gamma - 1} C_{\gamma,t}^{-\gamma}.$$

When all agents are identical, $C_\gamma = Y$, and so marginal utility can be written as:

$$\mathrm{MU}_{\gamma,t} = e^{-\delta\, t} e^{-\left(\gamma - \frac{1}{\psi}\right)\omega_t} e^{-\frac{1}{\psi} y_t}.$$

Thus, the instantaneous interest rate in the deterministic version of the economy is $r_t = -\ln MU_t = \delta + \left(\gamma - \frac{1}{\psi}\right) \frac{d\omega_t}{dt} + \frac{1}{\psi} \frac{dy_t}{dt}$. From (3), we can see that in the deterministic version of the economy, ω possesses a steady state. At the steady state, $\frac{d\omega_t}{dt} = 0$, so

$$\frac{\partial\, r_t|_{\frac{d\omega_t}{dt}=0}}{\partial \left(\frac{dy_t}{dt}\right)} = \frac{1}{\psi}.$$

One might be tempted to think of ψ as the elasticity of intertemporal substitution, but this interpretation would be accurate only in a model with internal habit.

By using the converse of this result, we make it possible to recover the Pareto-Negishi density from equilibrium variables.

3 Distributional Dynamics and the Distribution of Agents

In this section, we show how the Pareto-Negishi density can be recovered from the distributional dynamics of consumption simply by inverting a cumulant generating function. This result forms the basis for our methodology for recovering the Pareto-Negishi density from (i) the term structure of dividend yields and (ii) the wealth distribution. The asset-pricing implied density and the wealth-distribution implied density will not be identical, but ought to be so if aggregate asset prices and the wealth distribution are driven by a common set of state variables. Using US data on dividend yields from Giglio et al. (2021) and on the wealth distribution from Saez and Zucman (2016), we show that the slope of the dividend yield and the growth in stock holdings of the top 1% individual equity holders in the US are indeed correlated (see Figure 1).

It is important to understand that in contrast to an economy with a finite number of households, our model features a continuum of heterogeneous households. The consumption flow of an individual atomistic household is negligible. Instead of examining the consumption flows of individual households, we must therefore look at the cross-sectional consumption flow density, denoted by $C_{\gamma,t}$. To understand the role of this density function, observe that the date-t consumption flow of agents with relative risk aversion within the range $\left(\gamma - \frac{1}{2}\epsilon, \gamma + \frac{1}{2}\epsilon\right)$ is given by

$$\int_{\gamma-\frac{1}{2}\epsilon}^{\gamma+\frac{1}{2}\epsilon} C_{x,t} dx.$$

Naturally, if we compute the date-t consumption flow of all agents, $\int_0^\infty C_{\gamma,t} d\gamma$, market clearing in the consumption good market implies this must equal aggregate output flow, i.e. (5). The density function for the consumption share of agents with relative risk aversion parameter γ is defined by

$$\nu_{\gamma,t} = \frac{C_{\gamma,t}}{Y_t}.$$

We now present a solution to the social planner's problem for a general Pareto-Negishi density. This result is novel, because it expresses the optimal consumption sharing rule, $\nu_{\gamma,t}$, in terms of the inverse of a cumulant generating function, which summarizes the distribution of the Pareto-Negishi weights.

Proposition 1 *The cross-sectional consumption share density is given by*

$$\nu_{\gamma}(\omega_t) = f(\gamma)^{\frac{1}{\gamma}} e^{-\omega_t - \frac{1}{\gamma}\eta(\omega_t)},$$

where the function $\eta(\omega_t)$ is independent of any particular value of γ and defined by

$$\eta(\omega_t) = -m^{-1}(\omega_t),$$

where $m(\cdot)$ is the following cumulant generating function

$$m(x) = \ln\left(\int_0^{\infty} j(u) e^{xu} du\right),$$

and the density function, $j(\cdot)$, is a transformation of the Pareto-Negishi density, given by

$$j(u) = u^{-2}\left[f(u^{-1})\right]^{u},\ u \in [0, \infty).$$

Proposition 1 starts by giving the cross-sectional consumption share density in terms of an unknown function of $\eta(\omega_t)$ and then shows how to obtain $\eta(\omega_t)$ in terms of the Pareto-Negishi density. The function $\eta(\omega_t)$ captures how aggregate shocks impact the cross-sectional consumption share density.

The converse of the above Proposition 1 will provide us with a recovery theorem, making it possible to extract the Pareto-Negishi density, given a cross-sectional consumption share density. To understand the statement of the theorem it will be useful to recall the definition of the Laplace transform and its inverse.

Definition 1 *The Laplace transform of a function $h(t)$, defined for all real numbers $t \geq 0$, is the function $H(s)$, defined by*

$$H(s) = \mathcal{L}\{h\}(s) = \int_0^{\infty} h(t) e^{-st}\, dt,$$

where $s \in \mathbb{C}$. We denote the inverse Laplace transform by $\mathcal{L}^{-1}$, which is given by the Bromwich integral

$$h(t) = \mathcal{L}^{-1}\{H\}(t) = \frac{1}{2\pi i} \lim_{T\to\infty} \int_{\epsilon - iT}^{\epsilon + iT} e^{st} H(s)\, ds,$$

where ϵ is a real number so that the contour path of integration is in the region of convergence of $H(s)$.

We are now ready to state the recovery theorem for the Pareto-Negishi density.

Theorem 1 *Suppose we have a consumption sharing rule*

$$\nu_\gamma(\omega_t) = f(\gamma)^{\frac{1}{\gamma}} e^{-\omega_t - \frac{1}{\gamma}\eta(\omega_t)}, \tag{6}$$

where the function $\eta(\omega_t)$ is known. The Pareto-Negishi density $f(\gamma)$ is given by

$$f(\gamma) = [\gamma^{-2} j(\gamma^{-1})]^\gamma,$$

where the density $j(u)$ is the inverse Laplace transform of $\exp(\eta^{-1}(x))$, i.e.

$$j(u) = \mathcal{L}^{-1}\{\exp(\eta^{-1})\}(u).$$

The above theorem comes with a caveat grounded in practical considerations. To apply the theorem, the cross-sectional consumption share density must be observable as function of ω. In other words, we would need to observe the time series of the cross-sectional consumption share density with enough variation in the consumption surplus, ω. Of course, this appears impossible to achieve in practice.

Fortunately, asset prices and empiricists come to our rescue. Asset prices are observable – if we can find asset prices that reveal the form of the function $\eta(\omega_t)$, we can then use Theorem 1 to recover the Pareto-Negishi density. Recent empirical work by Saez and Zucman (2016) has made the wealth distribution observable, which provides us with a second path to recovery.

4 Recovering the Pareto-Negishi Density from the Term Structure of Dividend Yields

In this section, we use time series data on the slope of the term structure of dividend yields to recover the Pareto-Negishi density.

4.1 Pricing Dividend Strips

We price dividend strips using the equilibrium stochastic discount factor for our model and show how the dividend yield slope is related to aggregate risk aversion.

We start by assuming that dividends are log normally distributed as follows

$$D_t = A_t Y_t^a, \tag{7}$$

where

$$\frac{dA_t}{A_t} = \mu_A dt + \sigma_A dZ_{A,t},$$

and

$$E_t[dZ_{A,t} dZ_t] = \rho dt.$$

The constant $a > 1$ provides a reduced form way to model leverage as in Abel (1999). The stochastic process A introduces another shock, so that aggregate dividends and consumption flows are not perfectly correlated.

Our model provides an equilibrium stochastic discount factor (SDF), Λ, which we use to price dividend strips and hence compute dividend yields. In the first proposition of this Section, we show that the volatility of the SDF depends on the aggregate risk aversion in the economy, which is the negative of the derivative of the function $\eta(\omega_t)$, the function which governs how aggregate risk impacts the consumption distribution. We shall use the following consumption-based definition of aggregate risk aversion.

Definition 2 *The aggregate risk aversion in the economy at time-t is defined to be*

$$R(\omega_t) = \frac{1}{\int_0^\infty \frac{1}{\gamma}\nu_\gamma(\omega_t) d\gamma}. \tag{8}$$

We can see that $R(\omega_t)$ is the weighted harmonic mean of the consumers' relative risk aversions, where the weighting function is the consumption-share density. Aggregate risk aversion depends on the aggregate state of the economy via ω_t.

Proposition 2 *The equilibrium stochastic discount factor is given by* Λ*, where*

$$\frac{d\Lambda_t}{\Lambda_t} = -r_t dt - \Theta_t dZ_t, \tag{9}$$

and

$$r_t = \delta + \frac{1}{\psi}\mu_y + \left(R(\omega_t) - \frac{1}{\psi}\right)\lambda(\overline{\omega} - \omega_t) - \frac{1}{2}R(\omega_t)^2(1 + R(\omega_t)V(\omega_t))\sigma^2,$$

$$\Theta_t = R(\omega_t)\sigma,$$

where

$$R(\omega_t) = -\eta'(\omega_t),$$

and $V(\omega_t)$ *is the consumption-weighted variance of relative risk aversion, defined by*

$$V(\omega_t) = \int_{\underline{\gamma}}^{\overline{\gamma}} \frac{1}{\gamma^2}\nu_\gamma(\omega_t)d\gamma - \left(\int_{\underline{\gamma}}^{\overline{\gamma}} \frac{1}{\gamma}\nu_\gamma(\omega_t)d\gamma\right)^2,$$

and given in terms of $\eta(\omega)$ *by*

$$V(\omega_t) = R(\omega_t)^{-3}\eta''(\omega_t).$$

We can see that negative shocks to current consumption growth decrease the consumption surplus ω_t and lead to a greater demand for risk-free bonds which decreases the real risk-free rate, if aggregate risk aversion is higher than $1/\psi$. When the consumption surplus ω is at it's long run mean, $\overline{\omega}$, the risk-free rate is given by

$$r_t|_{\omega_t=\overline{\omega}} = \delta + \frac{1}{\psi}\mu_y - \frac{1}{2}R(\overline{\omega})(P(\overline{\omega}) - 1)\sigma^2.$$

We can now see that when the $\omega_t = \overline{\omega}$, the sensitivity of the risk-free rate to changes in the growth of log consumption is given by $1/\psi$. The price of risk, Θ_t, increases when there are negative shocks to current consumption growth, because there is a transfer of wealth from the less risk averse to the more risk averse, leading to an increase in aggregate risk aversion.

A dividend strip is a security which pays the cashflow D_T at time T. The time-t price of this dividend strip is $D_t p_{D,T-t}(\omega_t)$, where

$$p_{D,T-t}(\omega_t) = E_t\left[\frac{\Lambda_T D_T}{\Lambda_t D_t}\right],$$

from which we can obtain the date-t yield, $y_{D,T-t}(\omega_t)$, via

$$p_{D,T-t}(\omega_t) = e^{-y_{D,T-t}(\omega_t)(T-t)}.$$

We therefore have

$$y_{D,T-t}(\omega_t) = -\frac{1}{T-t} \ln p_{D,T-t}(\omega_t).$$

Similarly, we can define the time-t yield-to-maturity for a real zero coupon bond, paying off 1 unit of consumption at time T:

$$y_{B,T-t}(\omega_t) = -\frac{1}{T-t} \ln E_t \left[\frac{\Lambda_T}{\Lambda_t} \right].$$

The following proposition gives closed form expressions for the short-term and long-term dividend yields.

Proposition 3 *The short-term dividend yield is given by*

$$k_t = y_{D,0}(\omega_t) = r(\omega_t) + R(\omega_t) Cov_t \left(\frac{dD_t}{D_t}, \frac{dC_t}{C_t} \right) - \mu_D$$

and the long-term dividend yield by the constant

$$\hat{k} = \lim_{\tau \to \infty} y_{D,\tau}(\omega_t) = \hat{r} + \frac{1}{\psi} Cov_t \left(\frac{dD_t}{D_t}, \frac{dC_t}{C_t} \right) - \mu_D,$$

where $\hat{r}$ is the long-term risk-free rate, given by

$$\hat{r} = \delta + \frac{1}{\psi}\mu_Y - \frac{1}{2}\frac{1}{\psi}\left(1 + \frac{1}{\psi}\right)\sigma^2,$$

and the covariance between shocks to dividend growth and consumption growth given by

$$Cov_t \left(\frac{dD_t}{D_t}, \frac{dC_t}{C_t} \right) = \sigma(\rho\sigma_A + a\sigma),$$

and

$$\mu_D = \frac{1}{dt} E_t \left[\frac{dD_t}{D_t} \right].$$

Aggregate risk aversion is directly proportional the dividend yield slope, $\hat{k} - k_t$, as shown below

$$R(\omega_t) = \frac{1}{\psi} + \frac{-(\hat{k} - k_t) + \hat{r} - r_t}{Cov_t \left(\frac{dD_t}{D_t}, \frac{dC_t}{C_t} \right)}. \tag{10}$$

Fluctuations in aggregate risk aversion are driven by fluctuations in the slope of the dividend yield term structure. This result will allow us to estimate $R(\omega_t)$ as function of ω_t, which gives $\eta(\omega_t)$ and hence the Pareto-Negishi density via Theorem 1.

4.2 Asset-Pricing Implied Pareto-Negishi Density

Data for aggregate dividends is obtained from Robert Shiller's website (`http://www.econ.yale.edu/~shiller/data.htm`). The monthly times series data we use for the dividend yield slope is from Giglio et al. (2021), who estimate the dividend yield based on the universe of stocks from CRSP and COMPUSTAT and covers the period from September 1974 to August 2020. Our times series for real bond yields are based on quarterly data from Chernov and Philippe (2012), which runs from the third quarter of 1971 to the fourth quarter of 2002. For the period from 2002 onwards, we use daily data from Gürkaynak et al. (2010). In both cases, we convert the real yield data to monthly frequency. For $\hat{r}$ we used ten year real yields and for r_t two year real yields.

Table 4.2 shows the values of the different parameters that we use in the model. We obtain the log-consumption volatility σ and the log-consumption drift μ_y by fitting a monthly discretization of the following equation

$$dy_t = \mu_y dt + \sigma dZ_t,$$

to the logarithm of the consumption time series. We then choose ψ to ensure that the right-hand side of (10) is positive. We estimate the covariance between dividend growth and consumption growth to be 0.0049 in annualized units. Finally, λ is chosen to maximize the correlation between recessions and negative shocks to ω_t.

δ	ψ	σ	μ_y	$Cov_t(dD_t/D_t, dC_t/C_t)$	λ
0.03	0.0204	0.0292	0.0311	0.0049	1.29

Table 1: Parameter values

Notes: We obtain the log-consumption volatility σ and the log-consumption drift μ_y by fitting a monthly discretization of Equation (1) to the logarithm of the consumption time series. We then choose ψ to ensure that the right-hand side of (10) is positive. Finally, λ is chosen to maximize the correlation between recessions and negative shocks to ω_t. The parameters σ, μ_y, and λ are given in per annum units.

We then run a regression of the empirical times series for $\ln\left(\frac{1}{\psi}+\frac{-(\hat{k}-k_t)+\hat{r}-r_t}{Cov_t\left(\frac{dD_t}{D_t},\frac{dC_t}{C_t}\right)}\right)$ against the model-implied times series for surplus consumption as shown in Figure 4.2 below.

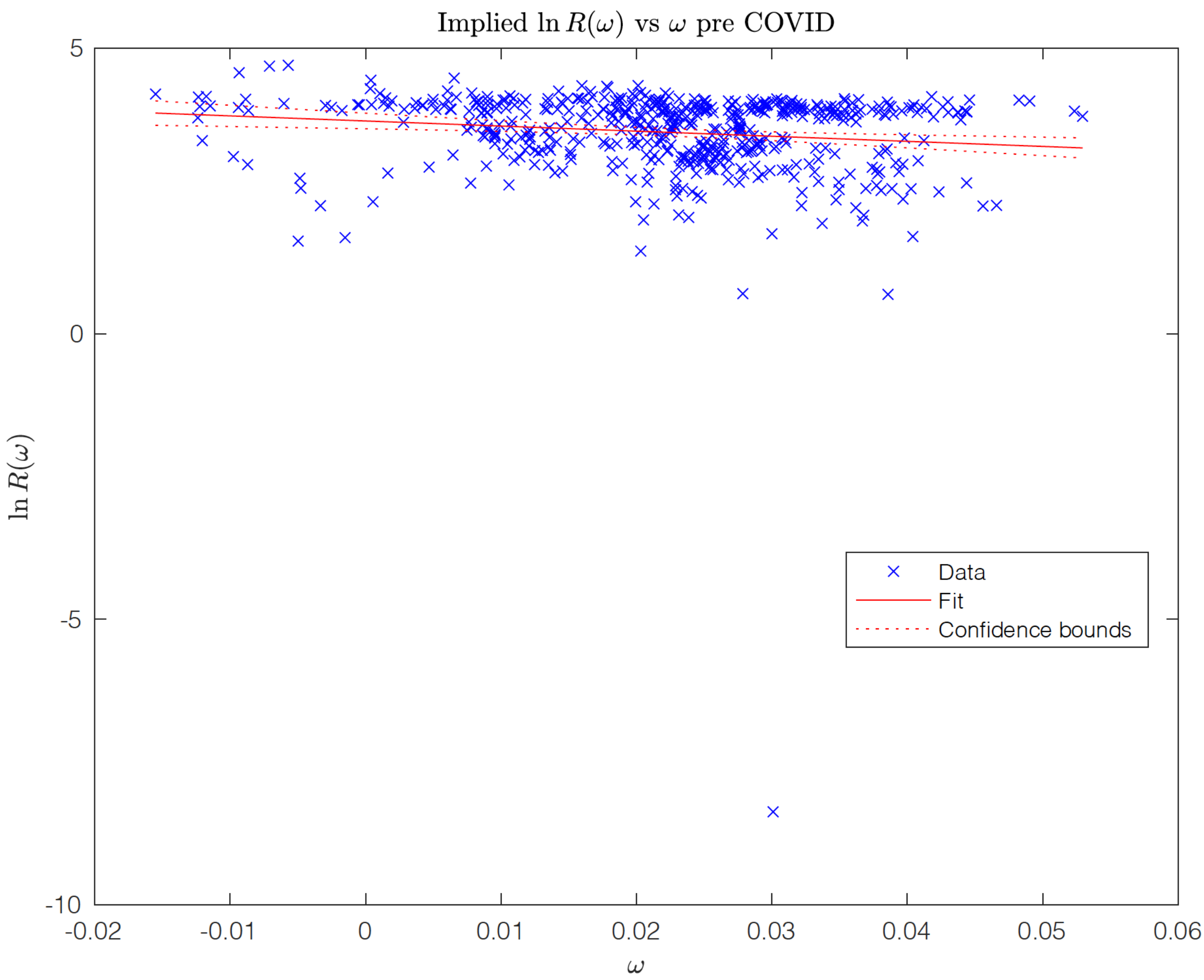

Figure 3: Log Aggregate Risk Aversion and the Consumption Surplus

Notes: We show a scatterplot of $\ln\left(\frac{1}{\psi}+\frac{-(\hat{k}-k_t)+\hat{r}-r_t}{Cov_t\left(\frac{dD_t}{D_t},\frac{dC_t}{C_t}\right)}\right)$ versus ω from September 1974 to February 2020 (just prior to the COVID-19 shock). The plot shows the results of a linear regression of $\ln\left(\frac{1}{\psi}+\frac{-(\hat{k}-k_t)+\hat{r}-r_t}{Cov_t\left(\frac{dD_t}{D_t},\frac{dC_t}{C_t}\right)}\right)$ versus ω. The blue crosses denote the data points to be fitted, the red line is the fitted linear model, and the dotted lines are the 95% confidence bounds. We estimate that $\ln R(\omega_t) = b_0 - b_1\omega_t$ where $b_0 = 3.7$ and $b_1 = 8.9$.

Coefficients	Estimate	Standard Error	t-stat	p-value
b_0	3.7	0.07	53.6	8×10^{-218}
b_1	8.9	2.7	3.4	0.001

Table 2: Regression Results: Log Risk Aversion and the Consumption surplus

Notes: We run a linear regression of $\ln\left(\frac{1}{\psi} + \frac{-(\hat{k}-k_t)+\hat{r}-r_t}{Cov_t\left(\frac{dD_t}{D_t}, \frac{dC_t}{C_t}\right)}\right)$ versus ω using data from September 1974 to February 2020 (just prior to the COVID-19 shock).

This allows us to estimate the log of aggregate risk aversion as a linear function of ω. We find that a log linear specification for aggregate risk aversion works well and adding higher order terms does not significantly improve the fit. We therefore assume that

$$\ln R(\omega_t) = b_0 - b_1\omega_t.$$

The results of our regression are summarized in Table 4.2. We find that $b_0 = 3.7$ and $b_1 = 8.9$. We can see from Figure 4.2 that a log-linear functional form for aggregate risk aversion in terms of ω offers a good fit of the data. Using this estimated functional form, we can apply Theorem 1 and obtain the asset-pricing implied Pareto-Negishi density in closed form, as shown below.

Proposition 4 *If*

$$\ln R(\omega_t) = b_0 - b_1\omega_t,\, b_1 > 0, \tag{11}$$

then the asset-pricing implied Pareto-Negishi density is given (up to an arbitrary constant) by

$$f_A(\gamma) \propto \gamma^{-\alpha\gamma} e^{\beta\gamma}, \tag{12}$$

with

$$\alpha = 1 + \frac{1}{b_1},$$

$$\beta = \ln\left(e^{\frac{b_0}{b_1}} b_1^{-\frac{1}{b_1}} \Gamma(b_1^{-1})\right),$$

where $\Gamma(\cdot)$ is the Gamma function.

The asset-pricing implied consumption sharing rule is given by

$$\nu_\gamma(\omega_t) = \frac{b_1^{-\frac{1}{b_1}} \gamma^{-\frac{1}{b_1}-1} R(\omega_t)^{\frac{1}{b_1}} e^{-\frac{R(\omega_t)}{b_1\gamma}}}{\Gamma\left(\frac{1}{b_1}\right)}. \tag{13}$$

To visualize the above proposition, we start by showing a plot of the asset-pricing implied Pareto-Negishi density normalized to integrate to one in Figure 4.2. The distribution is positively skewed with a mean relative risk aversion of 1.5, standard deviation of 0.9, as shown in Table 4.2.

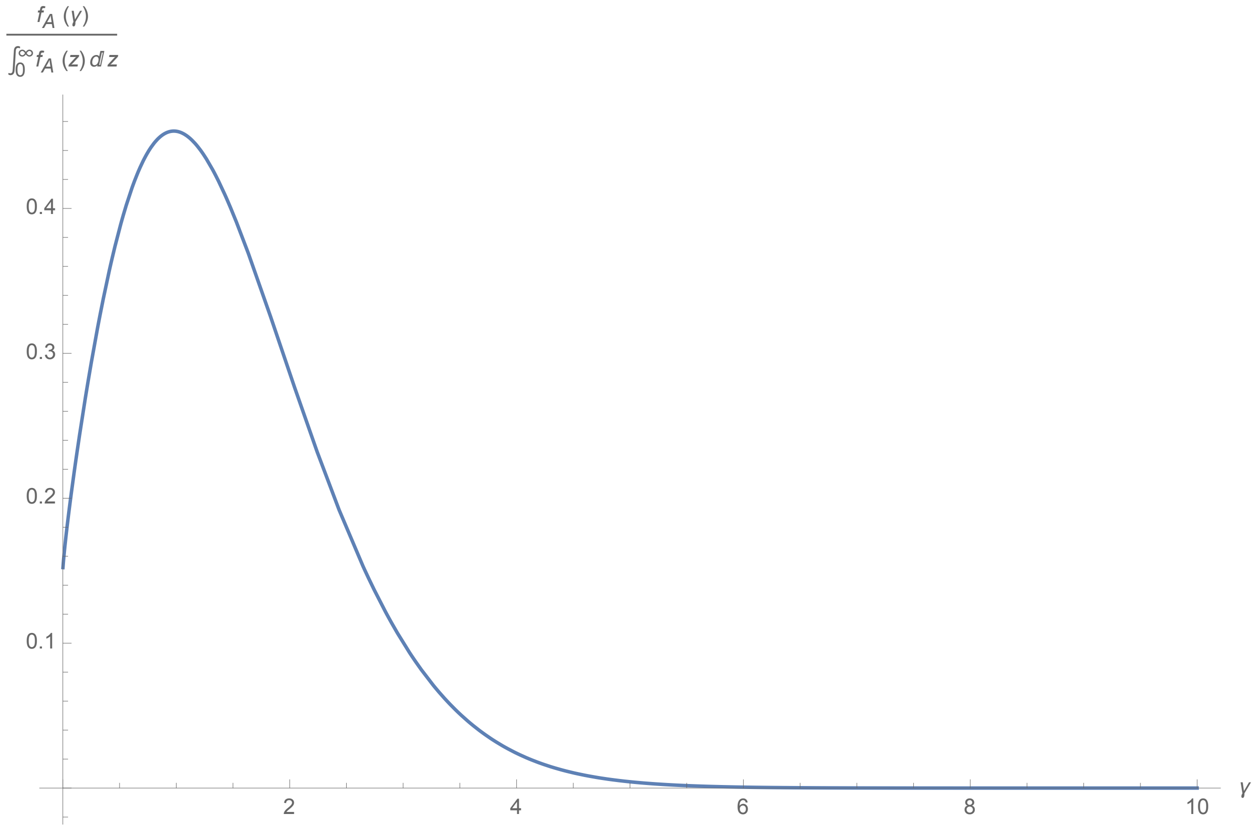

Figure 4: Asset-pricing implied Pareto-Negishi Density

Notes: This figure shows the asset-pricing implied Pareto-Negishi density normalized to integrate to one, i.e. $f_A(\gamma)/\int_0^\infty f_A(z)dz$ as a function of relative risk aversion γ.

Panel A: Raw moments		Panel B: Standardized moments	
1st moment	1.8	mean	1.8
2nd moment	4.2	standard deviation	1.1
3rd moment	12.3	skewness	0.8
4th moment	42.1	kurtosis	3.6

Table 3: Moments of the Asset-pricing implied Pareto-Negishi Density.

Notes: Panel A shows the raw moments of the normalized asset-pricing implied Pareto-Negishi density and Panel B shows the standardized moments as described.

5 Recovering the Pareto-Negishi Density from the Dynamics of the Wealth Distribution

In this section, we show how to derive the dynamics of the theoretical wealth distribution implied by our model for a general Pareto-Negishi density. We then use cross-sectional data on the US wealth distribution to estimate the Pareto-Negishi density.

5.1 Wealth Distribution Dynamics

We derive the cross-sectional density function for the wealth share, $\zeta_\gamma(\omega_t)$, which gives the fraction of aggregate wealth owned by agents with relative risk aversion equal to γ. We first consider the amount of wealth held by agents with risk aversion equal to γ, i.e. $W_{\gamma,t}$, where

$$\begin{aligned} W_{\gamma,t} &= E_t\left[\int_t^\infty \frac{\Lambda_u}{\Lambda_t}\nu_{\gamma,u}Y_u du\right] \\ &= e^{y_t}\nu_{\gamma,t}E_t\left[\int_t^\infty \frac{\Lambda_u}{\Lambda_t}\frac{\nu_{\gamma,u}Y_u}{\nu_{\gamma,t}Y_t}du\right] \end{aligned}$$

We define the cross-sectional density for the wealth-consumption ratio

$$h_\gamma(\omega_t) = E_t\left[\int_t^\infty \frac{\Lambda_u}{\Lambda_t}\frac{\nu_{\gamma,u}Y_u}{\nu_{\gamma,t}Y_t}du\right].$$

Hence, we can write the wealth density function as

$$W_{\gamma,t} = e^{y_t}\nu_\gamma(\omega_t)h_\gamma(\omega_t).$$

The cross-sectional density function for the wealth share is given by $\zeta_\gamma(\omega_t)$, where

$$\zeta_\gamma(\omega_t) = \frac{W_{\gamma,t}}{\int_0^\infty W_{\gamma,t}d\gamma}.$$

The following proposition characterizes the cross-sectional density function for the wealth share in terms of the cross-sectional density for the consumption share, $\nu_\gamma(\omega_t)$, the cross-sectional density for the wealth-consumption ratio, $h_\gamma(\omega_t)$, and the wealth-consumption ratio for the aggregate economy, $p_C(\omega_t) = \int_0^\infty W_{\gamma,t}d\gamma/Y_t$.

Proposition 5 *The cross-sectional density function for the wealth share is given by*

$$\zeta_\gamma(\omega_t) = \frac{\nu_\gamma(\omega_t)h_\gamma(\omega_t)}{p_C(\omega_t)}, \tag{14}$$

and the dynamics of the wealth density are given by

$$d\ln\zeta_\gamma(\omega) - E_t[d\ln\zeta_\gamma(\omega)] = (\phi_\gamma(\omega) - 1)\sigma_R(\omega_t)dZ_t.$$

where $\phi_\gamma(\omega)$ *is the fraction of personal financial wealth invested in risky assets for agents with relative risk aversion equal to* γ *and is given in equilibrium by*

$$\phi_\gamma(\omega) = \frac{\frac{R(\omega)}{\gamma} + \frac{h'_\gamma(\omega)}{h_\gamma(\omega)}}{1 + \frac{p'_C(\omega)}{p_C(\omega)}}. \tag{15}$$

We can see immediately from the above proposition that the impact of shocks to the aggregate state of the economy on the fraction of aggregate wealth held by households is amplified, because the volatility of stock returns is larger than the volatility of aggregate consumption growth.

We can also see that the fraction of aggregate wealth held by households with levered positions in risky assets, i.e. the risky share, increases when there is a positive aggregate shock. The risky share is important along two dimensions. Agents with lower risk aversion will have a higher risky share, giving them larger exposure to aggregate risk. This generates heterogeneity in portfolio returns, where less risk averse agents have a greater expected portfolio return, but also more volatile returns. Over time, if there are more positive aggregate shocks than negative, as is the case empirically, less risk averse agents will accumulate more wealth, creating positive skewness in the wealth density $\zeta_\gamma(\omega)$ as a function of relative risk aversion, γ.

Furthermore, time variation in the risky share governs the dynamics of the wealth distribution in the following way. An increase in financial leverage at a particular point in the distribution will mean that part of the distribution will change more as aggregate shocks arrive.

5.2 Empirical Evidence: The Wealth Distribution and the Pareto-Negishi Density

In this section, we build empirical estimates of the wealth distribution and the Pareto-Negishi Density implied by the wealth data. We focus on the US and rely on three main data sources: the updated individual wealth data series from Saez and Zucman (2016), the implied equity risk premium from Damodaran (2020) and the return volatility of S&P 500 from Bloomberg Finance. All three data sources are available for the period 1968-2019.

Given that the aim of the paper is to understand the feedback between stock market fluctuations and the dynamics of wealth inequality, we restrict our empirical analysis to the distribution of financial wealth. Financial wealth is the sum of equities, fixed-income assets and funded pension wealth. Since we have the implied equity risk premium, the return volatility and we can calculate the share of financial wealth in risky assets using the wealth series, we can derive the coefficient of relative risk aversion for every individual and year in the wealth distribution following Merton (1969) and inverting the expression for the risky share, given by

$$\phi_\gamma(\omega) = \frac{1}{\gamma}\frac{\mu_R(\omega) - r(\omega)}{\sigma_R(\omega)^2} + \frac{h'_\gamma(\omega)}{h_\gamma(\omega)}\frac{\sigma}{\sigma_R(\omega)},$$

where the first term is myopic demand and the second term is hedging demand and $\frac{\sigma}{\sigma_R(\omega)}$ is the covariance between shocks to the risky return and shocks to the consumption surplus ratio.

Figure 5 shows the evolution of the three series used to derive the coefficient of relative risk aversion: the equity risk premium, the return volatility of the S&P 500 and the share in risky assets in the US over the period 1968-2019. The equity risk premium has fluctuated between 2% and 7% over the whole period. In 2019 it reached 4.7% compared to 2.9% in 1968. The return volatility of S&P 500 has fluctuated from 8% to 50% over the whole period. In 2019 it reached 14.9%, compared to 8.7% in 1968. Finally, the risky share followed a U-shape pattern over the whole period. In the late 1960s it was more than 40%, it steadily declined during the 1970s and 1980s and it started to increase since 1990s, slightly declined during the early 2000s and the global financial crisis, reaching 27% in 2019.

Figure 6 shows the evolution of the share of total financial wealth by level of relative risk aversion in the US in every ten years between 1979 and 2019. In 2019, the share of wealth is highly concentrated around individuals with a low level of risk aversion (between 2 and 3). The density varies across time, as the results are based on a cross-section sample of individuals (individuals leave and enter the sample every year). In 1999 and 2009, the share of financial wealth was more concentrated at lower levels of risk aversion, as these dates were the ends of economic expansions – less risk averse individuals would have benefited more from the series of positive aggregate shocks, leaving them with a greater share of aggregate wealth.

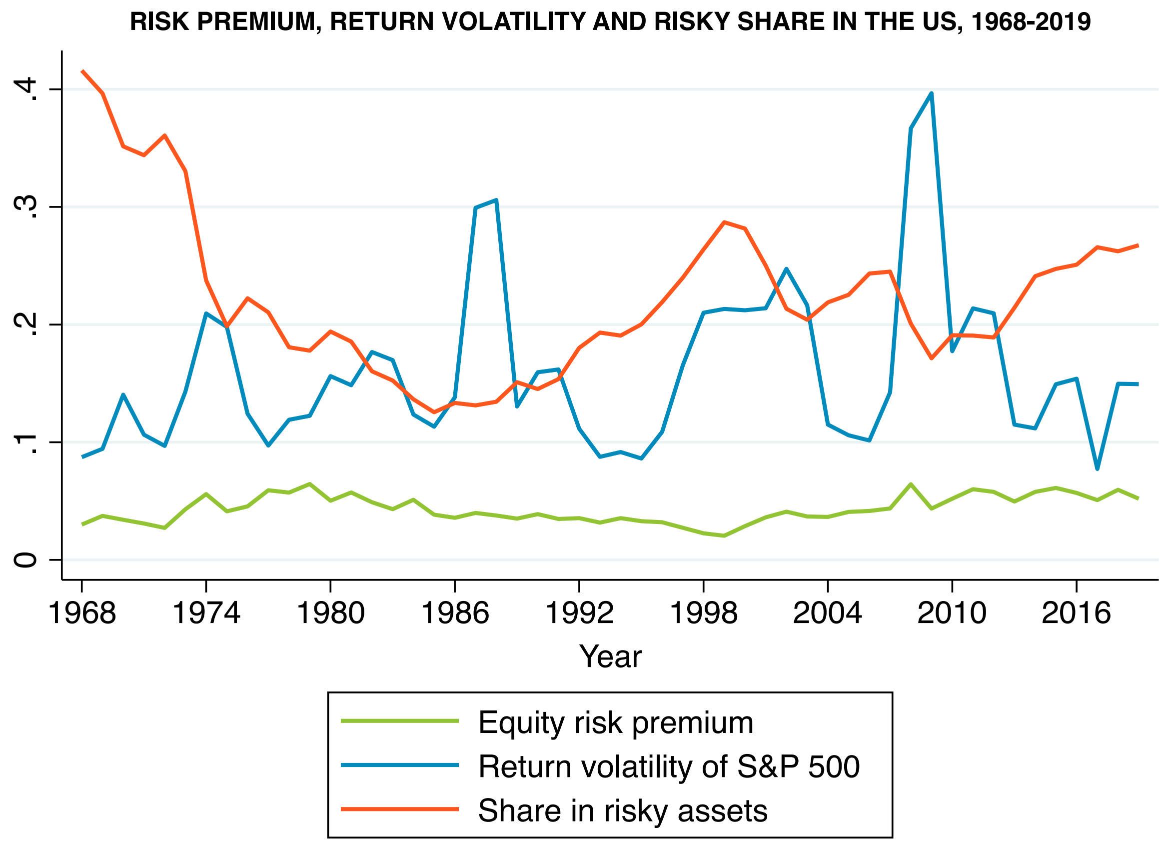

Figure 5: Risk Premium, Return Volatility and Risky Share in the US, 1968-2019

Notes: This figure depicts the equity risk premium, the return volatility and the share in risky assets that we use to back out the distribution of risk aversion over the period 1968-2019. We rely on the equity risk premium estimates of Damodaran (2020) and the return volatility of S&P 500 from Bloomberg. To derive the risky share, we use the microfiles of Saez and Zucman (2016) (`https://gabriel-zucman.eu/usdina/`). We rank the US population by their level of financial wealth (deposits, fixed-income securities, stocks and pension funds) and consider as risky asset the stock holdings. The equity risk premium is depicted in per-annum units and so is the return volatility. The risky share is graphed as a fraction.

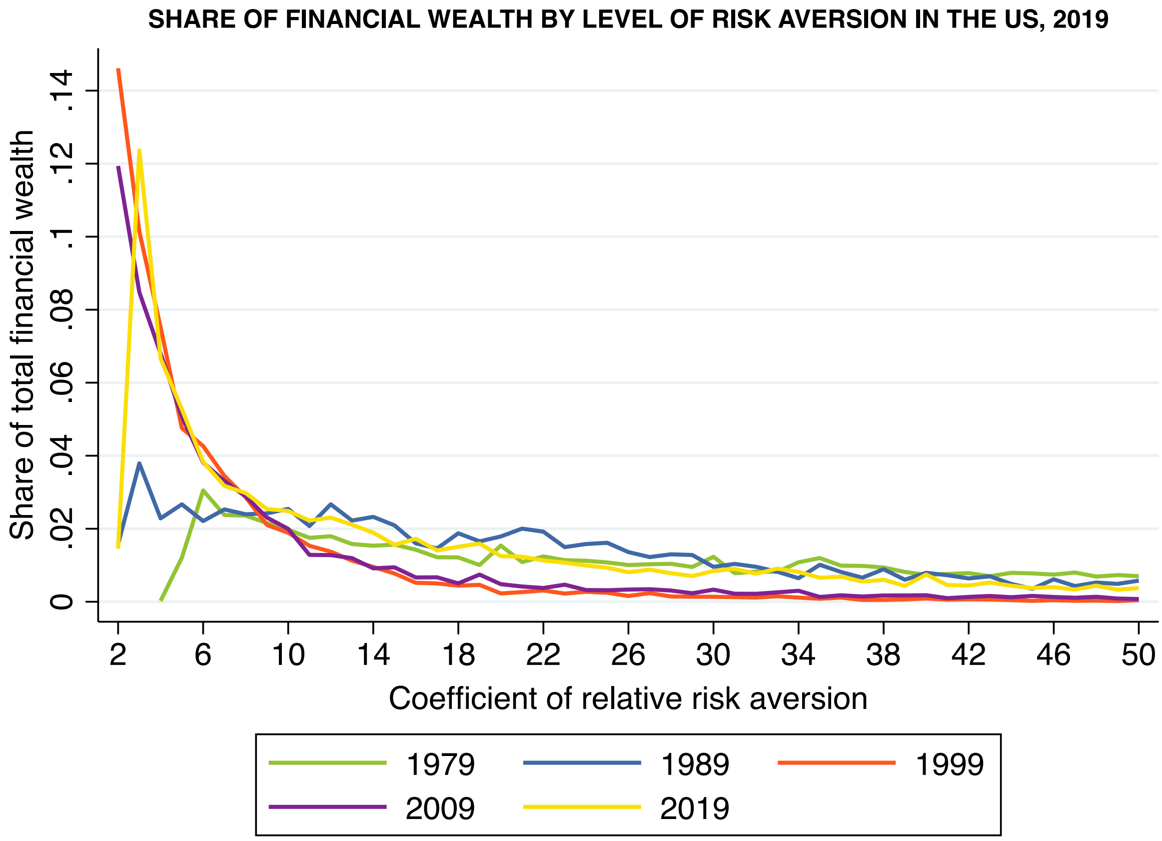

Figure 6: Share of Financial Wealth by Level of Risk Aversion in the US, 1979-2019

Notes: This figure depicts the share of total financial wealth by the level of risk aversion for different years between 1979 and 2019. We have backed out the coefficient of relative risk aversion using Merton (1969)'s formula above by relying on data for the equity risk premium, the return volatility and the share in risky assets depicted in Figure 5.

We now show how our model matches the quantiles of the cross-sectional empirical density function for the wealth share $\zeta_\gamma(\omega_t)$. We define the following wealth-weighted measure of aggregate risk aversion:

$$R_{\text{wealth}}(\omega_t) = \left(\int_0^\infty \frac{1}{\gamma} \zeta_\gamma(\omega_t) d\gamma \right)^{-1}. \tag{16}$$

We note that $R_{\text{wealth}}(\omega_t)$ is the harmonic mean of relative risk aversion, but weighted via the wealth-share density $\zeta_\gamma(\omega_t)$, as opposed to the consumption-share density used in (8). The p'th quantile of the wealth-share density, $\zeta_\gamma(\omega_t)$, is denoted by $\gamma_p(\omega_t)$ and defined by the integral

$$\int_0^{\gamma_p(\omega_t)} \zeta_\gamma(\omega_t) d\gamma = \frac{p}{100},\ p \in [1, 100].$$

We parametrize the underlying Pareto-Negishi density using Equation (12), leaving the coefficients b_0 and b_1 free to be identified from the wealth data.[8]

We compute the empirical quantiles of the wealth-share density for every year in our sample period. We then average these quantiles across time to obtain a measure of the average quantiles of the wealth share density function. These quantities are shown in Table 4.

Table 4: Average Empirical Quantiles of the Empirical Wealth-Share Density

Percentile	70	65	64	63	60	50	40	35	30	10	1
Time Series Mean of $\gamma_p(\omega_t)$	43.3	36	34.7	33.5	30.3	21.7	15.4	12.9	10.7	4.53	3

Notes: This table shows the quantiles of the empirical wealth share density.

Furthermore, using Equation (16), we compute the average empirical aggregate risk aversion obtaining a value of 12.08. We then want to find values for the parameters b_0 and b_1 that define the wealth-share density function in our model to match the empirical quantiles and empirical aggregate risk aversion of 12.08. To do so, we proceed as follows: using Equation (14), we compute the wealth-share density in our model as a function of b_0 and b_1; for every empirical quantile, we then pick b_0 and b_1 so that we best approximate the empirical quantile and the empirical aggregate risk aversion with the corresponding model-implied values. The exact criterion we minimize is the

[8] We explored using alternative functional forms for the density, but doing so made little difference to our quantitative results.

root mean squared error, defined by

$$\epsilon(p) = \sqrt{(\gamma_p(\hat{\omega})|_{\text{model}} - \gamma_p|_{\text{data}})^2 + (R_{\text{wealth}}(\hat{\omega})|_{\text{model}} - R_{\text{wealth}}(\hat{\omega})|_{\text{data}})^2}, \tag{17}$$

where $\hat{\omega}$ is the time series mean of ω for our sample. Table 5 shows for selected quantiles p, which values of b_0 and b_1 minimize the error, $\epsilon(p)$.

Table 5: Parameter Estimates for Wealth-Distribution implied Pareto-Negishi Density

Percentile, p	b_0	b_1
65	2.6	4.7
64	2.7	9.3
63	2.8	13.8
60	2.8	14.2
50	3.0	22.8
35	3.0	23.0
30	3.0	23.0
10	3.0	22.8
1	3.0	22.8

Notes: Using the parametric form for the Pareto-Negishi Density given in Proposition 4, we estimate b_0 and b_1 to minimize $\epsilon(p)$, defined in (17) for the percentiles, $p \in \{65, 64, 63, 60, 50, 35, 30, 10, 1\}$.

The dynamics of the wealth-share density are governed by the dependence on b_1, while the average form of the density is determined by b_0. Both the dividend-yield data and wealth data give very similar estimates for b_0 and hence the average form of the wealth share density. When it comes to dynamics, the qualitative variations in wealth-share density implied by two approaches are also very similar –we see that b_1 is always positive. The wealth data provides a range of estimates for the size of the swings in the density, which are centered around the estimate we obtain from the dividend-yield data. We interpret these results are showing that is strong feedback between the dynamics of the wealth distribution and asset price dynamics. However, some of the variation must be driven by factors outside our model and unrelated to the consumption surplus. Measures of uncertainty and changes in expecations are natural candidates for seeking to further explain the joint variation in wealth and asset price dynamics.

To more precisely quantity the differences between the asset-price implied Pareto-Negishi density and the wealth distribution implied densities, we compute the relative entropy of the wealth distribution density from the asset-price implied density for each percentile we have tried to match. The relative entropy of the wealth distribution density from the asset-price implied density is given by

$$KL[A|W] = \int_0^\infty \hat{f}_A(x) \ln \frac{\hat{f}_A(x)}{\hat{f}_W(x)} dx,$$

where the hat symbol ˆ denotes a density that has been normalized so it integrates to one. Table 6 shows our results. We see that for the 64th percentile the relative entropy is particularly small. Our results suggest that the sensitivity of the wealth-share density to changes in the consumption surplus ratio is on average very similar, regardless of whether we use dividend yield data or wealth data. However, the sensitivity of the wealth-share density based on the wealth data to changes in the consumption surplus ratio varies across risk aversion in way which is not seen when we use asset prices to recover the wealth-share density. In particular agents holding a share of aggregate wealth are highly sensitive to aggregate shocks, suggesting that richer households find it easier to rebalance their portfolios.

Table 6: The Relative Entropy of the Wealth Distribution implied Density from the Asset-Price implied Density

Percentile, p	$KL[A\|W]$
65	0.6898
64	0.0050
63	0.2283
60	0.2705
50	1.7164
35	1.7577
30	1.7577
10	1.7164
1	1.7164

Notes: This table shows the relative entropy of the wealth distribution implied density from the asset-price implied density for each percentile we have tried to match for the percentiles, $p \in \{65, 64, 63, 60, 50, 35, 30, 10, 1\}$.

6 Conclusion

This paper examines the feedback between asset prices and wealth inequality dynamics. First, we build a dynamic, consumption-based equilibrium model in which heterogeneity in risk aversion for a continuum of households translates into different exposures to aggregate consumption risk and this in turn drives the dynamics of both endogenous asset prices and wealth inequality. We then solve the model analytically for the distribution of consumption across households in terms of the social planner's Pareto weights and the difference between log aggregate consumption and its history. Finally, we take our model to the data and obtain estimates for the social planner's Pareto weights using both data for the US on asset prices (i.e., the dividend yield slope) and individuals' household wealth from Saez and Zucman (2016). The estimates of the risk aversion distribution based on asset prices and wealth dynamics considerably overlap. Our results thus provide evidence for significant feedback between asset price dynamics and wealth distribution dynamics.

The literatures on asset pricing and inequality have remained separate, despite the importance of the stock market for differences in rates of wealth accumulation and the role of portfolio heterogeneity in determining stock prices. This study is a step forward in developing a unified framework aimed at understanding from a theoretical and quantitative standpoint the interactions between wealth inequality and asset prices. Central to our paper is the role of heterogeneity in risk exposure in generating wealth disparities along the business cycle. We hope these findings will open up new avenues for future empirical and theoretical research on the interactions between wealth inequality and asset price dynamics.

A Proofs

Proof of Proposition 1

In this section, we derive consumption-share density, i.e. the density function for the consumption share of agents with relative risk aversion parameter γ, i.e.

$$\nu_{\gamma,t} = \frac{C_{\gamma,t}}{Y_t},$$

where $C_{\gamma,t}$ is the consumption density function. Observe that the date-t consumption flow of agents with relative risk aversion within the range $\left(\gamma - \frac{1}{2}\epsilon, \gamma + \frac{1}{2}\epsilon\right)$ is given by

$$\int_{\gamma - \frac{1}{2}\epsilon}^{\gamma + \frac{1}{2}\epsilon} C_{x,t} dx.$$

Markets are dynamically complete, so given aggregate output flow, we can use a social planner to derive optimal allocations. The social planner's objective function is to maximize

$$\int_{\underline{\gamma}}^{\overline{\gamma}} f(\gamma) u_\gamma(C_{\gamma,t}) H_{\gamma,t}^{-(1-\gamma)} d\gamma,$$

subject to the constraint

$$\int_{\underline{\gamma}}^{\overline{\gamma}} C_{\gamma,t} d\gamma = Y_t,$$

where $\underline{\gamma} = 0$ and we shall let $\overline{\gamma} \to \infty$.

Defining consumption shares

$$\nu_{\gamma,t} = \frac{C_{\gamma,t}}{Y_t},$$

the social planner's problem becomes

$$\int_{\underline{\gamma}}^{\overline{\gamma}} f(\gamma) \frac{Y_t^{1-\gamma} \nu_{\gamma,t}^{1-\gamma} - 1}{1-\gamma} H_{\gamma,t}^{-(1-\gamma)} d\gamma,$$

subject to the constraint

$$\int_{\underline{\gamma}}^{\overline{\gamma}} \nu_{\gamma,t} d\gamma = 1. \tag{A1}$$

The Lagrangian for this problem is

$$\mathcal{L} = \int_{\underline{\gamma}}^{\overline{\gamma}} f(\gamma) \frac{Y_t^{1-\gamma} \nu_{\gamma,t}^{1-\gamma} - 1}{1-\gamma} H_{\gamma,t}^{-(1-\gamma)} d\gamma + \hat{\Psi}_t \left(1 - \int_{\underline{\gamma}}^{\overline{\gamma}} \nu_{\gamma,t} d\gamma \right),$$

where the Lagrange multiplier $\hat{\Psi}_t$ is independent of individual agent type.

The FOC's are

$$f(\gamma) Y_t^{1-\gamma} \nu_{\gamma,t}^{-\gamma} H_{\gamma,t}^{-(1-\gamma)} = \hat{\Psi}_t,\ \gamma \in [\underline{\gamma}, \overline{\gamma}].$$

We now simplify the above FOC's to obtain

$$f(\gamma) Y_t^{1-\gamma} X_t^{\gamma-1/\psi} \nu_{\gamma,t}^{-\gamma} = \hat{\Psi}_t,\ \gamma \in [\underline{\gamma}, \overline{\gamma}],$$

and hence

$$f(\gamma) e^{\left(1-\frac{1}{\psi}\right) y_t - \left(\gamma - \frac{1}{\psi}\right)\omega_t} \nu_{\gamma,t}^{-\gamma} = \hat{\Psi}_t,\ \gamma \in [\underline{\gamma}, \overline{\gamma}].$$

Solving for the consumption shares, we obtain

$$\nu_{\gamma,t} = [f(\gamma)]^{\frac{1}{\gamma}} e^{\frac{1}{\gamma}\left(1-\frac{1}{\psi}\right) y_t - \frac{1}{\gamma}\left(\gamma - \frac{1}{\psi}\right)\omega_t} \hat{\Psi}_t^{-\frac{1}{\gamma}}.$$

The Lagrange multiplier $\hat{\Psi}_t$ is determined by the constraint (A1), so we obtain

$$\int_{\underline{\gamma}}^{\overline{\gamma}} f(\gamma)^{\frac{1}{\gamma}} e^{\frac{1}{\gamma}\left(1-\frac{1}{\psi}\right) y_t - \frac{1}{\gamma}\left(\gamma - \frac{1}{\psi}\right)\omega_t} \hat{\Psi}_t^{-\frac{1}{\gamma}} d\gamma = 1.$$

Therefore

$$e^{-\omega_t} \int_{\underline{\gamma}}^{\overline{\gamma}} f(\gamma)^{\frac{1}{\gamma}} \left(e^{-\left[\left(1-\frac{1}{\psi}\right) y_t + \frac{1}{\psi}\omega_t\right]} \hat{\Psi}_t \right)^{-\frac{1}{\gamma}} d\gamma = 1.$$

We see that $\Psi_t = e^{-\left[\left(1-\frac{1}{\psi_i}\right) y_t + \frac{1}{\psi_i}\omega_t\right]} \hat{\Psi}_t$ is independent of agent-type. Hence,

$$\int_{\underline{\gamma}}^{\overline{\gamma}} f(\gamma)^{\frac{1}{\gamma}} \Psi_t^{-\frac{1}{\gamma}} d\gamma = e^{\omega_t}.$$

Defining

$$\eta_t = \ln \Psi_t,$$

we obtain

$$\int_{\underline{\gamma}}^{\overline{\gamma}} f(\gamma)^{\frac{1}{\gamma}} e^{-\frac{1}{\gamma}\eta_t} d\gamma = e^{\omega_t}. \tag{A2}$$

and

$$\nu_\gamma(\omega_t) = f(\gamma)^{\frac{1}{\gamma}} e^{-\omega_t - \frac{1}{\gamma}\eta(\omega_t)}.$$

From the above integral equation (A2), we can see that η_t is a function of ω_t, i.e. $\eta_t = \eta(\omega_t)$, and so

$$\int_{\underline{\gamma}}^{\overline{\gamma}} f(\gamma)^{\frac{1}{\gamma}} e^{-\frac{1}{\gamma}\eta(\omega_t)} d\gamma = e^{\omega_t}. \tag{A3}$$

We define

$$u = \frac{1}{\gamma},$$

and so

$$\int_{\frac{1}{\overline{\gamma}}}^{\frac{1}{\underline{\gamma}}} u^{-2} \left[a_i f(u^{-1})\right]^u e^{-u\,\eta(\omega_t)} du = e^{\omega_t}.$$

Simplifying further, we obtain

$$\int_{\frac{1}{\overline{\gamma}}}^{\frac{1}{\underline{\gamma}}} u^{-2} \left[a_i f(u^{-1})\right]^u e^{-u\,\eta(\omega_t)} du = e^{\omega_t}.$$

Define the density function

$$j(u) = u^{-2} \left[f(u^{-1})\right]^u, \; u \in [1/\overline{\gamma}, 1/\underline{\gamma}].$$

Observe that $j(u)$ is not a probability density function, because it does not integrate to one. It follows that

$$M(-\eta(\omega_t)) = e^{\omega_t},$$

where

$$M(x) = \int_{\frac{1}{\overline{\gamma}}}^{\psi_i} j(u) e^{xu} du$$

is the moment generating function of for the density function $j(\cdot)$.

The corresponding cumulant generating function is defined by

$$m(x) = \ln M(x),$$

and so

$$m(-\eta(\omega_t)) = \omega_t.$$

Inverting the cumulant generating function, we obtain

$$\eta(\omega_t) = -m^{-1}(\omega_t).$$

■

Proof of Theorem 1

■

Proposition A1 *The function $\eta(\omega_t)$ is given explicitly in terms of ω_t by*

$$\eta(\omega_t) = -\sum_{n=0}^{\infty} J_n \frac{(e^{\omega_t} - I_0)^n}{n!},$$

where

$$J_1 = \frac{1}{I_1}$$

$$J_n = \frac{1}{I_1^n} \sum_{k=1}^{n-1} (-)^k n^{(k)} B_{n-1,k}\left(\widehat{I}_1, \ldots, \widehat{I}_{n-k}\right)$$

$$\widehat{I}_k = \frac{\widehat{I}_{k+1}}{(k+1)I_1},$$

and

$$I_n = \int_{\frac{1}{\gamma}}^{\psi_i} j(u) u^n du,\ n \in \{0, 1, 2, \ldots\},$$

and $B_{n,k}(x_1, x_2, \ldots, x_{n-k+1})$ is the exponential Bell polynomial defined as follows

$$B_{n,k}(x_1, x_2, \ldots, x_{n-k+1}) = \sum \frac{n!}{j_1! j_2! \cdots j_{n-k+1}!} \left(\frac{x_1}{1!}\right)^{j_1} \left(\frac{x_2}{2!}\right)^{j_2} \cdots \left(\frac{x_{n-k+1}}{(n-k+1)!}\right)^{j_{n-k+1}}, \quad \text{(A4)}$$

where where the sum is taken over all sequences $j_1, j_2, j_3, \ldots, j_{n-k+1}$ of non-negative integers such that these two conditions are satisfied: $j_1 + j_2 + \cdots + j_{n-k+1} = k$,$j_1 + j_2 + \cdots + j_{n-k+1} = k$, and $j_1 + 2j_2 + 3j_3 + \cdots + (n-k+1)j_{n-k+1} = n$.

Proof of Proposition A1

Expanding the moment generating function, we have

$$M(x) = \sum_{n=0}^{\infty} I_n \frac{x^n}{n!},$$

where

$$I_n = \int_{\frac{1}{\overline{\gamma}}}^{\frac{1}{\gamma}} j(u) u^n du, \; n \in \{0, 1, 2, \ldots\}.$$

Hence, we have the explicit expression

$$\sum_{n=0}^{\infty} I_n(-\eta(\omega_t))^n = e^{\omega_t}. \tag{A5}$$

We can use (A5) to compute $\eta(\omega_t)$ explicitly. We rewrite (A5) as

$$\sum_{n=1}^{\infty} I_n(-\eta(\omega_t)) = e^{\omega_t} - I_0.$$

We can then use Lagrange's Inversion Theorem (see Bhamra and Uppal, 2014) to show that

$$-\eta(\omega_t) = \sum_{n=0}^{\infty} J_n \frac{(e^{\omega_t} - I_0)^n}{n!},$$

where

$$J_1 = \frac{1}{I_1}$$

$$J_n = \frac{1}{I_1^n} \sum_{k=1}^{n-1} (-)^k n^{(k)} B_{n-1,k}\left(\widehat{I}_1, \ldots, \widehat{I}_{n-k}\right)$$

$$\widehat{I}_k = \frac{\widehat{I}_{k+1}}{(k+1)I_1},$$

and $B_{n,k}(x_1, x_2, \ldots, x_{n-k+1})$ is the exponential Bell polynomial defined in (A4). ■

Definition A1 *The aggregate prudence in the economy at time-t is defined to be*

$$P(\omega_t) = A(\omega_t) \int_{\underline{\gamma}}^{\overline{\gamma}} (1+\gamma) w_\gamma(\omega_t) d\gamma, \tag{A6}$$

where $w_\gamma(\omega_t)$ is a density function for γ

$$w_\gamma(\omega_t) = \frac{\left(\frac{R(\omega_t)}{\gamma}\right)^2 \nu_\gamma(\omega_t)}{\int_{\underline{\gamma}}^{\overline{\gamma}} \left(\frac{R(\omega_t)}{\gamma}\right)^2 \nu_\gamma(\omega_t) d\gamma},$$

and

$$A(\omega_t) = \int_{\underline{\gamma}}^{\overline{\gamma}} \left(\frac{R(\omega_t)}{\gamma}\right)^2 \nu_\gamma(\omega_t) d\gamma.$$

Proposition A2 *The dynamics of the consumption-share density are given by*

$$d\nu_\gamma(\omega_t) = E_t[d\nu_\gamma(\omega_t)] - \left(1 - \frac{1}{\gamma}R(\omega_t)\right)\nu_\gamma(\omega_t)\sigma dZ_{1,t},$$

where

$$\frac{1}{\nu_\gamma(\omega_t)}E_t\left[\frac{d\nu_\gamma(\omega_t)}{dt}\right] = -\left(1 - \frac{1}{\gamma}R(\omega_t)\right)E_t\left[\frac{d\omega_t}{dt}\right] + \frac{1}{2}\left[\left(\frac{R(\omega_t)}{\gamma} - 1\right)^2 - (P(\omega_t) - 1 - R(\omega_t))\frac{R(\omega_t)}{\gamma}\right]\sigma^2.$$

Aggregate risk aversion is given by $-\eta'(\omega_t)$ and the dynamics of aggregate risk aversion are given by

$$dR(\omega_t) = E_t[dR(\omega_t)] - R^3(\omega_t)V(\omega_t)\sigma dZ_{1,t} \tag{A7}$$

where

$$V(\omega_t) = \int_{\underline{\gamma}}^{\overline{\gamma}} \frac{1}{\gamma^2}\nu_\gamma(\omega_t)d\gamma - \left(\int_{\underline{\gamma}}^{\overline{\gamma}} \frac{1}{\gamma}\nu_\gamma(\omega_t)d\gamma\right)^2$$

is the time-t variance of risk tolerance and

$$E_t[dR(\omega_t)] = -R^3(\omega_t)V(\omega_t)E_t[d\omega_t] + \frac{1}{2}R''(\omega_t)\sigma^2 dt,$$

where

$$R''(\omega_t) = R(\omega_t) - 3R^3(\omega_t)\mathcal{M}_2(\omega_t) + 3R^5(\omega_t)\mathcal{M}_2^2(\omega_t) - R^4(\omega_t)\mathcal{M}_3(\omega_t),$$

and

$$\mathcal{M}_n(\omega_t) = \int_{\underline{\gamma}}^{\overline{\gamma}} \frac{1}{\gamma^n}\nu_\gamma(\omega_t)d\gamma,\ n \in \mathbb{Z}^+,$$

is the n'th moment of risk tolerance.

The following results hold

1. $d\nu_\gamma(\omega_t) - E_t[d\nu_{\gamma,t}] > 0$ *if and only if* $dZ_{1,t} > 0$ *and* $\gamma < R(\omega_t)$ *or* $dZ_{1,t} < 0$ *and* $\gamma > R(\omega_t)$

2. $dR(\omega_t) - E_t[dR(\omega_t)] < 0$ *if and only if* $dZ_{1,t} > 0$

Proof of Proposition A2

By differentiating the integral equation (A3) implicitly with respect to ω_t, we obtain

$$-\eta'(\omega_t)\int_{\underline{\gamma}}^{\overline{\gamma}} \frac{1}{\gamma} f(\gamma)^{\frac{1}{\gamma}} e^{-\frac{1}{\gamma}\eta(\omega_t)} d\gamma = e^{\omega_t}, \tag{A8}$$

and so

$$-\eta'(\omega_t)\int_{\underline{\gamma}}^{\overline{\gamma}} \frac{1}{\gamma}\nu_\gamma(\omega_t) d\gamma = 1,$$

implying that

$$-\eta'(\omega_t) = \frac{1}{\int_{\underline{\gamma}}^{\overline{\gamma}} \frac{1}{\gamma}\nu_\gamma(\omega_t) d\gamma}.$$

Using (2), we therefore denote $-\eta'(\omega_t)$ via $R(\omega_t)$, i.e.

$$R(\omega_t) = -\eta'(\omega_t).$$

Observe that

$$\frac{\partial \nu_\gamma(\omega_t)}{\partial \omega_t} = -\left(1 - \frac{1}{\gamma}R(\omega_t)\right)\nu_\gamma(\omega_t),$$

and $\nu_\gamma(\omega_t) > 0$. We thus see that $\frac{\partial \nu_\gamma(\omega_t)}{\partial \omega_t} > 0$ if and only if $\gamma < R(\omega_t)$. From Ito's Lemma

$$d\nu_\gamma(\omega_t) - E_t[d\nu_\gamma(\omega_t)] = \frac{\partial \nu_\gamma(\omega_t)}{\partial \omega_t}\sigma dZ_t,$$

where $d\nu_\gamma(\omega_t) - E_t[d\nu_\gamma(\omega_t)]$ is the unexpected change in the consumption-share density. Hence, for a given positive aggregate shock, $dZ_{1,t} > 0$, $d\nu_\gamma(\omega_t) - E_t[d\nu_\gamma(\omega_t)] > 0$ if and only if $\gamma < R(\omega_t)$. Furthermore,

$$\begin{aligned} E_t\left[\frac{d\nu_\gamma(\omega_t)}{dt}\right] &= \frac{\partial \nu_\gamma(\omega_t)}{\partial \omega_t} d\omega_t + \frac{1}{2}\frac{\partial^2 \nu_\gamma(\omega_t)}{\partial \omega_t^2}(d\omega_t)^2 \\ &= -\left(1 - \frac{1}{\gamma}R(\omega_t)\right)\nu_\gamma(\omega_t) d\omega_t + \frac{1}{2}\left[-\left(1 - \frac{1}{\gamma}R'(\omega_t)\right)\nu_\gamma(\omega_t) - \left(1 - \frac{1}{\gamma}R(\omega_t)\right)\nu'_\gamma(\omega_t)\right]\sigma^2 dt \\ &= -\left(1 - \frac{1}{\gamma}R(\omega_t)\right)\nu_\gamma(\omega_t) d\omega_t + \frac{1}{2}\left[-\left(1 - \frac{1}{\gamma}R'(\omega_t)\right)\nu_\gamma(\omega_t) + \left(1 - \frac{1}{\gamma}R(\omega_t)\right)^2\nu_\gamma(\omega_t)\right]\sigma^2 dt \end{aligned}$$

$$\frac{1}{\nu_\gamma(\omega_t)}E_t\left[\frac{d\nu_\gamma(\omega_t)}{dt}\right] = -\left(1 - \frac{1}{\gamma}R(\omega_t)\right) d\omega_t + \frac{1}{2}\left[-\left(1 - \frac{1}{\gamma}R'(\omega_t)\right) + \left(1 - \frac{1}{\gamma}R(\omega_t)\right)^2\right]\sigma^2 dt.$$

We now differentiate (A8) implicitly wrt ω to obtain

$$-\eta''(\omega_t)\int_{\underline{\gamma}}^{\overline{\gamma}}\frac{1}{\gamma}f(\gamma)^{\frac{1}{\gamma}}e^{-\frac{1}{\gamma}\eta(\omega_t)}d\gamma+(\eta'(\omega_t))^2\int_{\underline{\gamma}}^{\overline{\gamma}}\frac{1}{\gamma^2}f(\gamma)^{\frac{1}{\gamma}}e^{-\frac{1}{\gamma}\eta(\omega_t)}d\gamma=e^{\omega_t},$$

and so

$$-\eta''(\omega_t)R(\omega_t)^{-1}+(R(\omega_t))^2\int_{\underline{\gamma}}^{\overline{\gamma}}\frac{1}{\gamma^2}\nu_\gamma(\omega_t)d\gamma=1,$$

which implies that

$$\eta''(\omega_t)=R(\omega_t)^3V(\omega_t),$$

where

$$V(\omega_t)=\int_{\underline{\gamma}}^{\overline{\gamma}}\frac{1}{\gamma^2}\nu_\gamma(\omega_t)d\gamma-\left(\int_{\underline{\gamma}}^{\overline{\gamma}}\frac{1}{\gamma}\nu_\gamma(\omega_t)d\gamma\right)^2,$$

is the variance of $\frac{1}{\gamma}$ weighted using the consumption-share density. Therefore,

$$R'(\omega_t)=-R(\omega_t)^3V(\omega_t). \tag{A9}$$

Hence,

$$\frac{1}{\nu_\gamma(\omega_t)}E_t\left[\frac{d\nu_\gamma(\omega_t)}{dt}\right]=-\left(1-\frac{1}{\gamma}R(\omega_t)\right)E_t\left[\frac{d\omega_t}{dt}\right]+\frac{1}{2}\left[-\left(1+\frac{1}{\gamma}R(\omega_t)^3V(\omega_t)\right)+\left(1-\frac{1}{\gamma}R(\omega_t)\right)^2\right]\sigma^2.$$

From the definition of aggregate prudence in (A6), it follows that

$$\begin{aligned}
P(\omega_t)&=\int_{\underline{\gamma}}^{\overline{\gamma}}(1+\gamma)\left(\frac{R(\omega_t)}{\gamma}\right)^2\nu_{\gamma,t}d\gamma\\
&=R(\omega_t)^2\int_{\underline{\gamma}}^{\overline{\gamma}}\frac{1+\gamma}{\gamma^2}\nu_{\gamma,t}d\gamma\\
&=R(\omega_t)^2\left(\int_{\underline{\gamma}}^{\overline{\gamma}}\frac{1}{\gamma}\nu_{\gamma,t}d\gamma+\int_{\underline{\gamma}}^{\overline{\gamma}}\frac{1}{\gamma^2}\nu_{\gamma,t}d\gamma\right)\\
&=R(\omega_t)^2\left(R(\omega_t)^{-1}+\int_{\underline{\gamma}}^{\overline{\gamma}}\frac{1}{\gamma^2}\nu_{\gamma,t}d\gamma\right)\\
&=R(\omega_t)+R(\omega_t)^2\int_{\underline{\gamma}}^{\overline{\gamma}}\frac{1}{\gamma^2}\nu_{\gamma,t}d\gamma\\
&=1+R(\omega_t)+R(\omega_t)^2\int_{\underline{\gamma}}^{\overline{\gamma}}\frac{1}{\gamma^2}\nu_{\gamma,t}d\gamma-1
\end{aligned}$$

$$= 1 + R(\omega_t) + R(\omega_t)^2 \left[\int_{\underline{\gamma}}^{\overline{\gamma}} \frac{1}{\gamma^2} \nu_{\gamma,t} d\gamma - R(\omega_t)^{-2} \right]$$

$$= 1 + R(\omega_t) + R(\omega_t)^2 V(\omega_t)$$

$$= 1 + R(\omega_t)(1 + R(\omega_t)V(\omega_t)). \tag{A10}$$

Therefore,

$$R(\omega_t)^3 V(\omega_t) = P(\omega_t) - (1 + R(\omega_t)),$$

Hence,

$$\frac{1}{\nu_\gamma(\omega_t)} E_t \left[\frac{d\nu_\gamma(\omega_t)}{dt} \right] = - \left(1 - \frac{1}{\gamma} R(\omega_t) \right) E_t \left[\frac{d\omega_t}{dt} \right] + \frac{1}{2} \left[\left(\frac{R(\omega_t)}{\gamma} - 1 \right)^2 - (P(\omega_t) - 1 - R(\omega_t)) \frac{R(\omega_t)}{\gamma} \right] \sigma^2.$$

Applying Ito's Lemma to $R(\omega_t)$ gives

$$dR(\omega_t) = R'(\omega_t) d\omega_t + \frac{1}{2} R''(\omega_t)(d\omega_t)^2$$

$$= E_t[dR(\omega_t)] + R'(\omega_t)\sigma dZ_{1,t}$$

$$= E_t[dR(\omega_t)] - R^3(\omega_t)V(\omega_t)\sigma dZ_{1,t},$$

where we have used (A9), and

$$E_t[dR(\omega_t)] = -R^3(\omega_t)V(\omega_t)E_t[d\omega_t] + \frac{1}{2} R''(\omega_t)\sigma^2 dt.$$

From (A9) it follows that

$$R'(\omega_t) = -R^3(\omega_t) \left[\mathcal{M}_2(\omega_t) - R^{-2}(\omega_t) \right]$$

$$= R(\omega_t) - R^3(\omega_t)\mathcal{M}_2(\omega_t),$$

where

$$\mathcal{M}_n(\omega_t) = \int_{\underline{\gamma}}^{\overline{\gamma}} \frac{1}{\gamma^n} \nu_\gamma(\omega_t) d\gamma, \; n \in \mathcal{Z}^+.$$

Observe that

$$\mathcal{M}_n'(\omega_t) = \int_{\underline{\gamma}}^{\overline{\gamma}} \frac{1}{\gamma^n} \nu_\gamma'(\omega_t) d\gamma$$

$$= \int_{\underline{\gamma}}^{\overline{\gamma}} \frac{1}{\gamma^n} \left(\frac{R(\omega_t)}{\gamma} - 1 \right) \nu_\gamma(\omega_t) d\gamma$$

$$= R(\omega_t)\mathcal{M}_{n+1}(\omega_t) - \mathcal{M}_n(\omega_t).$$

Therefore

$$\begin{aligned}
R''(\omega_t) &= R'(\omega_t) - 3R^2(\omega_t)R'(\omega_t)\mathcal{M}_2(\omega_t) - R^3(\omega_t)\mathcal{M}_2'(\omega_t) \\
&= R'(\omega_t) - 3R^2(\omega_t)R'(\omega_t)\mathcal{M}_2(\omega_t) - R^3(\omega_t)[R(\omega_t)\mathcal{M}_3(\omega_t) - \mathcal{M}_2(\omega_t)] \\
&= R'(\omega_t) - 3R^2(\omega_t)R'(\omega_t)\mathcal{M}_2(\omega_t) - R^4(\omega_t)\mathcal{M}_3(\omega_t) + R^3(\omega_t)\mathcal{M}_2(\omega_t) \\
&= R(\omega_t) - R^3(\omega_t)\mathcal{M}_2(\omega_t) - 3R^2(\omega_t)R'(\omega_t)\mathcal{M}_2(\omega_t) - R^4(\omega_t)\mathcal{M}_3(\omega_t) + R^3(\omega_t)\mathcal{M}_2(\omega_t) \\
&= R(\omega_t) - 3R^2(\omega_t)R'(\omega_t)\mathcal{M}_2(\omega_t) - R^4(\omega_t)\mathcal{M}_3(\omega_t) \\
&= R(\omega_t) - 3R^2(\omega_t)[R(\omega_t) - R^3(\omega_t)\mathcal{M}_2(\omega_t)]\mathcal{M}_2(\omega_t) - R^4(\omega_t)\mathcal{M}_3(\omega_t) \\
&= R(\omega_t) - 3R^3(\omega_t)\mathcal{M}_2(\omega_t) + 3R^5(\omega_t)\mathcal{M}_2^2(\omega_t) - R^4(\omega_t)\mathcal{M}_3(\omega_t)
\end{aligned}$$

Since the variance of risk tolerance is strictly positive when there is heterogeneity in γ, i.e. $V(\omega_t) > 0$, it follows from (A7) that $dR(\omega_t) - E_t[dR(\omega_t)] < 0$ if and only if $dZ_{1,t} > 0$

■

Based on Proposition A2, we can think of households as being divided endogenously into two subclasses: (i) those for whom the benefits of positive aggregate shocks are amplified – the 'winners', and those for whom the benefits of positive aggregate shocks are muted – the 'left-behind'. We can then define the proportion of agents consisting of the 'left-behind' as follows:

Definition A2

$$L(\omega_t) = \int_{R(\omega_t)}^{\overline{\gamma}} \hat{f}(\gamma)d\gamma, \tag{A11}$$

where $\hat{f}(\gamma) = \frac{f(\gamma)}{\int_{\underline{\gamma}}^{\overline{\gamma}} f(\gamma)d\gamma}$.

The following proposition gives the dynamics of the 'left-behind' proportion of agents.

Proposition A3 *The the proportion of agents consisting of the 'left-behind' changes as follows*

$$dL(\omega_t) = \hat{f}(\gamma)(-E_t[dR(\omega_t)] + R^3(\omega_t)V(\omega_t)\sigma dZ_{1,t}). \tag{A12}$$

We can see that a positive shock to aggregate consumption leads to an unexpected increase in the proportion of 'left-behind' agents. The size of this unexpected increase is larger when aggregate risk aversion and the variance of risk tolerance are high.

Proof of Proposition A3

Applying Ito's Lemma to (A11) gives (A12). ■

Proof of Proposition 2

Markets are effectively dynamically complete. Therefore, each individual agent's stochastic discount factor (SDF) is the same at each time and for each state, and so

$$\begin{aligned}\Lambda_t &= f(\gamma)e^{-\delta t}u_C(C_{\gamma,t}/H_{\gamma,t}) \\ &= f(\gamma)e^{-\delta t}C_{\gamma,t}^{-\gamma}H_{\gamma,t}^{\gamma-1} \\ &= f(\gamma)e^{-\delta t}C_{\gamma,t}^{-\gamma}e^{\left(\gamma-\frac{1}{\psi}\right)x_t},\end{aligned} \tag{A13}$$

which reduces to

$$\Lambda_t = f(\gamma)e^{-\delta t}\nu_\gamma(\omega_t)^{-\gamma}Y_t^{-\gamma}e^{\left(\gamma-\frac{1}{\psi}\right)x_t}.$$

Substituting in our expression for $\nu_\gamma(\omega_t)$ from , we obtain

$$\Lambda_t = f(\gamma)e^{-\delta t-\frac{1}{\psi}y_t-\left(\gamma-\frac{1}{\psi}\right)\omega_t}\nu_\gamma(\omega_t)^{-\gamma},$$

and so

$$\begin{aligned}\Lambda_t &= f(\gamma)e^{-\delta t-\frac{1}{\psi}y_t-\left(\gamma-\frac{1}{\psi}\right)\omega_t}f(\gamma)^{-1}e^{\gamma\omega_t+\eta(\omega_t)} \\ &= e^{-\delta t-\frac{1}{\psi}(y_t-\omega_t)+\eta(\omega_t)}\end{aligned} \tag{A14}$$

Applying Ito's Lemma, we see that

$$\frac{d\Lambda_t}{\Lambda_t} = -\delta - \frac{1}{\psi}dy_t - \left(R(\omega_t) - \frac{1}{\psi}\right)d\omega_t + \frac{1}{2}R(\omega_t)^2\sigma^2dt - \frac{1}{2}R'(\omega_t)\sigma^2dt - R(\omega_t)\sigma dZ_t.$$

Using (A9) we see that

$$\begin{aligned}\frac{d\Lambda_t}{\Lambda_t} &= -\delta - \frac{1}{\psi}dy_t - \left(R(\omega_t) - \frac{1}{\psi}\right)d\omega_t + \frac{1}{2}R(\omega_t)^2\sigma^2dt + \frac{1}{2}R^3(\omega_t)V(\omega_t)\sigma^2dt - R(\omega_t)\sigma dZ_t \\ &= -\delta - \frac{1}{\psi}dy_t - \left(R(\omega_t) - \frac{1}{\psi}\right)d\omega_t + \frac{1}{2}R(\omega_t)\left(P(\omega_t)-1\right)\sigma^2dt - R(\omega_t)\sigma dZ_t,\end{aligned}$$

where we have used (A10) in the previous step. Therefore, we obtain the following dynamics for the SDF

$$\frac{d\Lambda_t}{\Lambda_t} = -\left(\delta + R(\omega_t)\mu_Y - \lambda\omega_t\left(R(\omega_t) - \frac{1}{\psi}\right) - \frac{1}{2}R(\omega_t)P(\omega_t)\sigma^2dt\right) - R(\omega_t)\sigma dZ_t.$$

Using the fact that

$$\frac{d\Lambda_t}{\Lambda_t} = -r_t dt - \Theta_t dZ_t,$$

where r_t is the date-t risk-free interest rate and Θ_t is the date-t price of risk, we see that

$$r_t = \delta + R(\omega_t)\mu_Y - \lambda\omega_t\left(R(\omega_t) - \frac{1}{\psi}\right) - \frac{1}{2}R(\omega_t)P(\omega_t)\sigma^2, \tag{A15}$$

$$\Theta_t = R(\omega_t)\sigma.$$

We can rewrite (A15) as

$$r_t = \delta + \frac{1}{\psi}\mu_y + \left(R(\omega_t) - \frac{1}{\psi}\right)\lambda(\overline{\omega} - \omega_t) - \frac{1}{2}R(\omega_t)(P(\omega_t) - 1)\sigma^2.$$

■

Proof of Proposition 3

We use the SDF given in (A14) to price a consumption strip, i.e.

$$\begin{aligned}\frac{P_{T-t}(\omega_t)}{Y_t} &= E_t\left[\frac{\Lambda_T}{\Lambda_t}\frac{Y_T}{Y_t}\right] \\ &= e^{-\delta(T-t)}E_t\left[\exp\left[\left(1 - \frac{1}{\psi}\right)(y_T - y_t) + \frac{1}{\psi}(\omega_T - \omega_t) + \eta(\omega_T) - \eta(\omega_t)\right]\right]\end{aligned}$$

Now observe that

$$\begin{aligned}e^{\left(1-\frac{1}{\psi}\right)(y_T-y_t)} &= e^{\left(1-\frac{1}{\psi}\right)[\mu_y(T-t)+\sigma(Z_T-Z_t)]} \\ &= e^{\left(1-\frac{1}{\psi}\right)\left[\left\{\mu_y+\frac{1}{2}\left(1-\frac{1}{\psi}\right)\sigma^2\right\}(T-t)+\sigma(Z_T-Z_t)\right]}e^{-\frac{1}{2}\left(1-\frac{1}{\psi}\right)^2\sigma^2(T-t)+\left(1-\frac{1}{\psi}\right)\sigma(Z_T-Z_t)} \\ e^{\left(1-\frac{1}{\psi}\right)(y_T-y_t)} &= e^{\left(1-\frac{1}{\psi}\right)[\mu_y(T-t)+\sigma(Z_T-Z_t)]} \\ &= e^{\left(1-\frac{1}{\psi}\right)\left[\left\{\mu_y+\frac{1}{2}\left(1-\frac{1}{\psi}\right)\sigma^2\right\}(T-t)\right]}\frac{M_T\left(\left(1-\frac{1}{\psi}\right)\sigma\right)}{M_t\left(\left(1-\frac{1}{\psi}\right)\sigma\right)}\end{aligned}$$

where

$$M_t(a) = e^{-\frac{1}{2}a^2t+aZ_t},$$

is an exponential martingale under the physical measure $\mathbb{P}$. Using this exponential martingale, we can define the new probability measure $\mathbb{P}^a$. Hence, We start with the time-t price of a bond which pay out a unit of consumption at time $T > t$, i.e.

$$B_{T-t}(\omega_t) = E_t\left[\frac{\Lambda_T}{\Lambda_t}\right].$$

Now we observe that

$$\begin{aligned}
\frac{\Lambda_u}{\Lambda_t} &= e^{-\delta(u-t)}e^{-\frac{1}{\psi_i}(y_u-y_t)}e^{\frac{1}{\psi}(\omega_u-\omega_t)+\eta(\omega_u)-\eta(\omega_t)} \\
&= e^{-\delta(u-t)}e^{-\frac{1}{\psi_i}\left[(\mu-\frac{1}{2}\sigma^2)(u-t)+\sigma(Z_u-Z_t)\right]}e^{\frac{1}{\psi}(\omega_u-\omega_t)+\eta(\omega_u)-\eta(\omega_t)} \\
&= e^{-\delta(u-t)}e^{-\frac{1}{\psi_i}\left[(\mu-\frac{1}{2}\sigma^2)(u-t)\right]}e^{\frac{1}{\psi_i}(\omega_u-\omega_t)+\eta(\omega_u)-\eta(\omega_t)}e^{-\frac{1}{\psi}\sigma(Z_u-Z_t)} \\
&= e^{-\hat{r}(u-t)}e^{-\frac{1}{2}\left[\left(\frac{1}{\psi}\right)^2\sigma^2\right](u-t)}e^{-\frac{1}{\psi}\sigma(Z_u-Z_t)}e^{\frac{1}{\psi}(\omega_u-\omega_t)+\eta(\omega_u)-\eta(\omega_t)},
\end{aligned}$$

where

$$\hat{r} = \delta + \frac{1}{\psi_i}\mu - \frac{1}{2}\frac{1}{\psi_i}\left(1+\frac{1}{\psi}\right)\sigma^2.$$

We define the following exponential martingale under the physical measure $\mathbb{P}$:

$$M_t(a) = e^{-\frac{1}{2}a^2t+aZ_t}.$$

Using the exponential martingale, we can define the new probability measure $\mathbb{P}^a$. Therefore,

$$B_{T-t}(\omega_t) = e^{-\hat{r}(T-t)}E_t\left[\frac{M(-\frac{1}{\psi_i}\sigma)_T}{M(-\frac{1}{\psi_i}\sigma)_t}e^{\frac{1}{\psi_i}(\omega_u-\omega_t)+\eta(\omega_u)-\eta(\omega_t)}\right],$$

where $M(-\frac{1}{\psi_i}\sigma)$ is the martingale $M(a)$ with $a = -\frac{1}{\psi_i}\sigma$. By changing the probability measure from $\mathbb{P}$ to $\mathbb{P}^{-\frac{1}{\psi_i}\sigma}$, we obtain

$$B_{T-t}(\omega_t) = e^{-\hat{r}(T-t)}E_t^{\mathbb{P}^{-\frac{1}{\psi_i}\sigma}}\left[e^{\frac{1}{\psi_i}(\omega_u-\omega_t)+\eta(\omega_u)-\eta(\omega_t)}\right].$$

We define the real bond yield $y_{T-t}(\omega_t)$ via

$$B_{T-t}(\omega_t) = e^{-y_{T-t}(\omega_t)(T-t)}.$$

Therefore

$$y_{T-t}(\omega_t) = -\frac{1}{T-t}\ln B_{T-t}(\omega_t).$$

Hence

$$\lim_{\tau\to\infty} y_\tau(\omega_t) = \hat{r}.$$

.

From Girsanov's Theorem it follows that under $\mathbb{P}^{\left(-\frac{1}{\psi_i}\right)\sigma}$, we have

$$d\omega_t = \lambda\left(\overline{\omega} + \frac{\left(-\frac{1}{\psi_i}\right)\sigma^2}{\lambda} - \omega_t\right)dt + \sigma dZ_t^{\left(-\frac{1}{\psi_i}\right)\sigma},$$

where $Z^{\left(-\frac{1}{\psi_i}\right)\sigma}$ is a standard Brownian motion under $\mathbb{P}^{\left(-\frac{1}{\psi_i}\right)\sigma}$.

We want to evaluate

$$p_{D,T-t}(\omega_t) = E_t\left[\frac{\Lambda_T D_T}{\Lambda_t D_t}\right].$$

We do so by a change of probability measure.

Observe that by applying Ito's Lemma to $\Lambda_t D_t$ and integrating the resulting stochastic differential equation, we can show that

$$\frac{\Lambda_T D_T}{\Lambda_t D_t} = e^{-\int_t^T k(\omega_u)du}\frac{M_{D,T}}{M_{D,t}},$$

where

$$k(\omega_t) = r(\omega_t) + R(\omega_t)\sigma(\rho\sigma_A + a\sigma) - \mu_D,$$

and M_D is an exponential martingale under $\mathbb{P}$, defined by

$$\frac{dM_{D,t}}{M_{D,t}} = [a - R(\omega_t))]\sigma dZ_t + \sigma_A dZ_{A,t}.$$

The new probability measure $\mathbb{P}_D$ is defined by M_D. Therefore,

$$p_{D,T-t}(\omega_t) = E_t^{\mathbb{P}_D}\left[e^{-\int_t^T k(\omega_u)du}\right]. \tag{A16}$$

We now derive the short-term dividend yield $y_{D,0}(\omega_t) = \lim_{\tau\to\infty} y_{D,\tau}(\omega_t)$. It follows from (A16) that

$$p_{D,\Delta t}(\omega_t) = e^{-k(\omega_t)\Delta t} + o(\Delta t).$$

Therefore,

$$y_{D,\Delta t}(\omega_t) = \frac{k(\omega_t)\Delta t + o(\Delta t)}{\Delta t}.$$

Letting $\Delta t \to 0$, we obtain

$$y_{D,0}(\omega_t) = k(\omega_t).$$

Using the Feynman-Kac Theorem, it follows from (A16) that

$$\frac{1}{2}\sigma^2 \partial_\omega^2 p_{D,\tau}(\omega) + E_t^{\mathbb{P}_D}\left[\frac{d\omega_t}{dt}\right] \partial_\omega p_{D,\tau}(\omega) - k(\omega) p_{D,\tau}(\omega) - \partial_\tau p_{D,\tau}(\omega) = 0, \tag{A17}$$

where

$$p_{D,0}(\omega) = 1.$$

To determine $E_t^{\mathbb{P}_D}\left[\frac{d\omega_t}{dt}\right]$, we apply Girsanov's Theorem:

$$E_t^{\mathbb{P}_D}[d\omega_t] = E_t[d\omega_t] + E_t\left[d\omega_t \frac{dM_{D,t}}{M_{D,t}}\right].$$

Therefore

$$E_t^{\mathbb{P}_D}\left[\frac{d\omega_t}{dt}\right] = \lambda\left(\frac{\mu_y + \{[a - R(\omega))]\sigma^2 + \rho\sigma_A\sigma\}}{\lambda} - \omega\right).$$

Therefore (A17) reduces to

$$\frac{1}{2}\sigma^2 \partial_\omega^2 p_{D,\tau}(\omega) + \lambda(\overline{\omega}_D - \omega)\partial_\omega p_{D,\tau}(\omega) - k(\omega) p_{D,\tau}(\omega) - \partial_\tau p_{D,\tau}(\omega) = 0,$$

where

$$\overline{\omega}_D = \overline{\omega} + \frac{[a - R(\omega))]\sigma^2 + \rho\sigma_A\sigma}{\lambda}$$

is the long-run mean of ω under $\mathbb{P}_D$.

We also want to evalulate the price-dividend ratio for the aggregate stock market. This is just the date-t price-dividend ratio on a claim which pays out the dividend flow D_u for $u \geq t$, i.e.

$$p_D(\omega_t) = E_t\left[\int_t^\infty \frac{\Lambda_u D_u}{\Lambda_t D_t} du\right].$$

Changing the probability measure from $\mathbb{P}$ to $\mathbb{P}_D$ gives

$$p_D(\omega_t) = E_t^{\mathbb{P}_D}\left[\int_t^\infty e^{-\int_t^u k(\omega_s)ds} du\right].$$

Using the Feynman-Kac Theorem, we obtain

$$\frac{1}{2}\sigma^2 p_D''(\omega_t) + \lambda\left(\frac{\mu_y + \{[a - R(\omega_t))]\sigma^2 + \rho\sigma_A\sigma\}}{\lambda} - \omega_t\right) p_D'(\omega_t) - k(\omega_t) p_D(\omega_t) + 1 = 0.$$

To find the long-term dividend yield, we perform a different change of measure. We observe that

$$
\begin{aligned}
&\frac{\Lambda_u D_u}{\Lambda_t D_t} \\
&= e^{-\delta(u-t)} e^{-\frac{1}{\psi_i}(y_u - y_t)} e^{\frac{1}{\psi_i}(\omega_u-\omega_t)+\eta(\omega_u)-\eta(\omega_t)} e^{\left(\mu_D - \frac{1}{2}\sigma_D^2\right)(u-t)+\sigma_D(Z_{D,u}-Z_{D,t})} \\
&= e^{-\delta(u-t)} e^{-\frac{1}{\psi_i}\left[(\mu-\frac{1}{2}\sigma^2)(u-t)+\sigma(Z_u-Z_t)\right]} e^{\left(\mu_D - \frac{1}{2}\sigma_D^2\right)(u-t)+\sigma_D(Z_{D,u}-Z_{D,t})} e^{\frac{1}{\psi_i}(\omega_u-\omega_t)+\eta(\omega_u)-\eta(\omega_t)} \\
&= e^{-\delta(u-t)} e^{-\frac{1}{\psi_i}\left[(\mu-\frac{1}{2}\sigma^2)(u-t)\right]} e^{\left(\mu_D - \frac{1}{2}\sigma_D^2\right)(u-t)} e^{\frac{1}{\psi_i}(\omega_u-\omega_t)+\eta(\omega_u)-\eta(\omega_t)} e^{-\frac{1}{\psi_i}\sigma(Z_u-Z_t)+\sigma_D(Z_{D,u}-Z_{D,t})} \\
&= e^{-\hat{k}(u-t)} e^{-\frac{1}{2}\left[\left(\frac{1}{\psi_i}\right)^2\sigma^2 - 2\frac{1}{\psi_i}\rho_D\sigma\sigma_D + \sigma_D^2\right](u-t)} e^{-\frac{1}{\psi_i}\sigma(Z_u-Z_t)+\sigma_D(Z_{D,u}-Z_{D,t})} e^{\frac{1}{\psi_i}(\omega_u-\omega_t)+\eta(\omega_u)-\eta(\omega_t)},
\end{aligned}
$$

where

$$\hat{k} = \delta + \frac{1}{\psi_i}\mu - \frac{1}{2}\frac{1}{\psi_i}\left(1+\frac{1}{\psi_i}\right)\sigma^2 + \frac{1}{\psi_i}\rho_D\sigma\sigma_D - \mu_D.$$

The exponential martingale we use to define the probability measure $\mathbb{L}$ is

$$\frac{dM_{L,t}}{M_{L,t}} = -\frac{1}{\psi_i}\sigma dZ_t + \sigma_D dZ_{D,t}.$$

Therefore, we can write

$$\frac{\Lambda_u D_u}{\Lambda_t D_t} = e^{-\hat{k}(u-t)}\frac{M_{L,u}}{M_{L,t}}.$$

Using the new probability measure $\mathbb{L}$, we obtain

$$p_{D,\tau}(\omega_t) = e^{-\hat{k}\tau} E_t^{\mathbb{L}}\left[e^{\frac{1}{\psi_i}(\omega_{t+\tau}-\omega_t)+\eta(\omega_{t+\tau})-\eta(\omega_t)}\right].$$

From the definition $y_{D,\tau}(\omega_t) = -\frac{1}{\tau}\ln p_{D,\tau}(\omega_t)$, it follows that

$$y_{D,\tau}(\omega_t) = \hat{k} - \frac{1}{\tau}\ln E_t^{\mathbb{L}}\left[e^{\frac{1}{\psi_i}(\omega_{t+\tau}-\omega_t)+\eta(\omega_{t+\tau})-\eta(\omega_t)}\right]$$

The long-term dividend yield is thus given by

$$
\begin{aligned}
\lim_{\tau\to\infty} y_{D,\tau}(\omega_t) &= \hat{k} \\
&= \delta + \frac{1}{\psi_i}\mu - \frac{1}{2}\frac{1}{\psi_i}\left(1+\frac{1}{\psi_i}\right)\sigma^2 + \frac{1}{\psi_i}\sigma(\rho\sigma_A + a\sigma) - \mu_D.
\end{aligned}
$$

We also observe that from Girsanov's Theorem, we obtain

$$E_t^{\mathbb{L}}\left[\frac{d\omega_t}{dt}\right] = \overline{\omega}_L - \omega_t,$$

where

$$\overline{\omega}_L = \frac{\mu_y - \frac{1}{\psi_i}\sigma^2 + \rho\sigma_A\sigma + a\sigma^2}{\lambda}.$$

■

Proposition A4 *Aggregate dividend dynamics are given by*

$$\frac{dD_t}{D_t} = \mu_D dt + \sigma_D dZ_{D,t},$$

where

$$\begin{aligned}
\mu_D &= \mu_A + a\left[\mu + \frac{1}{2}(a-1)\sigma^2 + \rho\sigma\sigma_A\right] \\
\sigma_D &= \sqrt{\sigma_A^2 + 2a\rho\sigma_A\sigma + a^2\sigma^2} \\
dZ_{D,t} &= \frac{\sigma_A}{\sigma_D}dZ_{A,t} + \frac{a\sigma}{\sigma_D}dZ_t \\
E_t[dZ_{D,t}dZ_t] &= \rho\frac{\sigma_A}{\sigma_D} + \frac{a\sigma}{\sigma_D}.
\end{aligned}$$

Proof of Proposition A4

Aggregate dividends are defined by (7). Using Ito's Lemma gives

$$\begin{aligned}
\frac{dD_t}{D_t} &= \frac{dA_t}{A_t} + \frac{d(Y_t^a)}{(Y_t^a)} + \frac{dA_t}{A_t}\frac{d(Y_t^a)}{(Y_t^a)} \\
&= \mu_A dt + \sigma_A dZ_{A,t} + a\left[\mu + \frac{1}{2}(a-1)\sigma^2\right]dt + a\sigma dZ_t + \rho a\sigma\sigma_A dt \\
&= \left\{\mu_A + a\left[\mu + \frac{1}{2}(a-1)\sigma^2\right] + \rho a\sigma\sigma_A\right\}dt + \sigma_A dZ_{A,t} + a\sigma dZ_t \\
&= \left\{\mu_A + a\left[\mu + \frac{1}{2}(a-1)\sigma^2 + \rho\sigma\sigma_A\right]\right\}dt + \sigma_A dZ_{A,t} + a\sigma dZ_t \\
&= \mu_D dt + \sigma_A dZ_{A,t} + a\sigma dZ_t \\
&= \mu_D dt + \sigma_D dZ_{D,t}
\end{aligned}$$

where μ_D, σ_D, and $dZ_{D,t}$ are given in the statement of the proposition.

Proof of Proposition 4

From

$$R(\omega_t) = -\eta'(\omega_t) = \frac{\partial}{\partial \omega} m^{-1}(\omega_t),$$

it follows that

$$\int_{\omega_0}^{\omega} R(x)dx = m^{-1}(\omega),$$

and so

$$m\left(\int_{\omega_0}^{\omega} R(x)dx\right) = \omega$$

We have

$$R(\omega) = \exp(b_0 - b_1\omega),$$

where $b_0 > 0$ and $b_1 > 0$. Therefore,

$$\int_{\omega_0}^{\omega} R(x)dx = A(1 - e^{-b_1(\omega-\omega_0)}),$$

where

$$A = \frac{e^{b_0 - b_1\omega_0}}{b_1} > 0.$$

Hence,

$$m\left(A(1 - e^{-b_1(\omega-\omega_0)})\right) = \omega.$$

Now define

$$z = A(1 - e^{-b_1(\omega-\omega_0)}),$$

which implies

$$\omega = \omega_0 + \ln\left(1 - \frac{z}{A}\right)^{-\frac{1}{b_1}}.$$

Hence,

$$m(z) = \omega_0 + \ln\left(1 - \frac{z}{A}\right)^{-\frac{1}{b_1}}.$$

Therefore, we obtain the moment generating function

$$M(z) = \exp(m(z)) = e^{\omega_0}\left(1 - \frac{z}{A}\right)^{-\frac{1}{b_1}}.$$

Recall that

$$M(x) = \int_0^\infty j(u)e^{xu}du,$$

which implies that

$$\int_0^\infty j(u)e^{-xu}du = e^{\omega_0}\left(1 + \frac{z}{A}\right)^{-\frac{1}{b_1}}$$

$$j(u) = u^{-2}\left[f(u^{-1})\right]^u,\ u \in [1/\overline{\gamma}, \psi].$$

Observe that $j(u)$ is not a probability density function, because it does not integrate to one. It follows that

$$M(-\eta(\omega_t)) = e^{\omega_t},$$

where

$$M(x) = \int_0^\infty j(u)e^{xu}du,$$

which is a Laplace transform. Inverting the Laplace transform gives

$$j(u) = \omega_0 e^{-Au}\frac{(Au)^{\frac{1}{b_1}}}{u\Gamma\left[\frac{1}{b_1}\right]}.$$

Hence,

$$f(u^{-1}) = \left(u^2\omega_0 e^{-Au}\frac{(Au)^{\frac{1}{b_1}}}{u\Gamma\left[\frac{1}{b_1}\right]}\right)^{\frac{1}{u}}$$

Therefore $f(x)$ is given by

$$f(x) = x^{-\frac{(b_1+1)x}{b_1}}A^{\frac{x}{b_1}}e^{x\omega_0 - A}\Gamma\left(\frac{1}{b_1}\right)^{-x},$$

where

$$A = \frac{e^{b_0 - b_1\omega_0}}{b_1} > 0.$$

Equation (12) follows.

Substituting (12) and (11) into (6) and simplifying gives (13).

As a check, observe that

$$\int_0^{\infty} \gamma^{-1} \nu_\gamma(\omega_t) d\gamma = \frac{1}{R(\omega_t)}.$$

■

Proof of Proposition 5

From (A13), we see that

$$C_{\gamma,t} = \left(e^{\delta t} e^{-\left(\gamma - \frac{1}{\psi}\right) x_t} \Lambda_t \right)^{-\frac{1}{\gamma}}.$$

Therefore

$$\Lambda_t C_{\gamma,t} = \left(e^{\delta t} e^{-\left(\gamma - \frac{1}{\psi}\right) x_t} \right)^{-\frac{1}{\gamma}} \Lambda_t^{1-\frac{1}{\gamma}}.$$

Taking logs, we obtain

$$\ln(\Lambda_t C_{\gamma,t}) = -\frac{\delta}{\gamma} t + \frac{1}{\gamma}\left(\gamma - \frac{1}{\psi}\right) x_t + \left(1 - \frac{1}{\gamma}\right) \ln \Lambda_t.$$

We now apply Ito's Lemma and use the result (9) to obtain

$$d \ln(\Lambda_t C_{\gamma,t}) = -\left\{ \frac{1}{\gamma}\left[\delta + \left(\gamma - \frac{1}{\psi}\right)\omega_t\right] + \left(1 - \frac{1}{\gamma}\right)\left(r_t + \frac{1}{2} R_t^2 \sigma^2\right)\right\} - \left(1 - \frac{1}{\gamma}\right) R_t \sigma dZ_t.$$

Therefore,

$$\begin{aligned}
\frac{d(\Lambda_t C_{\gamma,t})}{\Lambda_t C_{\gamma,t}} &= d \ln(\Lambda_t C_{\gamma,t}) + \frac{1}{2}(d \ln(\Lambda_t C_{\gamma,t}))^2 \\
&= -\left\{ \frac{1}{\gamma}\left[\delta + \left(\gamma - \frac{1}{\psi}\right)\omega_t\right] + \left(1 - \frac{1}{\gamma}\right)\left(r_t + \frac{1}{2\gamma} R_t^2 \sigma^2\right)\right\} - \left(1 - \frac{1}{\gamma}\right) R_t \sigma dZ_t.
\end{aligned}$$

We now define the following exponential martingale under $\mathbb{P}$:

$$\frac{dM_{\gamma,t}}{M_{\gamma,t}} = -\left(1 - \frac{1}{\gamma}\right) R(\omega_t) \sigma dZ_t.$$

Therefore, we can write

$$\frac{d(\Lambda_t C_{\gamma,t})}{\Lambda_t C_{\gamma,t}} = -l_{\gamma,t} dt + \frac{dM_{\gamma,t}}{M_{\gamma,t}},$$

where

$$l_{\gamma,t} = \frac{1}{\gamma}\left[\delta + \left(\gamma - \frac{1}{\psi}\right)\omega_t\right] + \left(1 - \frac{1}{\gamma}\right)\left(r_t + \frac{1}{2\gamma}R_t^2\sigma^2\right).$$

From

$$h_\gamma(\omega_t) = E_t\left[\int_t^\infty \frac{\Lambda_u C_{\gamma,t}}{\Lambda_t C_{\gamma,t}} du\right],$$

we obtain

$$h_\gamma(\omega_t) = E_t^{\mathbb{P}^\gamma}\left[e^{-\int_t^u l(\omega_u)ds} du\right],$$

where $\mathbb{P}^\gamma$ is the probability measure defined by the exponential martingale $\mathbb{P}^\gamma$.

Under the measure $\mathbb{P}^\gamma$, we have

$$d\omega_t = \lambda\left[\frac{\mu_y + \left(1 - \frac{1}{\gamma}\right)R(\omega_t)\sigma^2}{\lambda} - \omega_t\right]dt + \sigma dZ_t,$$

It follows from the Feynman-Kac Theorem that

$$0 = \frac{1}{2}\sigma^2 h_\gamma''(\omega) + \lambda\left[\frac{\mu_y + \left(1 - \frac{1}{\gamma}\right)R(\omega_t)\sigma^2}{\lambda} - \omega\right]h_\gamma'(\omega) - l_\gamma(\omega)h_\gamma(\omega) + 1.$$

Using $h_\gamma(\omega)$, we can obtain

$$W_{\gamma,t} = W_{\gamma,t}(y_t, \omega_t) = e^{y_t}\nu_\gamma(\omega_t)h_\gamma(\omega_t).$$

Note that

$$\frac{C_{\gamma,t}}{W_{\gamma,t}} = \frac{1}{h_\gamma(\omega_t)}.$$

An individual agent faces the following stochastic optimal control problem.

$$J_{\gamma,t} = \sup_{(C_{\gamma,t})_{t\in[0,\infty)},(\phi_{\gamma,t})_{t\in[0,\infty)}} E_t\int_t^\infty e^{-\delta(u-t)}\frac{C_{\gamma,u}^{1-\gamma}}{1-\gamma}X_u^{\gamma-\frac{1}{\psi}}du,$$

subject to

$$dW_{\gamma,t} = W_{\gamma,t}[r_t dt + \phi_{\gamma,t}(dR_t - r_t dt)] - C_{\gamma,t}dt,$$

where dR_t is the return on the claim to aggregate consumption over the interval $[t, t+dt)$ and $\phi_{\gamma,t}$ is the fraction of wealth $W_{\gamma,t}$ invested in this claim at time t. We know that

$$dx_t = \omega_t dt,$$

where

$$d\omega_t = \lambda(\overline{\omega} - \omega_t)dt + \sigma dZ_t,$$

so we can see that the exogenous state variables for this problem are x and ω.

The Hamilton-Jacobi-Bellman equation for an individual agent's problem is

$$0 = \sup_{C,\phi} \frac{C^{1-\gamma}}{1-\gamma} X^{\gamma-\frac{1}{\psi}} - \delta J + J_W E_t[dW] + J_\omega E_t[d\omega] + J_x dx$$

$$+ \frac{1}{2} J_{WW} E_t[(dW)^2] + J_{W\omega} E_t[dW d\omega] + \frac{1}{2} J_{\omega\omega} E_t[(d\omega)^2],$$

where we omit the index γ for ease of notation. We use the Ansatz

$$J = V(W, \omega) X^{\gamma-\frac{1}{\psi}}$$

to reduce the Hamilton-Jacobi-Bellman equation to

$$0 = \sup_{C,\phi} \frac{C^{1-\gamma}}{1-\gamma} + \lambda\omega\left(\gamma - \frac{1}{\psi}\right) V - \delta V + V_W E_t[dW] + V_\omega E_t[d\omega] +$$

$$\frac{1}{2} V_{WW} E_t[(dW)^2] + V_{W\omega} E_t[dW d\omega] + \frac{1}{2} V_{\omega\omega} E_t[(d\omega)^2].$$

We now use the standard Ansatz

$$V = g(\omega)^\gamma \frac{W_\gamma^{1-\gamma}}{1-\gamma}$$

to further reduce the Hamilton-Jacobi-Bellman equation to

$$0 = \sup_{C,\phi} \frac{C^{1-\gamma}}{1-\gamma} + (1-\gamma)V\left[r + \phi\left(\mu_R + \gamma\sigma\sigma_R \frac{g_\omega}{g}\right) - \frac{C}{W}\right] + \lambda(\overline{\omega} - \omega)\gamma\frac{g_\omega}{g} V -$$

$$\frac{1}{2}\sigma_R^2 \phi^2 \gamma(1-\gamma)V + \frac{1}{2}\sigma^2\gamma V\left[\frac{g_{\omega\omega}}{g} + (\gamma - 1)\left(\frac{g_\omega}{g}\right)^2\right],$$

where

$$\mu_R = E_t\left[\frac{dR}{dt}\right]$$

$$\sigma_R = E_t\left[\frac{(dR)^2}{dt}\right]$$

The FOC for consumption is

$$C^{-\gamma} = (1-\gamma)\frac{V}{W},$$

which implies that

$$\frac{W_{\gamma,t}}{C_{\gamma,t}} = g_\gamma(\omega_t),$$

where we have reintroduced the γ index for clarity. The above equation implies that

$$g_\gamma(\omega_t) = h_\gamma(\omega).$$

The optimal portfolio choice problem reduces to

$$\sup_\phi \phi\left(\mu_R + \gamma\sigma\sigma_R\frac{g_\omega}{g}\right) - \frac{1}{2}\sigma_R^2\phi^2\gamma,$$

which has the unique solution

$$\phi_\gamma(\omega) = \frac{1}{\gamma}\frac{\mu_R(\omega) + \gamma\sigma\sigma_R(\omega)\frac{g_\gamma'(\omega)}{g_\gamma(\omega)} - r(\omega)}{\sigma_R(\omega)^2}.$$

Therefore

$$\phi_\gamma(\omega) = \frac{1}{\gamma}\frac{\mu_R(\omega) + \gamma\sigma\sigma_R(\omega)\frac{h_\gamma'(\omega)}{h_\gamma(\omega)} - r(\omega)}{\sigma_R(\omega)^2}.$$

From

$$P_{C,t} = p_{C,t}Y_t,$$

and using Ito's Lemma, we obtain

$$dR_t = \frac{dY_t}{Y_t} + \frac{p_C'(\omega_t)}{p_C(\omega)}d\omega_t + \frac{1}{2}\frac{p_C''(\omega_t)}{p_C(\omega)}\sigma^2 dt + \frac{1}{p_C(\omega)}dt.$$

Therefore,

$$\sigma_R(\omega) = \left(1 + \frac{p_C'(\omega)}{p_C(\omega)}\right)\sigma.$$

From the basic asset pricing equation

$$E_t\left[dR_t - r_t dt\right] = -E_t\left[\frac{d\Lambda_t}{\Lambda_t} dR_t\right]$$
$$= \left(1 + \frac{p_C'(\omega)}{p_C(\omega)}\right) R(\omega)\sigma^2 dt,$$

and so

$$\mu_R(\omega) - r(\omega) = \left(1 + \frac{p_C'(\omega)}{p_C(\omega)}\right) R(\omega)\sigma^2.$$

Therefore, we obtain (15).

The amount of wealth held in risky assets by agents with relative risk aversion γ is given by $W_{\gamma,t}\phi_{\gamma,t}$. Their amount of wealth held in the risk-free bond is $W_{\gamma,t}(1-\phi_{\gamma,t})$. The fraction of aggregate wealth held by agents with relative risk aversion γ in risky assets is $\phi_{\gamma,t}\zeta_t$.

Define $\gamma^*(\omega)$ via

$$\phi_{\gamma^*(\omega)}(\omega) = 1.$$

Households with $\gamma < \gamma^*(\omega)$ have levered positions in the risky asset, i.e. $\phi_{\gamma,t} > 1$ and are borrowers in the bond market. Households with $\gamma > \gamma^*(\omega)$ have $\phi_{\gamma,t} < 1$ and are lenders in the bond market. Open interest in the bond market is given by

$$-\int_0^{\gamma^*(\omega)} W_{\gamma,t}(1-\phi_{\gamma,t})d\gamma,$$

which is equal to $\int_{\gamma^*(\omega)}^{\infty} W_{\gamma,t}(1-\phi_{\gamma,t})d\gamma$ by virtue of bond market clearing, because the bond is in zero net-supply. Open-interest in the bond market relative to aggregate wealth is given by

$$O(\omega_t) = -\int_0^{\gamma^*(\omega_t)} \zeta_{\gamma,t}(1-\phi_{\gamma,t})d\gamma.$$

Define $\widehat{\gamma}(\omega)$ via

$$\phi_{\widehat{\gamma}(\omega)}(\omega) = 0.$$

Households with $\gamma < \widehat{\gamma}(\omega)$ have short positions in the risky asset. The total value of these short positions is

$$\int_{\widehat{\gamma}(\omega)}^{\infty} W_{\gamma,t}\phi_{\gamma,t}d\gamma.$$

The value of short positions in the risky asset relative to aggregate wealth is given by

$$S(\omega_t) = \int_{\widehat{\gamma}(\omega_t)}^{\infty} \zeta_{\gamma,t}\phi_{\gamma,t} d\gamma.$$

Observe that

$$\ln \zeta_\gamma(\omega_t) = \ln \nu_\gamma(\omega_t) + \ln h_\gamma(\omega_t) - \ln p_C(\omega_t),$$

and so

$$d\ln \zeta_\gamma(\omega_t) = d\ln \nu_\gamma(\omega_t) + d\ln h_\gamma(\omega_t) - d\ln p_C(\omega_t).$$

Hence,

$$d\ln \zeta_\gamma(\omega_t) = d\ln \nu_\gamma(\omega_t) + \frac{dh_\gamma(\omega_t)}{h_\gamma(\omega_t)} - \frac{1}{2}\left(\frac{dh_\gamma(\omega_t)}{h_\gamma(\omega_t)}\right)^2 - \frac{dp_C(\omega_t)}{p_C(\omega_t)} + \frac{1}{2}\left(\frac{dp_C(\omega_t)}{p'_C(\omega_t)}\right)^2.$$

Observe that

$$\begin{aligned}
dp_C(\omega_t) &= p'_C(\omega_t)d\omega_t + \frac{1}{2}p''_C(\omega_t)(d\omega_t)^2 \\
&= p'_C(\omega_t)d\omega_t + \frac{1}{2}p''_C\sigma^2 dt \\
&= p'_C(\omega_t)d\omega_t - \lambda\left(\frac{\mu_y + (1-R(\omega))\sigma^2}{\lambda} - \omega_t\right)p'_C(\omega_t)dt + k(\omega)p_C(\omega)dt - dt \\
&= p'_C(\omega_t)[\lambda\left(\lambda^{-1}\mu_y - \omega_t\right)dt + \sigma dZ_t] - \lambda\left(\frac{\mu_y + (1-R(\omega))\sigma^2}{\lambda} - \omega_t\right)p'_C(\omega_t)dt + k(\omega)p_C(\omega)dt - dt \\
&= [(R(\omega)-1)\sigma^2 p'_C(\omega_t) + k(\omega)p_C(\omega) - 1]dt + p'_C(\omega_t)\sigma dZ_t
\end{aligned}$$

where

$$k(\omega) = r(\omega) + R(\omega)\sigma^2 - \mu_Y.$$

Similarly, we have

$$dh_\gamma(\omega) = \left[\left(\frac{1}{\gamma} - 1\right)R(\omega)\sigma^2 h'_\gamma(\omega_t) + l_\gamma(\omega)h_\gamma(\omega) - 1\right]dt + h'_\gamma(\omega_t)\sigma dZ_t,$$

where

$$k(\omega) = r(\omega) + R(\omega)\sigma^2 - \mu_Y.$$

Also,

$$d\ln \nu_\gamma(\omega_t) = -\left(1 - \frac{1}{\gamma}R(\omega_t)\right)E_t\left[\frac{d\omega_t}{dt}\right] - \frac{1}{2}(P(\omega_t) - 1 - R(\omega_t))\frac{R(\omega_t)}{\gamma}\sigma^2 dt - \left(1 - \frac{1}{\gamma}R(\omega_t)\right)\sigma dZ_t$$

Therefore,

$$
\begin{aligned}
d\ln\zeta_\gamma(\omega_t) = &-\left(1-\frac{1}{\gamma}R(\omega_t)\right)E_t\left[\frac{d\omega_t}{dt}\right]-\frac{1}{2}(P(\omega_t)-1-R(\omega_t))\frac{R(\omega_t)}{\gamma}\sigma^2dt-\left(1-\frac{1}{\gamma}R(\omega_t)\right)\sigma dZ_t\\
&+\left[\left(\frac{1}{\gamma}-1\right)R(\omega)\sigma^2\frac{h'_\gamma(\omega_t)}{h_\gamma(\omega_t)}+l_\gamma(\omega)-\frac{1}{h_\gamma(\omega_t)}\right]dt+\frac{h'_\gamma(\omega_t)}{h_\gamma(\omega_t)}\sigma dZ_t-\frac{1}{2}\sigma^2\left(\frac{h'_\gamma(\omega_t)}{h_\gamma(\omega_t)}\right)^2dt\\
&-\left[(R(\omega)-1)\sigma^2\frac{p'_C(\omega_t)}{p_C(\omega)}+k(\omega)-\frac{1}{p_C(\omega)}\right]dt-\frac{p'_C(\omega_t)}{p_C(\omega)}\sigma dZ_t+\frac{1}{2}\sigma^2\left(\frac{p'_C(\omega_t)}{p_C(\omega)}\right)^2dt\\
=&-\left(1-\frac{1}{\gamma}R(\omega_t)\right)E_t\left[\frac{d\omega_t}{dt}\right]-\frac{1}{2}(P(\omega_t)-1-R(\omega_t))\frac{R(\omega_t)}{\gamma}\sigma^2dt\\
&+\left[\left(\frac{1}{\gamma}-1\right)R(\omega)\sigma^2\frac{h'_\gamma(\omega_t)}{h_\gamma(\omega_t)}+l_\gamma(\omega)-\frac{1}{h_\gamma(\omega_t)}\right]dt-\frac{1}{2}\sigma^2\left(\frac{h'_\gamma(\omega_t)}{h_\gamma(\omega_t)}\right)^2dt\\
&-\left[(R(\omega)-1)\sigma^2\frac{p'_C(\omega_t)}{p_C(\omega)}+k(\omega)-\frac{1}{p_C(\omega)}\right]dt+\frac{1}{2}\sigma^2\left(\frac{p'_C(\omega_t)}{p_C(\omega)}\right)^2dt\\
&-\left(1-\frac{1}{\gamma}R(\omega_t)\right)\sigma dZ_t+\frac{h'_\gamma(\omega_t)}{h_\gamma(\omega_t)}\sigma dZ_t-\frac{p'_C(\omega_t)}{p_C(\omega)}\sigma dZ_t\\
=&-\left(1-\frac{1}{\gamma}R(\omega_t)\right)E_t\left[\frac{d\omega_t}{dt}\right]-\frac{1}{2}(P(\omega_t)-1-R(\omega_t))\frac{R(\omega_t)}{\gamma}\sigma^2dt\\
&+\left[\left(\frac{1}{\gamma}-1\right)R(\omega)\sigma^2\frac{h'_\gamma(\omega_t)}{h_\gamma(\omega_t)}+l_\gamma(\omega)-\frac{1}{h_\gamma(\omega_t)}\right]dt-\frac{1}{2}\sigma^2\left(\frac{h'_\gamma(\omega_t)}{h_\gamma(\omega_t)}\right)^2dt\\
&-\left[(R(\omega)-1)\sigma^2\frac{p'_C(\omega_t)}{p_C(\omega)}+k(\omega)-\frac{1}{p_C(\omega)}\right]dt+\frac{1}{2}\sigma^2\left(\frac{p'_C(\omega_t)}{p_C(\omega)}\right)^2dt\\
&+(\phi_\gamma(\omega_t)-1)\sigma_R(\omega_t)dZ_t
\end{aligned}
$$

■

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