Imperfect Competition and the Financialization of Commodities Markets

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Abstract

I study a futures market model with imperfectly competitive traders, some precluded to trade spot (financial traders), some not (physical traders). I first show that, suprisingly, introducing futures makes physical traders worse off without financial traders, because physical traders seek to influence futures payoff by trading spot, and choose negative hedging ratios. Financial traders improve futures market liquidity, so that physical traders adopt positive hedging ratios when liquidity is sufficiently improved. However financial traders also raise prices when they are long, which benefits high-inventory physical traders at the expense of low-inventory physical traders. Overall, physical traders with high or very low inventory are better off with financial traders than without futures, while traders with intermediate inventory and trading in the same direction as financial traders lose. I also show that imperfect competition makes futures and spot market imperfect substitutes, implying a spot-futures basis.

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1 Introduction

In commodities futures markets, the participation of traders who are not involved in underlying physical markets (henceforth "financial traders") has long triggered debate, especially since mid-2000s when they have massively entered US markets.[1] Among other issues, "excessive speculation" by such traders has been charged with inflating commodity prices and with increasing volatility.[2] On the other hand, proponents of the financialization have emphasized that financial traders offer additional risk-bearing capacity to physical traders, so that they participate in a well-functioning commodities market.

In this paper, I examine these issues in a model with physical and financial traders who are all risk averse and imperfectly competitive. Information is symmetric. The model is otherwise standard. I show that, surprisingly, introducing futures without financial traders makes at least some, often all physical traders worse off, because they want to influence futures payoff by trading spot, and they end up with negative hedging ratios. Financial traders improve futures market liquidity and increase physical traders' incentives to actually hedge with futures, which is beneficial, but affect the terms of spot trade, raising prices if they are long futures. In equilibrium with long financial traders, physical traders with high inventory or with very low inventory end up better off than without futures, while traders with low but intermediate inventory end up worse off.

I derive additional asset pricing implications. First, because of imperfect competition, the classical redundancy result between spot and futures market breaks down, creating a futures-spot basis. Second, although long futures traders make spot and futures prices go up, there is no bubble in the model: financial traders 1) compress spot and futures inventory risk premia and 2) make prices more reactive to news about a future liquidity shock. The latter is because physical traders face both a quantity and a price risk at date 1, which leads them to react less than one-for-one to news about the liquidity trade at date 0. Financial traders, who face only a price risk as they do not trade in the physical market, trade more aggressively and make the date-0 spot price react more to news, although not excessively to avoid negative returns.

In the model, all traders have the same risk preferences, and physical traders are endowed with different inventories of the risky asset, with an arbitrary distribution;

[1]See the survey by Cheng and Xiong (2014).

[2]See US Senate (2009). Markham (2014) reports this expression in congressional hearings in the 1920s already, then in a 1947 episode of wheat price surge.

there is no pre-existing futures position. At date 0, all traders trade futures and physical traders trade spot. At date 1, futures mature and physical traders trade spot, together with unmodelled liquidity traders. Liquidity trades affect the date-1 spot price and terms of trade between long and short physical traders; they are unknown at date 0. Information is symmetric. The risky asset pays off an uncertain amount at date 2.

I first show that, given imperfect competition, futures and spot trades at date 0 are not equivalent, so that the spot price does not equal the futures price. The classical equivalence result under perfect competition stems from the fact that in the short run, for a low-inventory trader, buying one unit spot saves (*i.e.*, pays off) the date-1 spot price; while for futures, entering a long position pays off the same spot price. (The reasoning is symmetric for large-inventory traders.) Yet this argument relies on the fact that one less unit purchased at date 0 translates into one more unit purchased at date 1. This is not the case when at date 1, competition is imperfect: one less unit purchased at date 0 entails less than one unit purchased at date 1. Therefore, physical traders value spot and futures differently, quantities are determinate, and a spot-futures basis arises in equilibrium without commonly invoked reasons to explain it (interest rate, storage costs, ...).

Then I study the equilibrium with only physical traders, with and without futures. Crucially, physical traders take the effect that date-0 spot trades on date-1 price into account, given other traders' equilibrium trades. To do so, a trader contemplates two aspects of date-1 price: first, what the date-1 spot price, and especially the inventory risk premium, looks like if he/she does not participate, and second, the impact of his/her trade on this price.

The first aspect is the most important and works as follows: the level of the price without this trader's participation affects his/her willingness to trade spot and futures at date 0 (futures pay off the date-1 spot price minus the futures price). Heterogenous traders contemplate different such spot prices: a low-inventory trader contemplates a higher inventory risk premium (a lower price) than a high-inventory trader when they are not in the market. Therefore, without futures, a low inventory trader is less willing to purchase spot and a high inventory traders is less willing to sell.

Then the second aspect of the intertemporal price impact works as follows: as low-inventory buy more and large-inventory traders sell more, they also recognize that deferring more trades to date 1 makes the price move unfavorably, which limits their incentive to defer trades to date 1. In equilibrium, because of this intertemporal price impact, traders trade less at date 0 than in a static market.

With futures, physical traders trade off the same impact on their spot trading strategies and on their futures payoff. For futures, a low-inventory traders sees a lower payoff for a long futures position than a high-inventory trader, again because the he/she sees a higher inventory risk premium (lower spot price). Hence low-inventory traders may end up with a short position in futures, and high-inventory traders, a long position. But this conflicts with the spot trading strategy: if a low-inventory trader purchases little at date 0 and short-sells futures, he/she ends up with a very large exposure to the risk that the date-1 spot price is higher than expected.

Thus a trader has two options: either to favor the price impact of spot trades on futures payoff, so that a low-inventory trader enters a short position in futures and buys spot more aggressively than in the static market, and symmetrically for high-inventory traders; or to favor the impact on spot trading, so that low-inventory (high-inventory) traders buy (sell) less spot and enter long (short) futures position. Between the two options, however, favoring the impact of date-0 spot trades on futures payoff is more profitable than favoring spot trading strategy. The intuition is as follows: by deferring one unit of spot purchase, a low-inventory trader earns less than one unit times the impact on date-1 price, again because of imperfect competition at date 1; while by entering a long position by one unit, the same trader earns one unit times the impact on date-1 price. A trader also weighs the cost of trading spot and futures at date 0 (date-0 price impact, and variation in holding costs associated with a trading strategy): but given that at date 0, spot and futures are substitutes, the associated trading costs are comparable. Therefore, traders prefer to favor the impact of their date-0 spot trade on futures payoff. A corollary to this is that their futures position decreases as their hedging needs increase: as date-1 price volatility stemming from the liquidity trade grows, the impact on futures payoff becomes more uncertain and less often profitable.

Thus, in the equilibrium without financial traders, physical traders end up with futures positions that are opposite to their spot trading needs at date 1: date-1 sellers are long futures, and date-1 buyers are short, which is costly for them. Yet they also trade spot quantities at date 0 that are closer to the competitive quantity, which entails more efficient risk sharing in the spot market. How is the net of the two? Again date-1 imperfect competition entails a higher marginal benefit of trading futures to benefit from a favorable price: by concavity of wealth,a trader's optimal choice involves equating the marginal cost of trading futures to the marginal benefit,which is higher, and implies a higher futures position. Therefore a trader ends up with too high a futures position with respect to spot quantities, and thus, when the expectation of the supply shock is not extreme, all traders are worse off with futures. The expectation of the supply shock acts on the terms of trade between spot buyers and sellers: when it is high in any direction, one side of the spot market gains at the expense of the other side – therefore, at least some traders, often all of them, are worse off with futures.

Then I consider the impact of financial traders, using the approach developed by Malamud and Rostek (2017) to determine the equilibrium slopes of residual demand schedules ("price impacts") that traders face. On top of bringing additional risk-bearing capacity to physical traders, financial traders improve futures market liquidity without deteriorating the spot market liquidity: in equilibrium, physical traders trade spot in the same quantities as without financial traders, and now share inventory risk through futures market. Therefore, there is overall more efficient risk sharing between physical traders.

I also study the impact of financial traders on prices. First, they increase spot and futures prices by compressing inventory risk premia: this simply reflects the additional risk-bearing capacity they bring into the market, which is positive for all traders and does not affect the terms of trade between low-inventory and high-inventory traders.

Second, they make the date-0 spot price react more strongly to news about the liquidity shock, but this reaction is in no obvious way excessive in the model. In fact, without financial traders, the spot price exhibits momentum: news about the date-1 liquidity shock are incorporated less than one-for-one into the spot price. This is because the liquidity shock at date 1 creates both a price and a quantity risk for physical traders: if they expect liquidity traders to buy, they try to build up additional inventories at date 1 at low price, to re-sell at a high price (and conversely if they expect liquidity traders to sell); but if liquidity traders sell instead, physical traders end up with excess inventory, thus higher marginal holding costs for all their inventories. Thus they prefer not to react too much to news if the associated uncertainty is high. This implies that on average, news about the spot price is incorporated partly, then fully at date 1: there is momentum. Financial traders reduce momentum, as they make the date-0 spot price incorporate news about the liquidity shock up to one-for-one. They have the same risk aversion, but face only price risk on their futures position: therefore, they behave as ordinary mean-variance traders, which turn out to be more aggressive.

However, financial traders may disconnect prices from their fundamental if they have an inelastic component of demand. When financial traders want to buy irrespective of expected returns, spot and futures prices go up. This gives a cautionary note on the financialization of commodity markets: if financial traders may increase prices beyond physical market fundamentals, it is not because of speculation. This calls for a better understanding of the determinants of non-speculative demand by financial traders, whose empirical relevance is given in particular by Henderson et al. (2014).

I now discuss the model assumptions. Importantly, I assume imperfect competition from all market participants.[3] In commodities markets, this seems a reasonable assumption: physical traders are typically large trading houses, large mining companies or farming cooperatives, and large downstream industrial companies. Moreover, Markham (2014) reports several cases of "battles" between large traders on opposite sides of futures markets, while troubles in the London Metal Exchange's nickel market in March 2022 involved large traders on both sides.

Second, all information is symmetric. Therefore, this paper complements the existing literature on the financialization of commodities markets, which typically focuses on information asymmetries, but in a perfectly competitive setting. The inefficiency result in the present context shows that transparency is not the ultimate condition to have well-functioning commodities markets. Both frictions are relevant in real-world markets, and studying their interaction is an interesting avenue for future research.

Third, there are no financing constraints: this is for analytical simplicity, and highlights issues that arise when financing constraints do not bind. For instance, a futures-spot basis arises in the present setting without commonly assumed constraints, because imperfect competition makes spot and futures imperfect substitutes. Again, studying the interaction of imperfect competition and financing constraints is an interesting avenue for future research.

Literature review. Existing theoretical analyses of the financialization focus on information frictions (Sockin and Xiong 2015, Goldstein and Yang 2022) or agency

[3] This contrasts with many papers on futures market manipulation, which typically assume that one monopolist manipulates futures payoff at the expense of competitive traders. See, among others, Easterbrook (1986), Kumar and Seppi (1992), Pirrong (1993), Jarrow (1994), Jarrow and Li (2021).

conflicts between investors and fund managers (Basak and Pavlova 2016). These papers assume that 1) traders are perfectly competitive and 2) cannot trade in the spot market simultaneously to futures. The latter is necessary to break the redundancy between spot and futures markets. The present paper complements this literature by focusing on imperfect competition, and assumes perfect information and no agency friction. In this way it also shows that imperfect competition creates problems of its own, that increased transparency cannot solve.

This paper also connects to Malamud and Rostek (2017), henceforth MR, in two ways. First, I use methods from MR to derive equilibrium. Second, MR shows that under when traders have different risk aversions, fragmenting markets may Pareto-dominate. In the present paper, I present another suprising result, that introducing futures, a substitute for spot trades, decreases traders' welfare when financial traders are absent. Yet this result arises for a different reason: here the inefficiency stems from intertemporal price impact, while in MR, all traders trade in all markets simultaneously.

It also relates to Zhang (2021), which studies a model in which manipulation creates basis risk for hedgers and affects the informativeness of prices; futures can decrease welfare, and futures position limits solve the problem. Our settings differ in important respects: in Zhang (2021), there are information asymmetries and no simultaneous trading in spot and futures market is possible.

The present paper also connects to the literature on futures and swaps as non-redundant assets. Grossman (1977) and Bray (1981) the informational role of futures. My paper has symmetric information and is closer to Rostek and Yoon (2021), which shows that under imperfect competition, non-redundant derivative products endogenously emerge, with a welfare impact that can be positive or negative. The mechanisms are very different however: in Rostek and Yoon (2021), the crucial ingredient is that traders have limited ability to condition demand in one asset on prices of other assets. Derivative products are built as portfolios of underlying assets and have the same maturity (*e.g.*, like CDS), and can generally increase or decrease welfare. In the present paper, traders can condition their demand schedule in one asset on other asset prices, and non-redundant futures arise because they have shorter maturity than the underlying; the unambiguously negative welfare effect arises because futures' payoff depends on the underlying spot price (like options, but unlike CDSs). Other papers motivate futures/swaps trading by trader heterogeneity: Oehmke and Zawadowski (2015, 2016) emphasizes differences in trading horizon, Biais et al. (2016) and Biais et al. (2019), some traders are specialized in managing the underlying asset. In Biais et al. (2021) differences in preferences imply that derivatives are needed to implement optimal risk sharing.

Finally, this paper connects to the literature on dynamic trading with imperfectly competitive double auctions (see Vayanos 1999, Du and Zhu 2017, Rostek and Weretka 2015). This paper is to my knowledge the first to make forward/futures contracts emerge in this context.

The paper is organized as follows. Section 2 presents the setting. Section 3 derives the competitive case, where futures and date-0 spot are perfect substitutes. Section 4 shows that under imperfect competition, date-0 spot and futures markets are imperfect substitutes and derives the futures-spot basis. Section 5 studies equilibrium trades and welfare without financial traders. Section 6 studies the impact of financial traders on market liquidity and equilibrium quantities. Section **??** discusses the pricing implication of the financialization of futures markets. Section 7 concludes.

2 Setting

There are three dates $t = 0, 1, 2$. There is one risky asset that pays off at $t = 2$ an *ex ante* unknown amount

$$v = v_0 + \epsilon_1 + \epsilon_2$$

per unit, where ϵ_1 and ϵ_2 are independent and normally distributed with mean 0 and respective variances $\sigma_1^2 \geq 0$ and $\sigma_2^2 > 0$.[4] At date 1, ϵ_1 is realized and observed before any action takes place. All information about ϵ_1 or ϵ_2 is symmetric. It is also possible to borrow and save cash at the risk-free rate normalized to zero.

There are two types of traders, physical and financial. Physical traders can trade the risky asset in spot markets, and physical trader $n = 1, ..., N$ is endowed with inventory $I_n \in \mathbb{R}$ of the risky asset. Denoting $\bar{I}_0 = \sum_n I_n/N$ the average physical trader inventory, I refer to traders with inventory I_n greater (lower) than $\bar{I}_0$ as *high-inventory traders* (*low-inventory traders*). Physical traders with large inventories can be thought of as commodity producers, while physical traders with low inventory

[4] The normality assumption is for tractability. It implies that the payoff can be negative without lower bound, which is not consistent with real world limited liability: one could use truncated normal distributions instead. At least for date 1 trade and for small probabilities of negative v, results are approximately identical with and without lower truncation of the probability distribution.

look like downstream industrial companies who consume the commodity. Physical traders could also be commodity trading houses, who buy and sell physical stocks, while trading in futures markets. I assume that $N \geq 3$.[5] Inventories are publicly known before the date-0 market opens.

Financial traders cannot trade in the spot market, only invest in futures contracts to be defined shortly, and start with zero endowment in the underlying asset. There are $K \geq 0$ financial traders, indexed with k.

All traders have exponential utility (CARA) with risk aversion parameter γ and seek to maximize the expected utility of their terminal wealth.[6] Traders are forward-looking and fully rational.

At date 0, all traders meet in a centralized market where they simultaneously trade the risky asset and enter futures contracts. A long position in such contract pays off $p_1 - f_0$, where p_1 is the asset price at date 1 and f_0 is the futures price at date 0, both to be determined in equilibrium. For simplicity, I ignore margin constraints associated with futures.

Markets operate through uniform-price double auctions as in Kyle (1989) or Vayanos (1999). At date 0, physical trader n simultaneously posts demand schedules $q_{n,0}(p_0, f_0)$ for the risky asset and $x_n(p_0, f_0)$ for the futures contract, conditional on available information. Financial trader k posts a demand schedule $y_k(f_0)$ in the futures market. A walrasian auctioneer computes the equilibrium prices p_0^* and f_0^* that clear the asset and futures markets. Futures are in zero-net supply, and traders do not have pre-existing futures positions. Thus the market clearing conditions at date 0 are

$$\sum_{n=1}^{N} q_{n,0} = 0, \tag{2.1}$$

$$\sum_{n=1}^{N} x_n + \sum_{k=1}^{K} y_k = 0. \tag{2.2}$$

At date 1, trader n arrives in the date-1 market with inventory $I_{n,1} = I_n + q_{n,0}$. I denote $\bar{I}_1 = \frac{1}{N}\sum_{n=1}^{N} I_{n,1}$ the average inventory when traders start date 1. A liquidity shock Q is realized. The signing convention is that when $Q > 0$, some unmodelled

[5]The condition $N \geq 3$ ensure existence of equilibria in linear strategies. When there are only two traders, Du and Zhu (2017) show existence of equilibria in *non-linear* strategies.

[6]The results carry over if financial traders have a lower risk aversion parameter than physical traders; results also carry over with a higher risk aversion, provided it remains below some threshold.

liquidity traders are willing to sell the asset to physical traders. Conditional on date-0 information, Q is normally distributed with mean $\mathbb{E}_0[Q]$ and variance σ_Q^2. In the model I will use mostly the variance of Q/N, which I denote $\sigma_q^2 = \sigma_Q^2/N^2$. I also assume that Q is jointly normally distributed with ϵ_1 and ϵ_2, and that it is independent from ϵ_1, ϵ_2 – so that Q is a pure liquidity shock. The date-1 market again operates through a uniform-price double auction. Physical trader n posts demand schedule $q_{n,1}(p_1)$. In equilibrium, physical traders purchase quantities $q_{1,1}, ..., q_{N,1}$ that satisfy the market clearing condition:

$$\sum_{n=1}^{N} q_{n,1} = Q. \tag{2.3}$$

(2.3) pins down the equilibrium price p_1^*, and thus the futures payoff. The terminal wealths of a physical trader n and financial trader k are thus

$$W_n = I_n v + q_{n,0}(v - p_0) + q_{n,1}(v - p_1) + x_n(p_1 - f_0), \tag{2.4}$$

$$W_k = y_k(p_1 - f_0). \tag{2.5}$$

I look for subgame-perfect Nash equilibria in demand schedules.[7] As there is neither information asymmetry nor uncertainty on supply shocks when traders post their demand schedules, there are multiple equilibria (Klemperer and Meyer 1989). I use the trembling-hand stability criterion to select a unique equilibrium (see Vayanos 1999).[8] Specifically, I look for equilibria where demand schedules are linear, as in most of the literature in a CARA-normal framework.[9]

3 Perfect competition benchmark

In this section, I study the perfect competition benchmark. I assume that there are no financial traders ($K = 0$): futures and spot trades are then perfect substitutes

[7]The conditions for equilibrium are described in definitions 1, 2 and **??**.

[8]I could also assume, in the spirit of Klemperer and Meyer (1989), that traders face supply shocks at *both* periods that are revealed *after* traders have posted their demand schedules. This would complicate notations without any additional insight.

[9]See Kyle (1989), Vayanos (1999), Malamud and Rostek (2017) among many others. Subject to equilibrium selection by trembling-hand stability or through Klemperer and Meyer (1989)' procedure, such an equilibrium is unique under reasonable assumptions: Glebkin et al. (2022) establish it for $N \geq 3$ in the class of symmetric equilibria with strictly decreasing, continuously differentiable demands and arbitrage-free equilibrium prices. With $N = 2$ and normally distributed payoffs, (Du and Zhu 2017) show that equilibria exist with nonlinear demand schedules.

at date 0.

3.1 Equilibrium definition.

I look for competitive equilibria defined as sets of demand schedules $(q^c_{n,0}(p_0), q^c_{n,1}(p_1))$ and equilibrium prices p^c_0, p^c_1 (p^c_1 function of ϵ_1 and Q) such that (i) all traders are price-takers; (ii) trader n's date 1 demand schedule $q^c_{n,1}(p_1)$ maximizes his/her expected utility of terminal wealth W_n given information available at date 1; (iii) for each trader n, date 0 demand schedules $q^c_{n,0}(p_0, f_0)$ maximize their expected utility of terminal wealth W_n given information available at date 0 and anticipated equilibrium outcomes at date 1; the market clearing conditions (2.1) - (2.3) hold. I solve for equilibria by backward induction.

3.2 Date-1 equilibrium

Trader n maximize over $q_{n,1}$ his/her expected utility. Given that the only uncertainty is on the normally distributed variable ϵ_2, the certainty equivalent of wealth can be written as

$$\widetilde{W}_{n,1} = I_n v_1 + q_{n,0}(v_1 - p_0) + x_n(p_1 - f_0) + q_{n,1}(v_1 - p_1) - \frac{\gamma\sigma_2^2}{2}(I_{n,1} + q_{n,1})^2, \quad (3.1)$$

where $v_1 = v_0 + \epsilon_1$ is the expectation of payoff v conditional on date-1 information. From the first order condition of this maximization problem one easily derives the optimal competitive demand schedule:

$$q^c_{n,1}(p_1) = \frac{v_1 - p_1}{\gamma\sigma_2^2} - I_{n,1}. \quad (3.2)$$

Plugging optimal demands into the market clearing condition (2.3) pins down the equilibrium price:

$$p^c_1 = v_1 - \gamma\sigma_2^2 \frac{1}{N}\left(\sum_{n=1}^{N} I_{n,1} + Q\right). \quad (3.3)$$

Finally, plugging the equilibrium price (3.3) into demand schedule (3.2), one gets the equilibrium quantity purchased by trader n:

$$q^c_{n,1} = \bar{I}_1 - I_n - q_{n,0} + \frac{Q}{N}. \quad (3.4)$$

From (3.4), it is apparent that one extra unit $dq_{n,0}$ purchased by trader n at date 0 translates into exactly one unit less $dq^c_{n,1} = -dq_{n,0}$ purchased at date 1.

3.3 Date-0 equilibrium

Because trader n is sensitive to the risk on Q both through equilibrium price (3.3) and equilibrium quantity (3.4), physical trader n's preference and attitude towards the risk on Q is unusual, although perfectly standard.

Trader n's certainty equivalent of wealth after competitive trade is, plugging equilibrium price (3.3) and quantity (3.4) into (3.1):

$$\widetilde{W}^c_{n,1} = I_n v_1 + q_{n,0}(v_1 - p_0) - \frac{\gamma\sigma_2^2}{2}(I_n + q_{n,0})^2 + S^c_{n,1}, \tag{3.5}$$

$$\text{with } S^c_{n,1} = q^c_{n,1}(v_1 - p^c_1) - \frac{\gamma\sigma_2^2}{2}\left((I_{n,1} + q^c_{n,1})^2 - (I_{n,1})^2\right) \;=\; \frac{\gamma\sigma_2^2}{2}\left(q^c_{n,1}\right)^2. \tag{3.6}$$

The share $S_{n,1}$ of date-1 trading surplus accruing to trader n is simply the expected payoff $q^c_{n,1}(v_1 - p^c_1)$ of trader n's trade, minus the variation in risk holding cost induced by trade $q^c_{n,1}$. The second equality in (3.6) obtains using the date-1 demand schedule (3.2) at equilibrium price p^c_1, and shows the quadratic dependence of trader n's wealth on Q in a simple form.

Taking the certainty equivalent of (3.5) with respect to both ϵ_1 and Q using Lemma 5 in the appendix, one gets:

$$\begin{aligned}\widetilde{W}_{n,0} &= V_n + S_{n,0} + \widetilde{S}_{n,1} \\ &\quad + x_n(\widetilde{p}_1 - f_0) - \frac{\gamma}{2}\left(\sigma_1^2 + (1 - z_c)\sigma_2^2\right)\left((x_n)^2 + I_{n,1}x_n\right),\end{aligned} \tag{3.7}$$

where

$$\widetilde{p}_1 = v_0 - \gamma\sigma_2^2 z_c \left(\bar{I}_1 + \mathbb{E}_0\left[\frac{Q}{N}\right]\right)$$

is a risk-adjusted expected spot price at date-1. The second line of (3.7) is non-zero as long as trader n trades futures ($x_n \neq 0$). The first term is the risk-ajdusted expected payoff; the second term has a component in x_n^2 and reflects the cost of holding a futures position whose payoff is uncertain. The component in $I_{n,1}x_n$ reflects the substitutability between date-0 spot and futures trades. That the coefficients in front of these two components are equal suggests that the substitutability may be perfect, which we check shortly. On the first line of (3.7), the term

$$V_n = I_n v_0 - \frac{\gamma(\sigma_1^2+\sigma_2^2)}{2}(I_n)^2 \tag{3.8}$$

is the certainty equivalent of trader n's wealth when he/she does not trade. $S_{n,0}$ is the share of date-0 surplus accruing to trader n:

$$S_{n,0}(q_{n,0}) = q_{n,0}(v_0 - p_0) - \frac{\gamma}{2}(\sigma_1^2+\sigma_2^2)\left[(I_n+q_{n,0})^2 - (I_n)^2\right]. \tag{3.9}$$

where $q_{n,0}(v_0 - p_0)$ is the expected profit from date-0 transaction, and the second term is the variation in inventory holding costs induced by trading the quantity $q_{n,0}$.

The term $\widetilde{S}_{n,1}$ shows that although standard assumptions have been made about trader n's preferences, an unusual attitude towards risk of Q emerges: $\widetilde{S}_{n,1}$ is the certainty equivalent of (3.6) and has the same form, with expectation operators and a discount factor $z_c = \left(1+\frac{1}{N^2}\gamma^2\sigma_2^2\sigma_Q^2\right)^{-1}$ that is positive and below 1, decreases with the supply shock variance σ_Q^2 and with the number N of physical traders:

$$\widetilde{S}_{n,1}(q_{n,0}) = z_c \times \frac{\gamma\sigma_2^2}{2}\left(\mathbb{E}_0\left[q_{n,1}^c\right]\right)^2 - \frac{\ln z_c}{2\gamma}. \tag{3.10}$$

The factor z_c comes from the fact that date-1 trading profit is a function of the square of Q, and thus has a chi-squared distribution. Qualitatively, the uncertainty on Q entails no risk of loss: extreme values of Q means buying a large quantity at a low price, or selling at a high price; therefore there is no risk premium to be subtracted from the certainty equivalent. But $z_c < 1$ implies a lower weight of date-1 trading profit in trader n's optimization, reflecting the uncertainty associated with dynamic trading strategy. The term $-\ln z_c/2\gamma$ does not depend on $q_{n,0}$. $\widetilde{S}_{n,1}$ depends on $q_{n,0}$ through the equilibrium quantity $q_{n,1}^c$, and through the variation in holding costs, which involves $I_{n,1} \equiv I_n + q_{n,0}$. $\widetilde{S}_{n,1}$ also depends on p_1^c, which itself, given equation (3.3), which trader n takes as given in the competitive setting.

The redundancy of futures. Differentiating (3.7) with respect to both trade in the underlying asset $q_{n,0}$ and futures x_n, and treating prices p_0, f_0 and $\widetilde{p}_1$ as constants, I find the following first-order condition:

$$\begin{pmatrix} v_0 - p_0 \\ v_0 - f_0 \end{pmatrix} = \gamma\left(\sigma_1^2 + (1-z_c)\sigma_2^2\right)\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} q_{n,0} \\ x_n \end{pmatrix} + cst.$$

This equation cannot be inverted to get demand schedules: quantities traded are indeterminate and futures are redundant.

As a useful benchmark, the following proposition presents the equilibrium without futures: all gains from trade are realized at date 0. The proof is in the appendix.

Proposition 1. *Without futures, the equilibrium price is*

$$p_0^c = v_0 - \gamma(\sigma_1^2 + \sigma_2^2)\bar{I}_0 - \gamma\sigma_2^2 z_c \, \mathbb{E}_0 \left[\frac{Q}{N} \right]. \tag{3.11}$$

Equilibrium trades at date 0 and date 1 are

$$q_{n,0}^c = \bar{I}_0 - I_n, \qquad q_{n,1}^c = \frac{Q}{N}. \tag{3.12}$$

All traders end up with the Pareto optimal average inventory $\bar{I}_0$ at date 0.

4 Imperfect competition, the non-redundancy of futures and the futures-spot basis

This section shows that imperfect competition in the date-1 spot market makes futures and spot trade imperfect substitutes even without financial traders, contrary to the case with perfect competition. This implies that spot and futures prices are different, and also entails determinate predictions regarding spot and futures quantities that we derive in later sections. I again assume that $K = 0$.

4.1 Date-1 equilibrium

We know treat the date-1 equilibrium with imperfect competition, which is a simple version of earlier models by Kyle (1989), Vayanos (1999) or Malamud and Rostek (2017). At date 1, physical trader n maximizes the certainty equivalent of (2.4), which is (3.1) as in the competitive case. But here traders take the impact of their demand on the equilibrium price into account, taking the residual demand curve, which is the sum of all other traders' demand curves, as given. For a given quantity $q_{n,1}$ demanded by a trader n, this residual demand curve implies an equilibrium price p_1, so that a marginal increase in the quantity demanded by trader n implies a price impact $\partial p_1 / \partial q_{n,1}$. Differentiating the certainty equivalent of wealth

(3.1), the first order condition for a trader n is

$$v_1 - p_1 - q_{n,1}\frac{\partial p_1}{\partial q_{n,1}} = \gamma\sigma_2^2(I_{n,1} + q_{n,1}).$$

I look for equilibria in linear strategies: trader n expects a linear residual demand curve, with constant slope denoted $1/\lambda_{n,1}$, so that $\partial p_1/\partial q_{n,1} = \lambda_{n,1}$. Thus trader n's optimal demand schedule given this residual demand curve is

$$q^*_{n,1}(p_1, \lambda_{n,1}) = \frac{v_1 - p_1}{\lambda_{n,1} + \gamma\sigma_2^2} - \frac{\gamma\sigma_2^2}{\lambda_{n,1} + \gamma\sigma_2^2} I_{n,1}. \tag{4.1}$$

The quantity $\lambda_{n,1} + \gamma\sigma_2^2$ is the *trading costs* associated with buying or selling, which have two dimensions: buying 1) moves the spot price p_1 upwards by $\lambda_{n,1}$ per unit and 2) increases the marginal holding cost by $\gamma\sigma_2^2$ per unit. The assessment of date-0 trading costs drives much of the equilibrium trading strategies.

As all traders follow linear strategies given a linear residual demand curve, summing optimal demand (4.1) over traders other than n gives the residual demand curve. Requiring consistency of slopes of the residual demand curve faced by all traders with actual equilibrium schedules leads to the following system of equations:

$$\lambda_{n,1} = \left(\sum_{m=1, m\neq n}^{N} (\lambda_{m,1} + \gamma\sigma_2^2)^{-1}\right)^{-1}, \qquad n = 1, ..., N. \tag{4.2}$$

This equation points to a static price impact externality identified in the literature: as $\lambda_{n,1}$ increases with $\lambda_{m,1}$ in (4.2), further demand reduction by trader m leads to further demand reduction by trader n.

The date-1 equilibrium can now be formally defined and solved.

Definition 1. *Demand schedules $q^*_{n,1}(p_1)$ for $n = 1, ..., N$ and a price function $p^*_1(\epsilon_1, Q; (I_n, q_{n,0})_{n=1,...,N})$ form a date-1 equilibrium conditional on date-0 trades if:*

- *demand schedules $q^*_{n,1}(p^*_1)$ maximize* (3.1)*, given price impact $\lambda_{n,1}$;*
- *price impacts $\lambda_{n,1}$ satisfy* (4.2)*;*
- *the market clearing condition* (2.3) *holds.*

Proposition 2 (Vayanos (1999), Malamud and Rostek (2017))**.** *A date-1 equilibrium in linear strategies with imperfect competition exists and is unique. In this* *equilibrium, all physical traders have the same price impact*

$$\lambda_{n,1} = \frac{\gamma\sigma_2^2}{N-2},$$

so that equilibrium demand schedules are reduced with respect to the competitive case:

$$q_{n,1}^*(p_1) = \frac{N-2}{N-1}\left[\frac{v_1 - p_1}{\gamma\sigma_2^2} - I_n - q_{n,0}\right]. \tag{4.3}$$

The equilibrium quantity traded by trader n and equilibrium price are

$$q_{n,1}^* = \frac{N-2}{N-1}(\bar{I}_1 - I_{n,1}) + \frac{Q}{N}, \tag{4.4}$$

$$p_1^* = v_1 - \gamma\sigma_2^2\left(\bar{I}_1 + \frac{N-1}{N-2}\frac{Q}{N}\right), \tag{4.5}$$

where $\bar{I}_1 = \frac{1}{N}\sum_{n=1}^{N} I_{n,1}$ is the date-1 average inventory before date-1 trade.

The equilibrium quantity $q_{n,1}^*$ traded by trader n is reduced by a factor $(N-2)/(N-1)$ with respect to the competitive equilibrium, as shown in equation (4.4).

This also means that one unit $dq_{n,0}$ bought (sold) at date 0 implies that $\frac{N-2}{N-1}dq_{n,0}$ less units, strictly less than $dq_{n,0}$, are bought (sold) at date 1. Back to (**??**), one already foresees that spot and futures are not perfect substitute at date 0. We now check this by showing that at date 0, physical traders' valuations for futures and the underlying asset, are different, so that equilibrium spot and futures prices differ.

4.2 Date-0 equilibrium without financial traders: determinacy and the futures-spot basis

Here I assume that there are no financial traders ($K = 0$), and first derive the certainty equivalent of wealth for trader n.

Lemma 1. *Trader n's certainty equivalent of wealth at date 0 is:*

$$\begin{aligned}\widehat{W}_{n,0}(q_{n,0}, x_n) = V_n + S_{n,0}(q_{n,0}) + \widehat{S}_{n,1}(q_{n,0}) + (\widehat{p}_1 - f_0)x_n \\ - \gamma\left(\sigma_1^2 + (1-z)\sigma_2^2\right) I_{n,1}x_n - \frac{\gamma}{2}\left(\sigma_1^2 + \frac{1-z}{\alpha}\sigma_2^2\right) x_n^2,\end{aligned} \tag{4.6}$$

where $\alpha = 1-(N-1)^{-1}$, and the risk-adjusted expected spot price is

$$\widehat{p}_1 = v_0 - \gamma\sigma_2^2 z\left(\bar{I}_1^e + \frac{N-1}{N-2}\,\mathbb{E}_0\left[\frac{Q}{N}\right]\right).$$

$\widehat{W}_{n,0}$ is analogous to the competitive certainty equivalent (3.7). The main differences are with date-1 trading profit $\widehat{S}_{n,1}$ and the holding costs associated with futures (second line). Because of imperfect competition, date-1 trading profit $\widehat{S}_{n,1}$ is changed in three ways with respect to $\widetilde{S}_{n,1}$. First, the equilibrium quantity involved is now the imperfectly competitive one $q^*_{n,1}$, given by (4.4). Second, the discount factor is decreased to $z = \left(1+\left(\frac{N-1}{N-2}\right)^2 \frac{1}{N^2}\gamma^2\sigma_2^2\sigma_Q^2\right)^{-1}$, with properties similar to z_c's. Third, there is a factor $\frac{N}{N-2}$ reflecting date-1 imperfect competition:

$$\widehat{S}_{n,1}(q_{n,0}) = z\frac{N}{N-2}\frac{\gamma\sigma_2^2}{2}\left(\mathbb{E}_0\left[q^*_{n,1}\right]\right)^2 - \frac{\ln z}{2\gamma}. \tag{4.7}$$

On the second line of (4.6), the term in x_n^2 now differs from the hedging term in $I_{n,1}x_n$; both converge to the competitive coefficients as N grows to infinity.

It is convenient to express the certainty equivalent of wealth as a quadratic form, obtained by rearranging (4.6):

$$\begin{aligned}\widehat{W}_{n,0}(q_{n,0},x_n) = cst &+ \left(v_0 - \frac{N-2}{N-1}w_n - p_0\right)q_{n,0} + (v_0 - w_n - f_0)x_n \\ &- \frac{\gamma}{2}(I_n + q_{n,0}, x_n)\Sigma(I_n+q_{n,0},x_n)',\end{aligned} \tag{4.8}$$

where X' is the transpose of a matrix X, Σ is a symmetric matrix that determines equilibrium trades (see Section 5):

$$\Sigma = \begin{pmatrix}\Sigma_{11} & \Sigma_{12}\\ \Sigma_{21} & \Sigma_{22}\end{pmatrix} = \begin{pmatrix}\sigma_1^2 + \left(1-\frac{N-2}{N}z\right)\sigma_2^2 & \sigma_1^2+\left(1-\frac{N-1}{N}z\right)\sigma_2^2\\ \sigma_1^2+\left(1-\frac{N-1}{N}z\right)\sigma_2^2 & \sigma_1^2+\frac{1-z}{\alpha}\sigma_2^2\end{pmatrix}. \tag{4.9}$$

$1/2\gamma\Sigma_{11}(I_n+q_{n,0})$ is the marginal cost $1/2\gamma(\sigma_1^2+\sigma_2^2)(I_n+q_{n,0})$ associated with holding the risky asset until maturity, net of the benefit of trading some of this quantity at date 1 – the term proportional to $z\sigma_2^2$.

Another critical term, which comes from p_1^* that appears in trader n's wealth, is

$$w_n = \gamma\sigma_2^2 z\left(\sum_{m\neq n}\frac{I^e_{m,1}}{N} + \frac{N-1}{N-2}\,\mathbb{E}_0\left[\frac{Q}{N}\right]\right). \tag{4.10}$$

In (4.5), p_1^* exhibits an inventory risk premium

$$\gamma\sigma_2^2\left(\sum_{m\neq n}\frac{I_{m,1}}{N}+\frac{N-1}{N-2}\frac{Q}{N}\right)+\gamma\sigma_2^2\frac{I_{n,1}}{N}.$$

w_n is the certainty equivalent of the first term, while the second term ends up affecting Σ. As trader n is imperfectly competitive, he/she takes the impact of his/her own trade $q_{n,0}$ on date-1 price, through this inventory risk premium, which we will study more in Section 5. As we will see then, w_n acts as a demand shifter that is specific to trader n: if trader n is the only low-inventory trader, then w_n is positive (assume $\mathbb{E}_0[Q] \geq 0$ for simplicity) and (4.8) shows that this decreases trader n's marginal valuation of both the asset at date 0 and of futures. Traders with different inventories see have different w_n, which act as different demand shifters and create additional motives for trade: in the following sections, we will show how different w_n affects equilibrium trades, and therefore welfare.

For now, we notice that the term w_n appears twice in (4.8), once for spot and once for futures trades. For spot trade, $-\frac{N-2}{N-1}w_n$ is the marginal effect of date-0 spot trade on date-1 surplus $\widehat{S}_{n,1}$: if trader n expects to face high-inventory traders (sellers) at date 1, he/she prefers to buy less of the asset at date 0 – it will be more profitable to buy at date 1, because the price p_1^* will be lower. For futures trade, $-w_n$ is in the risk-adjusted expected futures payoff $\widehat{p}_1 - f_0$: if trader n expects to face a low date-1 price because other trader will have high inventory, the futures payoff is lower.

I now define date-0 equilibria and show that trader n's optimal demand for spot and futures are well-defined.

4.2.1 Equilibrium definition and determinacy

Given imperfect competition in both the underlying asset and futures markets, trader n takes the impact of trade in each market on both prices simultaneously. To solve the equilibrium, I apply methods from Malamud and Rostek (2017), which extends the method used at date 1 to a setting with several assets. A 2×2 matrix of price impacts

$$\Lambda_n \equiv \begin{pmatrix}\Lambda_{n,11} & \Lambda_{n,12}\\ \Lambda_{n,21} & \Lambda_{n,22}\end{pmatrix} = \begin{pmatrix}\partial p_0/\partial q_{n,0} & \partial f_0/\partial q_{n,0}\\ \partial p_0/\partial x_n & \partial f_0/\partial x_n\end{pmatrix}$$

replaces the scalar price impact $\lambda_{n,1}$ of the date-1 equilibrium, and the equilibrium condition becomes:

$$\Lambda_n = \left(\sum_{m=1, m\neq n}^{N} (\Lambda_m + \gamma\Sigma)^{-1} \right)^{-1} \qquad \text{for all } n. \tag{4.11}$$

Importantly, trader n's demand is a function of other traders' date-0 equilibrium trades $q^e_{m,0}$ ($m \neq n$) in the spot market:[10] this is because $\widehat{S}_{n,1}$ depends on p_1^*, which itself (see (4.5)) depends on $\bar{I}_1 = \sum_{m=1}^{N}(I_m + q_{m,0})$. This point is studied in more detail in Section 5. In equilibrium, one requires these quantities to coincide with actual equilibrium quantities:

$$q^e_{m,0} = q^*_{m,0}, \quad m = 1, ..., N. \tag{4.12}$$

Definition 2. *Demand schedules $q^*_{n,0}(p_0, f_0)$ in the underlying asset and $x^*_n(p_0, f_0)$ in futures contracts, and spot price p_0^* and futures price f_0^* are an equilibrium if:*

- *demand schedules $q^*_{n,0}(p_0, f_0)$ and $x^*_n(p_0, f_0)$ maximize* (4.6)*;*
- *price impacts matrices Λ_n ($n = 1, ..., N$) satisfy* (4.11)*;*
- *quantities $q^e_{n,0}$ satisfy* (4.12)*;*
- *market clearing conditions* (2.1) *and* (2.2) *hold.*

We now show that optimal demand schedules, and thus quantities, are well-defined. Demand schedules are determinate iff $\widehat{W}_{n,0}$ is a concave function of $(q_{n,0}, x_n)$. The following lemma shows that this is the case if the variance of the supply shock σ_Q^2 is not too low.

Lemma 2. *The certainty equivalent of wealth* (4.6) *is concave if and only if σ_Q^2 is above a threshold $\bar{s}(\sigma_1^2)$. This threshold decreases with σ_1^2.*

The proof of this lemma amounts to showing that Σ is positive definite for $\sigma_Q^2 > \bar{s}(\sigma_1^2)$, which implies the invertibility of Σ: therefore, demand schedules are well-defined. In what follows, we assume that $\sigma_Q^2 > \bar{s}(\sigma_1^2)$.

[10] For a given trader m, $q^e_{m,0}$ should depend on the trader n, since it is specific to trader n's optimization; but given that all traders' anticipations $q^e_{m,0}$ are in the end required to be correct, to ease notation I drop the dependence in trader n.

4.2.2 The equilibrium futures-spot basis

As shown in (4.8), the unit expected profits from buying spot and futures differ by a factor $\frac{N-2}{N-1}$ in their second terms. Each of these two terms come from different parts of $\widehat{W}_{n,0}$: for spot trades, it comes from date-1 spot trading profit $\widehat{S}_{n,1}$, as $\frac{N-2}{N-1}w_n$ is the marginal impact of trade date-0 spot trade on date-1 trading profit $\widehat{S}_{n,1}$ (minus the spot trade price impact). By contrast, w_n in the term in x_n comes from the futures expected futures $x_n(\widehat{p}_1 - f_0)$.

Given different marginal valuations for spot and futures trades, we expect different prices. The following proposition, proved in the appendix, shows that this is the case.

Proposition 3. *Equilibrium prices are:*

$$p_0^* = v_0 - \gamma(\sigma_1^2 + \sigma_2^2)\bar{I}_0 - \gamma\sigma_2^2 z\,\mathbb{E}_0\left[\frac{Q}{N}\right], \tag{4.13}$$

$$f_0^* = v_0 - \gamma(\sigma_1^2 + \sigma_2^2)\bar{I}_0 - \frac{N-1}{N-2}\gamma\sigma_2^2 z\,\mathbb{E}_0\left[\frac{Q}{N}\right], \tag{4.14}$$

so that there is a basis between the spot and futures price, unless $\mathbb{E}_0[Q] = 0$*:*

$$f_0^* - p_0^* = -\frac{\gamma\sigma_2^2 z}{N-2}\,\mathbb{E}_0\left[\frac{Q}{N}\right].$$

Given imperfect substitutability between spot and futures, and therefore different marginal valuations of spot and futures trades, the futures-spot basis is *not* a failure of the law of one price. Therefore, no spot-futures arbitrage is possible in this context, ultimately because of imperfect competition at date 1.

5 Inefficient futures trading without financial traders

We now come to the main result of the paper: under imperfect competition and without financial traders, introducing futures makes all physical traders worse off. This is because they seek to influence their futures positions' payoff by trading spot rather than hedge their spot positions with futures, which leads them to adopt negative hedging ratios.

To understand why, it is key to notice that date-0 trades have an impact on date-1 price p_1^*: from (4.5), one sees that the futures payoff $p_1^* - f_0$ involves the term

$$-\gamma\sigma_2^2 \bar{I}_1 = -\gamma\sigma_2^2 \left(\sum_{m=1, m\neq n}^{N} \frac{I_{m,1}}{N} + \frac{I_n + q_{n,0}}{N} \right), \tag{5.1}$$

which depends on date-0 spot trade $q_{n,0}$: thus trader n takes the impact of his/her date-0 spot trades on date-1 price (given other traders' strategies) into account.[11] This is done in two steps. First, at order 1, trader n contemplates the spot price p_1^* before his own impact (and given other traders strategies) and how it affects his/her spot and futures marginal demands – this is the first term in (5.1), of which w_n is the certainty equivalent. Second, at order 2, trader n considers the impact of different inventory choices $I_n + q_{n,0}$ on p_1^*, and thus on his/her spot and futures marginal demands – which affects Σ.

This is true without futures, since p_1^* naturally appears in date-1 trading profit $\widehat{S}_{n,1}$ (see equation (4.7)). But futures introduce another incentive to influence date-1 price, through their payoff $x_n(\widehat{p}_1 - f_0)$. I review both cases in what follows.

5.1 Equilibrium without futures

To present the intertemporal price impact in a simple context, I study the case where there are no futures, *i.e.* x_n is exogenously constrained to be zero. This is also a natural benchmark to assess traders' welfare effect of introducing futures. The date-1 equilibrium, given inventories after date-0 trade, is as in Section 4.1. Setting $x_n = 0$ in (4.8), it is easy to invert the first order condition to get trader n's optimal demand schedule:[12]

$$q_{n,0}^{\times}(p_0) = \frac{v_0 - p_0 - \frac{N-2}{N-1} w_n}{\lambda_{n,0} + \gamma\Sigma_{11}} - \frac{\gamma\Sigma_{11}}{\lambda_{n,0} + \gamma\Sigma_{11}} I_n. \tag{5.2}$$

In (5.2), the term in $-w_n$ reflects the incentive on trader n to postpone some of his/her trades to date 1. Suppose trader n has low inventory: then he/she sees a higher w_n than a high-inventory trader, thus a lower price (see (4.5)). Therefore,

[11]At this stage one may correctly see that with market clearing, $\bar{I}_1 = \sum_{n=1}^{N} I_{n,1}/N = \sum_{n=1}^{N} I_{n,0}/N$, so that the date-1 price does not depend on $q_{n,0}$ anymore. This is a knife-edge case however: when buyers and sellers have different risk aversions, the date-1 equilibrium price does not depend on the average inventory anymore, as shown by Malamud and Rostek (2017), and market clearing does not simplify its expression. Therefore, not applying market clearing directly and keeping this intertemporal price impact is more robust.

[12]One easily checks that $\widehat{W}_{n,0}(q_{n,0}, 0)$ is concave in $q_{n,0}$ ($\Sigma_{11} > 0$ since $z < 1$)

trader n's date-0 demand schedule is shifted downward more than a high-inventory trader's, reflecting his/her greater willingness to postpone purchases to date 1.

Similarly to date 1 and from (4.8), a unique equilibrium price impact exists:

$$\lambda_{n,0} = \frac{\gamma\Sigma_{11}}{N-2}. \tag{5.3}$$

Then aggregating traders' optimal demand schedules in the market clearing condition, one finds that the equilibrium price $p_1^{\times}$ (see the proof of Proposition 4) satisfies

$$v_0 - p_0^{\times} = \gamma\Sigma_{11}\bar{I}_0 - \gamma\sigma_2^2 z\left(\frac{N-2}{N}\bar{I}_1^e + \mathbb{E}_0\left[\frac{Q}{N}\right]\right).$$

Optimal price impact, demand schedules, and equilibrium price lead to the equilibrium quantity, still as a function of traders' anticipations of other traders' trades $q_{m,0}^e$. Intermediate computation steps are provided in the appendix.

Proposition 4. *Trader n's date-0 equilibrium quantity, given other traders' equilibrium trades, is:*

$$q_{n,0}^{\times} = \frac{N-2}{N-1}\left(\bar{I}_0 - I_n\right) + \kappa_{\times}\left(\bar{I}_1^e - I_{n,1}^e\right), \tag{5.4}$$

where $\bar{I}_1^e = \sum_m (I_m + q_{m,0}^e)/N$, $I_{n,1}^e = I_n + q_{n,0}^e$ is trader n's equilibrium inventory anticipated by other traders, and

$$\kappa_{\times} = \frac{-\frac{\gamma\sigma_2^2 z}{N}\frac{N-2}{N-1}}{\lambda_{n,0} + \gamma\Sigma_{11}} < 0. \tag{5.5}$$

In (5.4), the first term $\frac{N-2}{N-1}(\bar{I}_0 - I_n)$ is the quantity that would be traded if there was no date-1 trading round (see (4.4)). The second term is proportional to trader n's trade at date 1 $(\bar{I}_1^e - I_{n,1}^e)$, implied by all traders' anticipations of equilibrium trades, and comes from the aggregation of all w_m.

Intuitively, by taking his/her own date-1 impact into account, trader n first contemplates the aggregate date-1 market inventory without him/her (reflected in w_n), which ultimately impact the date-1 price. Heterogenous inventories imply that different traders contemplate different prices when they are not in the market, and thus different opportunities to trade at date 1: a low-inventory trader sees a higher inventory without him than a high inventory trader, so that the low-inventory trader sees a lower price than the high-inventory trader. Therefore, the low-inventory trader wishes to purchase less at date 0, and more at date 1, to benefit from a favorable price, and a high inventory trader symmetrically wants to postpone sales. This explains why $\kappa_\times < 0$, and implies that date-0 trading is lower than in the static benchmark.[13]

This effect increases with the absolute value of $\kappa_\times$, which is the ratio of the marginal incentive $\gamma\sigma_2^2 z/N$ to benefit from date-1 price (corrected for risk with z) by adjusting date-0 trade, multiplied by the marginal quantity traded at date 1, $\frac{N-2}{N-1}$, to marginal trading costs of purchasing at date 0, $\lambda_{n,0}+\gamma\Sigma_{11}$. Thus increasing trading costs, or lowering the quantity traded at date-1, or increasing the date-1 spot price risk premium, all lower the incentive to postpone trades to date 1. Imposing the equilibrium condition (4.12), we fully solve the model and confirm our intuition.

Proposition 5. *An equilibrium exists and is unique for all parameter values. Equilibrium quantities are:*

$$q_{n,0}^{\times} = \psi_\times(\bar{I}_0 - I_n), \qquad q_{n,1}^{\times} = \frac{N-2}{N-1}(1-\psi_\times)(\bar{I}_0 - I_n) + \frac{Q}{N}, \tag{5.6}$$

where $0 < \psi_\times < \frac{N-2}{N-1}$, *and* $\psi_\times = \psi(\kappa_\times)$ *decreases with* $\kappa_\times$. *Finally, the date-0 equilibrium price is*

$$p_0^{\times} = v_0 - \gamma(\sigma_1^2+\sigma_2^2)\bar{I}_0 - \gamma\sigma_2^2 z\,\mathbb{E}_0\left[\frac{Q}{N}\right]. \tag{5.7}$$

In equilibrium, only a fraction $\psi_\times$ of the competitive quantity is traded, and this fraction is lower than the fraction $\frac{N-2}{N-1}$ in a static market with no subsequent trading round, as occurs at date 1. At date 1, the fraction $\frac{N-2}{N-1}$ of the remaining inventory imbalance $(1-\psi_\times)(\bar{I}_0 - I_n)$ is traded, plus the supply shock.

In an apparent paradox, trader n trades *more* at date 0 (*i.e.*, $\psi_\times$ is greater) when the price impact $\lambda_{n,0}$ of such trade is *greater*: as shown by (5.5), the absolute value of $\kappa_\times$ is lower when the impact $\lambda_{n,0}$ of date-0 trade on date-0 price is greater. This is simply because with higher date-0 price impact, trader n cares less about the impact of current trade on date-1 surplus, and thus on other traders' equilibrium trades.

[13] There is also a feedback effect related to the $q_{m,1}^e$ that makes trades even slower: consider a trader n with low inventory at date 1, thus a buyer at date 1 (see (4.4)): (5.4) says trader n buys less at date 0 since $\kappa_\times < 0$. Doing so lowers other traders' anticipation of trader n's purchases, translating into a lower w_m ($m \neq n$): thus trader $m \neq n$ is more willing to purchase the asset, which raises w_n and lowers trader n's willingness to purchase the asset, and so on, and so forth. The reasoning works in a similar way for high-inventory traders, who are less willing to sell. Thus intertemporal price impact, in the setting without futures, begets current illiquidity.

5.2 Equilibrium trades with futures contracts

We come back to the equilibrium with futures. Compared with the case without futures, the perspective of a high or low date-1 price has an additional effect through the futures payoff: a low-inventory trader sees, when he/she is not in the date-1 market, a low payoff of a long futures position, because the date-1 spot price is low.

This effect shows up as the second occurence of w_n in (4.8), which now shifts trader n's futures demand schedule differently, depending on trader n's inventory position: a low-inventory trader has a higher w_n, thus sees a lower $\widehat{p}_1$ and a lower payoff, for a long futures position, than a high-inventory trader. This also shows up in trader n's demand schedule that we now derive. From (4.8), (4.9), (4.10) and given Λ_n, one easily derives trader n's optimal demand schedules:

$$\begin{pmatrix} q^*_{n,0}(p_0, f_0) \\ x^*_n(p_0, f_0) \end{pmatrix} = (\Lambda_n + \gamma\Sigma)^{-1} \left[\begin{pmatrix} v_0 - p_0 \\ v_0 - f_0 \end{pmatrix} - \gamma\Sigma \begin{pmatrix} I_n \\ 0 \end{pmatrix} - \begin{pmatrix} \frac{N-2}{N-1} w_n \\ w_n \end{pmatrix} \right] \tag{5.8}$$

The last term inside the brackets gathers the effects of date-0 spot trades on the marginal values of spot and futures trade. As without futures, a low inventory trader is less willing to buy spot at date 0 to benefit from a lower price at date 1 in the spot market; not buying spot at date 0 translates into buying $\frac{N-2}{N-1}$ units at date 1, because of imperfect competition at date 1. Considering futures, a low-inventory trader also expects a lower futures payoff than a high inventory-inventory trader when contemplating date-1 market before the impact of his/her date-0 spot trade on date-1 spot price. How the two considerations interact to determine each asset demand is determined by the trading costs matrix $\Lambda_n + \gamma\Sigma$: developing, one gets

$$\begin{pmatrix} q^*_{n,0}(p_0, f_0) \\ x^*_n(p_0, f_0) \end{pmatrix} = (\Lambda_n + \gamma\Sigma)^{-1} \left[\begin{pmatrix} v_0 - p_0 \\ v_0 - f_0 \end{pmatrix} - \gamma\Sigma \begin{pmatrix} I_n \\ 0 \end{pmatrix} \right] + \frac{w_n}{\gamma\sigma_2^2/N} \begin{pmatrix} \kappa_0 \\ h_0 \end{pmatrix} \tag{5.9}$$

where

$$\kappa_0 = \frac{-\gamma\sigma_2^2 z/N \left[\frac{N-2}{N-1}(\Lambda_{n,22} + \gamma\Sigma_{22}) - (\Lambda_{n,12} + \gamma\Sigma_{12})\right]}{|\Lambda_n + \gamma\Sigma|}, \tag{5.10}$$

$$h_0 = \frac{-\gamma\sigma_2^2 z/N \left[\Lambda_{n,11} + \gamma\Sigma_{11} - \frac{N-2}{N-1}(\Lambda_{n,21} + \gamma\Sigma_{21})\right]}{|\Lambda_n + \gamma\Sigma|}. \tag{5.11}$$

I now derive intuition for the last term. First, the matrix $\Lambda_n + \gamma\Sigma$ gives the variations of the marginal costs c_x of trading futures, and of the marginal cost c_q of trading the underlying asset, with spot and futures trade: a spot trade $q_{n,0}$ impacts both the spot price p_0 by $\Lambda_{n,11} q_{n,0}$ and the spot marginal holding cost $\gamma \Sigma_{11} q_{n,0}$, but also the futures price f_0 and the futures holding cost in a total amount $\Lambda_{n,21} + \gamma \Sigma_{21}$ per unit, and similarly for a futures trade, the impact on futures trading cost being $\Lambda_{n,22} + \gamma \Sigma_{22}$ and on spot trading costs, $\Lambda_{n,12} + \gamma \Sigma_{12}$.

Second, the numerator of κ_0 compares the benefits of delaying spot trade to date 1 (first term) to the benefit of taking a futures position strategically as described above (second term). Consider a low-inventory trader: when the benefit of trading futures strategically by short-selling them is stronger, this trader shifts his/her demand downwards *less* than a high-inventory trader, so that in equilibrium, he/she buys spot *more* than in a static market. This is for hedging reasons, as indicated by the cross-market trading cost $\Lambda_{n,12} + \gamma \Sigma_{12}$: if trader n both shorted the futures contract and delayed purchases, he/she would be too much exposed to the risk of a high date-1 price. A similar intuition holds for futures demand and h_0. Therefore, a low-inventory trader n either prefers to buy aggressively in the spot market, and to short-sell futures, privileging the benefit of the higher payoff of a short futures position, or to buy less spot and to buy futures. What actually holds in equilibrium depends on the price impact matrix and will be determined shortly. The following lemma completes the intuition by giving an explanation for all other terms in κ_0 and h_0, the proof being in the appendix.

Lemma 3. *κ_0 is the ratio of the marginal increase in the exposure to p_1^* when low-inventory (high-inventory) trader n buys (sells) more spot and sells (buys) futures to keep futures trading costs c_x constant, to the marginal increase in c_q under the same trading pattern.*

h_0 is the ratio of the marginal increase in the exposure to p_1^* when low-inventory (high-inventory) trader n buys (sell) futures and sells (buys) spot to keep spot trading costs constant, to the marginal increase in futures trading costs under the same pattern.

Compared to the equilibrium with futures, κ_0 and h_0 are the ratio of the benefit of trading to strategically anticipate date-1 price, to the date-0 cost of doing so. But this is done with two assets, so demand for each assets holds trading costs for the other constant.

Applying market clearing conditions (2.1) and (2.2) and plugging the resulting equilibrium prices (see proposition 3) into demand schedules (5.8), one gets the following proposition. Detailed computations are in the appendix.

Proposition 6. *The equilibrium quantities traded by trader n, as a function of the price impact matrix Λ_n and given other traders' spot quantities $q^e_{m,0}$, are*

$$\begin{pmatrix} q^*_{n,0} \\ x^*_n \end{pmatrix} = (\Lambda_n + \gamma\Sigma)^{-1}\gamma\Sigma \begin{pmatrix} \bar{I}_0 - I_n \\ 0 \end{pmatrix} - (\bar{I}^e_1 - I_{n,1}) \begin{pmatrix} \kappa_0 \\ h_0 \end{pmatrix} \tag{5.12}$$

where h_0 and κ_0 are given by (5.10) *and* (A.17).

Proposition 7. *If $\sigma^2_Q \geq \bar{s}(\sigma^2_1)$, an equilibrium exists and is unique. Equilibrium quantities, as a function of other traders' trades, satisfy*

$$q^*_{n,0} = \frac{N-2}{N-1}(\bar{I}_0 - I_n) + \kappa_0 \left(\bar{I}^e_1 - I^e_{n,1}\right), \tag{5.13}$$

$$x^*_n = h_0 \left(\bar{I}^e_1 - I^e_{n,1}\right). \tag{5.14}$$

where $\bar{I}^e_1 = \sum_m (I_n + q^e_{m,0})/N$, $I_{n,1} = I_n + q_{n,0}$. Moreover, $\kappa_0 > 0$ and $h_0 < 0$ and equilibrium quantities finally are:

$$q^*_{n,0} = \psi_0(\bar{I}_0 - I_n), \qquad q^*_{n,1} = \frac{N-2}{N-1}(1-\psi_0)(\bar{I}_0 - I_n) + \frac{Q}{N}, \tag{5.15}$$

$$x^*_n = h_0(1-\psi_0)(\bar{I}_0 - I_n) \tag{5.16}$$

where $\psi_0 = \psi(\kappa_0) > \frac{N-2}{N-1}$.

In equilibrium, a low-inventory trader buys more than in the static benchmark, where the fraction of gains from trades realized is $\frac{N-2}{N-1}$, and short sells futures, while high-inventory traders do the opposite. We will shortly see that this is inefficient.

Why does such an equilibrium occur? Consider, again from the perspective of low-inventory trader n, the marginal incentive to buy less at date 0, and to purchase futures to hedge: this yields a marginal benefit of $\frac{N-2}{N-1}p^*_1$, with the factor $\frac{N-2}{N-1}$ coming from imperfect competition at date 1 – trader n is unable to trade the whole unit that he/she has not traded at date 0. On the contrary, if this trader chooses to short futures and to buy spot aggressively to hedge the futures position, then the marginal payoff is p^*_1, which is greater. Therefore, trader n chooses the second option.[14]

A corollary to this strategic positioning on futures side is that, when the risk on Q is greater, the incentive to do so is lower. The following proposition confirms this on equilibrium quantities.

[14] By contrast, if trader n was not sensitive on his/her impact on futures payoff, he/she would seek to benefit from the low price only on the spot side. I show such an equilibrium in online appendix B, where I impose a fictitious tax scheme that offsets the incentive on futures side.

Proposition 8. *The quantity of futures traded* $|x_k^*|$ *decreases as* σ_Q^2 *increases, and shrinks to zero as* σ_Q^2 *diverges to infinity.* $|x_k^*|$ *also decreases as* σ_1^2 *increases.*

This proposition thus states that, counter-intuitively, traders trade *less* futures when they have *more* hedging needs. This is because they hedge a strategic futures position by trading spot, instead of hedging a strategic spot position with futures.

5.3 The inefficiency of futures

Here I derive equilibrium utilities for traders, and the positive and negative effects of introducing futures. I show that the negative effect dominates, so that introducing futures decreases traders' utilities. I also show that it remains rational for traders to participate in the market. I also discuss the welfare of liquidity traders.

Denoting $S_n(\psi)$ the sum of date-0 and date-1 spot trading surplus when the fraction of the competitive quantity traded at date 1 is ψ, trader n's welfare without futures is $\widehat{W}_{n,0}^{\times} = V_n + S_n(\psi_{\times})$, so that, also from (4.6), one has for an arbitrary futures position x_n,

$$\begin{aligned}\widehat{W}_{n,0}^{*} - \widehat{W}_{n,0}^{\times} = {} & V_n + S_n(\psi_0) - S_n(\psi_{\times}) \\ & + x_n\left(\widehat{p}_1 - f_0^* - \gamma(\sigma_1^2 + (1-z)\sigma_2^2)(I_n + q_{n,0}^*)\right) - \frac{\gamma}{2}\Sigma_{22}(x_n)^2. \quad (5.17)\end{aligned}$$

From propositions 4 and 6, the equilibrium spot proportion traded with futures, ψ_0, is greater than the proportion without futures $\psi_{\times}$. Intuitively, given that the spot price is unchanged by the introduction of futures, a greater spot quantity traded implies more efficient risk sharing. Computing the derivative of S_n with respect to ψ (see the appendix), one confirms this.

Proposition 9. $S_n(\psi)$ *increases with* ψ*.*

Spot trading surplus S_n increases for the following three reasons. First, large-inventory traders save holding costs from date 0 to date 1 that show up when $\sigma_1^2 > 0$. The second effect goes through uncertainty over the supply shock $\sigma_Q^2 > 0$ (which implies $z < 1$): it makes date-1 surplus more uncertain, thus less valuable. Reducing trade delay saves such cost. The third effect comes from imperfect competition: given that $\alpha < 1$, risk sharing is ultimately reduced given z when trade is delayed more.

On the futures side however, having a futures position opposite to date-1 spot trade is a welfare cost for trader n. To see this, it is interesting to compare two opposite positions $x_n > 0$ and $-x_n$, holding spot trades equal: the two have the same risk holding cost $\gamma\Sigma_{22}/2(x_n)^2$, so their impact on welfare differ only through the term in x_n. This term contains a risk-ajusted expected payoff $\widehat{p}_1 - f_0^*$ minus the marginal holding cost of the underlying asset after date-0 trade, $\gamma(\sigma_1^2 + (1-z)\sigma_2^2)(I_n + q_{n,0}^*)$: in equilibrium, using equilibrium futures price (4.14) and (4.6), this term turns out to equal (denoting $I_{n,1}^* = I_n + q_{n,0}^*$)

$$+x_n\gamma(\sigma_1^2 + (1-z)\sigma_2^2)(\bar{I}_1 - I_{n,1}^*),$$

whatever $\mathbb{E}_0[Q]$. Therefore, between x_n and $-x_n$, the position that maximizes welfare is the one that has the same sign as the date-1 quantity traded, which is proportional to $(\bar{I}_1 - I_{n,1}^*)$.

Is the welfare gain of faster spot trading strong enough to offset the cost of not hedging date-1 transaction? The answer is no – there is always one side of the market that loses, and often all physical traders lose.

Theorem 1. *The equilibrium without futures always Pareto-dominates the equilibrium with futures. More precisely, for all $N \geq 3$, $\sigma_1^2 \geq 0$, and all $\sigma_Q^2 \geq 0$ such that an equilibrium exists, there are $\underline{Q} < 0, \overline{Q} > 0$ such that:*

- *if $\underline{Q} < \mathbb{E}_0[Q] < \overline{Q}$, all traders are worse off with futures: $\widehat{W}_{n,0}^* < \widehat{W}_{n,0}^\times$.*
- *if $\mathbb{E}_0[Q] > \overline{Q}$, large-inventory traders are worse off with futures ($\widehat{W}_{n,0}^* < \widehat{W}_{n,0}^\times$) and low-inventory traders gain; and conversely if $\mathbb{E}_0[Q] < \underline{Q}$.*

It is however always rational to participate in spot and futures markets: $V_n < \widehat{W}_{n,0}^*$.

Why is the outcome always inefficient with futures? Consider the case where $\underline{Q} < \mathbb{E}_0[Q] < \overline{Q}$ first. The futures position is always too large with respect to the gain from faster spot trading, precisely because the marginal benefit of strategically positioning on futures is greater than the marginal benefit of strategically positioning on spot: by concavity of trader n's wealth and optimality of his/her demand schedule given other trader's strategies, the equilibrium marginal cost of trading futures is also greater. And since spot and futures are substitutes, their trading costs are comparable, so that the quantity of futures traded is comparatively large.

To understand the second part of the theorem, notice that, as shown by equilibrium spot prices (4.5) and (4.14), the supply shock Q and its expectation $\mathbb{E}_0[Q]$ affect the terms of trade between physical traders.[15] When liquidity traders sell

[15]This comes from the fact that they do not directly value anything correlated with Q.

more, low-inventory physical traders gain at the expense of large-inventory traders, and conversely. As futures accelerate spot trading, for expectations of large liquidity trades (purchases or sales), the gain in futures trade for one side of the market can offset the loss, but this is at the expense of the other side of the market.

5.3.1 The welfare of liquidity traders

The date 1 supply shock Q is a net demand posted by traders whose preferences are not modelled, which in principle precludes computation of their welfare. However, it is possible to run simple welfare comparisons for them, because their demand is inelastic: modelled traders always absorb all their quantities at a given price. Thus liquidity traders' welfare is measured by the price at which their trades are executed. The date-1 equilibrium price is unaffected by the presence of futures: thus date 1 liquidity traders' welfare is unchanged.

5.3.2 Heterogenous risk aversions

The model presented in this paper assumes that buyers and sellers have the same risk aversion parameter γ. In the online appendix I show numerically that the results presented here regarding negative hedging ratios appear very robust, and the results regarding welfare are robust or very robust, depending on the number of traders on each side.

6 The impact of financial traders

Financial traders have several impacts on futures market. First, they bring new risk-bearing capacity to the market, so that they compress risk premia. Second, by increasing competition, they increase liquidity in futures markets relative to the spot market: this implies that physical traders now want to share inventory risk through the futures market, which they don't without financial traders; Third, although they have the same risk preferences as physical traders, financial traders react more strongly to news about the futures supply shock: I study this effect in Section **??**.

In this section, I first show the second effect, on liquidity, by computing the price impact matrices. Then I show the consequences of this improved liquidity on equilibrium trades: physical traders trade futures also to share inventory risk between date 0 and date 1, both using financial traders' risk bearing capacity and even between themselves. I also discuss welfare.

6.1 Financial traders improve futures market liquidity

Financial trader k's certainty equivalent of wealth at date 0 is simply the mean-variance equivalent of (2.5):

$$\widehat{W}_{k,0} = y_k \left(\mathbb{E}_0[p_1^*] - f_0\right) - \frac{\gamma}{2} \operatorname{Var}_0(p_1^*) y_k^2.$$

In an analogous way to physical trader n at date 1, financial trader k cares about the impact of his/her futures trade y_k on futures price f_0, which I denote μ_k; as before, I look for equilibria in linear strategies, so that μ_k does not depend quantities. Trader k's optimal demand schedule given μ_k is simply:

$$y_k^*(f_0; \mu_k) = \frac{\mathbb{E}_0[p_1^*] - f_0}{\mu_k + \gamma \operatorname{Var}_0(p_1^*)}. \tag{6.1}$$

These additional demand schedules increase the slope of the residual curve in futures market faced by physical trader n; symmetrically, financial trader k faces all other financial traders and physical traders' demand schedules in the market. To be consistent with equilibrium slopes, price impacts $\Lambda_{n,K}$ for physical trader n and μ_k financial trader k, now solve the following (see Malamud and Rostek (2017)):

$$\begin{cases} \Lambda_{n,K} & = \left((N-1)\left(\Lambda_{n,K} + \gamma\Sigma\right)^{-1} + \frac{K}{\mu_k + \gamma\sigma^2} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right)^{-1} \\ \mu_k & = \left[\left(N \left(\Lambda_{n,K} + \gamma\Sigma\right)^{-1} + \frac{K-1}{\mu_k + \gamma\sigma^2} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right) \right]_{2,2} \end{cases} \tag{6.2}$$

where $\sigma^2 \equiv \operatorname{Var}_0(p_1^*)$ and for a matrix X, $[X]_{i,j}$ is the element on the ith row and jth column.[16] The following proposition, proved in the appendix, gives the solution to this system together with some of their properties.

Proposition 10. *The unique solution to* (6.3) *with positive price impacts is*

$$\mu_k = a\, \gamma \operatorname{Var}_0(p_1^*), \tag{6.3}$$

$$\Lambda_{n,K} = \frac{\gamma}{N + \widetilde{K} - 2} \Sigma + \gamma \begin{pmatrix} \Delta & 0 \\ 0 & 0 \end{pmatrix}, \tag{6.4}$$

[16] I follow Malamud and Rostek (2017), who use the natural procedure of filling the price impact matrices with zeros on rows and columns corresponding to markets where trader k is not present ("lifting"), and solve their model accordingly.

where $\Delta > 0$ iff $K \geq 1$, $\Delta = 0$ if $K = 0$, and $\widetilde{K}$ is an effective number of financial traders, equal to zero when $K = 0$ and increasing without bound with K. Moreover,

$$0 < a < \frac{1}{N+K-2} < b \equiv \frac{1}{N+\widetilde{K}-2}.$$

All coefficients in physical traders' price impacts matrix $\Lambda_{n,K}$ decrease with K.

Physical traders' price impact matrix (6.4) is composed of two terms. The first one is analogous to the case without financial traders, as it is proportional to Σ and is divided by the number of traders (although twisted from K to $\widetilde{K}$ for financial traders) minus 2. In particular, the more financial traders there are, the greater competition is, and the lower this term.

The second term in (6.4) reflects improvement by financial traders of futures market liquidity relative to the spot market: the coefficient Δ limits the improvement in spot market liquidity with respect to the first term. The off-diagonal terms are also zero, which means that financial traders decrease the cross-market price impacts $\partial f_0/\partial q_{n,0}$ and $\partial p_0/\partial x_n$ in the same way as the futures price impact of a futures trade.

As we will see, this improvement in futures market liquidity gives physical traders incentive to offload more of their inventory risk through futures market. The last part of the proposition states that this is not at the expense of spot market liquidity. In the next subsections, I show that spot trades are unchanged with financial traders, so that financialization indeed improves risk sharing between physical traders.

6.2 Equilibrium prices

Futures demand from financial traders is, from (6.1) and Proposition 10,

$$y_k^*(f_0) = \frac{\mathbb{E}[p_1^*] - f_0}{(1+a)\gamma \operatorname{Var}_0(p_1^*)}, \tag{6.5}$$

while the first order condition for physical trader n is (5.8) but with price impact matrix $\Lambda_{n,K}$. Putting the two into the market clearing conditions (2.1) and (2.2) and solving the system, one obtains the following proposition (detailed computations are in the appendix).

Proposition 11. *The equilibrium spot and futures prices at date 0 are:*

$$p_0^*(K) = p_0^*(0) + \frac{\widetilde{K}}{N}\frac{\Sigma_{21}}{\Sigma_{22}}(\mathbb{E}_0[p_1^*] - f_0^*(K)), \tag{6.6}$$

$$f_0^*(K) = f_0^*(0) + \frac{\widetilde{K}}{N}(\mathbb{E}_0[p_1^*] - f_0^*(K)), \tag{6.7}$$

where $p_0^*(0)$ and $f_0^*(0)$ are the spot and futures prices without financial traders, which are given in Proposition 3. The expected profit per unit of futures contract is:

$$\mathbb{E}_0[p_1^*] - f_0^*(K) = \gamma \frac{N}{N + \widetilde{K}} \left(\sigma_1^2 \bar{I}_0 - (1 - z)\frac{N-1}{N-2}\sigma_2^2 \, \mathbb{E}_0 \left[\frac{Q}{N} \right] \right). \tag{6.8}$$

Equations (6.6) and (6.7) show that, when financial traders are long futures, *i.e.* when the expected profit on a long futures position is positive, spot and futures price both rise. This is intuitive and echoes long-standing concerns about the effect of commodity futures financialization. The more financial traders there are and/or the less risk they perceive (increasing $\widetilde{K}$ in both cases), the larger the effect. Financial traders traders thus affect the terms of trade between high- and low-inventory physical traders in both markets, in a way that depends on equilibrium trades.

Financial traders could as well be short futures and decrease prices. But under plausible conditions in the model, financial traders are long: In commodities markets, one typically expects that the market is on average long ($\bar{I}_0 > 0$). On top of this, we may reasonably expect that future flows out of the market $-\mathbb{E}_0[Q]$ do not exceed existing inventories, so that $\mathbb{E}_0[Q] < \bar{I}_0$. If σ_1^2 is not too small with respect to σ_2^2, then (6.8) implies that futures expected profit is positive.

Finally, notice that the expected futures profit mainly depends on fundamentals of the spot market. The only dependence on the financial sector is through $\tilde{K}$: with more financial traders, the futures price increases and the expected profit decreases. We now study why prices are affected by financial traders.

Price volatility. Here I study how financial traders affect the volatility of prices by making spot and futures prices more reactive to news on the supply shock. News on payoff v are not affected, since the coefficient on v_0 in equilibrium prices is still one. To assess the impact on price volatility, I now consider $\mathbb{E}_0[Q]$ itself as a random variable with mean 0 and variance $\sigma_{EQ}^2 > 0$.

From (6.6) and (6.8), one has

$$p_0^*(K) = v_0 - \gamma s_K \bar{I}_0 - \left[z + (1-z)\frac{\widetilde{K}}{N+\widetilde{K}}\frac{\Sigma_{21}}{\Sigma_{22}}\frac{N-1}{N-2}\right]\gamma\sigma_2^2\,\mathbb{E}_0\left[\frac{Q}{N}\right]. \tag{6.9}$$

for some easily computed $s_K > 0$. Clearly, given that $1 - z > 0$, the coefficient in front of $\mathbb{E}_0[Q]$ is greater as K increases: the date-0 spot price is more reactive to news. However, given that $\widetilde{K}/(N+\widetilde{K}) < 1$ and $\Sigma_{21} < \Sigma_{22}$, one easily sees that this coefficient remains lower than that in front of Q/N in p_1^* in (4.5): therefore there is no over-reaction of date-0 spot price in that, setting $\sigma_1^2 = 0$ to leave inventory risk premia considerations aside, the date-0 price will never increase above (decrease below) the expected date-1 spot price.

The variance of p_0^* before news is:

$$\mathrm{Var}\left(p_0^*(K)\right) = \left[z + (1-z)\frac{\widetilde{K}}{N+\widetilde{K}}\frac{\Sigma_{21}}{\Sigma_{22}}\frac{N-1}{N-2}\right]^2 \gamma^2\sigma_2^4\frac{\sigma_{EQ}^2}{N^2},$$

which increases with $\widetilde{K}$, thus with the number of financial traders. One can apply the same steps to futures price to show a similar behavior of futures prices.

However, financial traders do not affect the unconditional distribution of Q, nor the distribution of the date-1 spot price.[17] Moreover, from (6.9) it can be inferred that the date 0 price never overreacts to news on Q, in the sense that the coefficient in front of $\mathbb{E}_0[Q/N]$ is always below that of Q/N in the date-1 spot price. Thus the following, that I prove formally in the appendix.

Lemma 4. *The variance of p_0^* before news $\mathbb{E}_0[Q]$ increases with financial traders, but the variance of $p_1^* - p_0^*$ decreases, while the variance of p_1^* is unchanged.*

6.3 Equilibrium trades and welfare discussion

Plugging equilibrium prices into demand schedules, one gets equilibrium quantities as follows, the proof being in the appendix.

[17]This may not hold under asymmetric information about $\mathbb{E}_0[Q]$. However, if the number of financial traders and their risk aversion were common knowledge (as the former can be from Commitments of Traders data), physical traders could likely undo the effect of a larger price movement in their update from spot price.

Proposition 12. *In equilibrium, financial trader k's futures trade is:*

$$y_k^* = \frac{N}{N+\widetilde{K}} \frac{\sigma_1^2 \bar{I}_0 - (1-z)\frac{N-1}{N-2}\sigma_2^2\, \mathbb{E}_0\left[Q/N\right]}{(1+a)\,\mathrm{Var}_0(p_1^*)} \tag{6.10}$$

For physical trader n, equilibrium spot trades are unchanged compared to the case without financial traders:

$$q_{n,0}^* = \psi_0 \left(\bar{I}_0 - I_n\right), \qquad q_{n,1}^* = \frac{N-2}{N-1}(1-\psi_0)\left(\bar{I}_0 - I_n\right) + \frac{Q}{N}, \tag{6.11}$$

and physical trader n's equilibrium futures trade is:

$$x_n^* = h_K\left(\bar{I}_1^e - I_{n,1}\right) + \Delta\frac{\Sigma_{21}}{|\Sigma|}\frac{N-2}{N-1}\left(\bar{I}_0 - I_n\right) - \frac{K}{N}\, y_k^* \tag{6.12}$$

$$= H_K\left(\bar{I}_0 - I_n\right) - \frac{K}{N}\, y_k^* \tag{6.13}$$

where $h_K < h_0 < 0$ for $K \geq 1$, but $H_K > h_0$.

Remarkably, spot trades are unchanged by the presence of financial traders, the fraction of gains from trade still being ψ_0. This however is not surprising, given the discussion of Section 5.2: DISCUSSION HERE.

Futures trades are changed in several ways. First, physical traders take the other side to financial traders, as shown by the term $-K/Ny_k^*$. In particular, physical traders as a whole now share risk with financial traders.

Second, because of the greater liquidity of the futures market (the liquidity gap being measured with Δ), physical traders have an incentive to share inventory risk through the futures market, irrespective of intertemporal price impact consideration: this is the second term in (6.12). Since spot trades are unchanged, this means that physical traders share more inventory risk among themselves, because of the greater futures market liquidity: this again increases their welfare.

Third, with financial traders, the quantity related to date-1 trade is *worse*, *i.e.* $h_K < h_0$. Yet this effect remains small enough so that it is more than offset with the increase in inventory risk sharing due to greater futures market liquidity: *i.e.*, $H_K > h_0$.

One may wonder whether the incentive to share inventory risk can dominate the intertemporal price impact consideration seen without financial traders. The following proposition, proved in the appendix, shows that this is the case: with sufficiently many financial traders, physical traders' use futures for hedging, *i.e.*

$H_K > 0$.

Proposition 13. *H_K increases with K. For all $N \geq 3$, there is K_N such that for all $K > K_N$, $H_K > 0$.*

Moreover, in the limit where the number of financial traders is infinite, H_K increases with σ_Q^2.

Therefore, with sufficiently many financial traders, physical traders actually share risk between themselves using futures. There is no need to have a very high number of financial traders to have $H_K > 0$: the relevant metric to assess it is $\widetilde{K}/(N+\widetilde{K}-2)$, which is below 1. Simulations show that for $N = 3$ or $N = 4$, having $\widetilde{K} = N$ ensures $H_K > 0$ for a broad range of parameters.

6.4 Welfare implications

By improving futures market liquidity, financial traders allow physical traders to restore futures positions in the "right"direction, *i.e.* allowing them to share risk. Trading futures in the wrong direction was a major cost to physical traders without financial traders. Since spot trades are unaffected by financial traders, and more gains from spot trade are realized with futures than without, one may wonder whether financial traders can lead to an equilibrium that is more efficient than without futures.

This is not obvious however, given that financial traders also harm some physical traders by raising spot and futures prices if they are long, which harms low-inventory traders, and lowering prices if they are short.

The following theorem gives the answer when there are many financial traders. Suppose that financial traders are long, then traders with large, or very low inventory always benefit from financialization, so much that they are better off with respect to the case without futures. Symmetrically, when financial traders are short, all low-inventory traders benefit from financialization, as are physical traders with very large inventory.

This result does not extend in general to physical traders with intermediate inventory: the theorem also exhibits values – when risk is low – for which physical traders with intermediate low inventory are worse off when financial traders are long, even with respect to the case with futures and without financial traders. This is because for them, the effect of terms of trade dominates the benefit of improved hedging. The proof is in appendix.

Theorem 2. *Suppose $H_K > 0$. Then there exists $I_{min} \leq I_{max} \leq \bar{I}_0$, dependent on K and other parameters, such that*

$$I_n < I_{min} \text{ or } I_n > I_{max} \quad \Rightarrow \quad \widehat{W}_{n,0}(K) > \widehat{W}^{\times}_{n,0}.$$

Suppose that 1) $K \to \infty$ and that 2) σ_1^2 is close to 0 and σ_Q^2 is close the lowest possible value such that an equilibrium exists (z is close to $\bar{z}(0)$. Then when financial traders are long, there exists $I_{min,2} < I_{max,2} \leq \bar{I}_0$ such that

$$I_{min,2} < I_n < I_{max,2} \Rightarrow \widehat{W}_{n,0}(K) < \widehat{W}_{n,0}(0) < \widehat{W}^{\times}_{n,0}$$

How does the second part of the theorem generalize? Numerical computations suggest that it generalizes quite well: for all N, there is a wide range of σ_1^2 and σ_Q^2 for which it holds, and for $N \geq 4$, it seems to be for all σ_1^2 and admissible σ_Q^2.

7 Conclusion

In this paper, I theoretically study the financialization of futures markets when traders are imperfectly competitive. I provide an equilibrium model of futures trading where some traders can trade spot without constraint (physical traders), and some cannot trade spot (financial traders). This model is otherwise fully standard and I assume symmetric information. Traders can trade spot and futures at date 0, physical traders trade spot at date 1, and the underlying asset pays off at date 2; futures mature at date 1, thus have a shorter maturity than the spot, as often observed in real-world markets. This model allows to study the effects of the financialization of commodities markets from the lens of imperfect competition, a plausible friction of real world commodities markets.

I first show that without financial traders, futures and spot are not perfect substitutes, because of imperfect competition at date 1: therefore, delaying one unit of spot sales yields less than one unit sold at the spot price, while buying one unit of futures at date 0 pays off the spot price times this unit. This implies that spot and futures price at date 0 differ: there is a futures-spot basis, without any violation of the law of one price.

Then I show that, surprisingly, futures trading is inefficient, as traders end up with lower welfare with futures than without. This is because traders' spot and futures trading strategies are driven essentially by price impact considerations.

Finally, I show that with financial traders, futures market liquidity is improved, giving physical traders incentive to use futures for risk sharing, with this motive dominating when there are sufficiently many financia traders in the market. I also discuss the pricing implications of financialization, by showing that financial traders reduce momentum and makes date-0 spot price more strongly react to news about futures supply shock.

Appendix

A Proofs

A.1 Proof of proposition 1

When $x_n = 0$, (3.5) becomes:

$$\widetilde{W}^c_{n,1} = V_n + S_{n,0} + \widetilde{S}_{n,1} \tag{A.1}$$

The optimal demand $q^c_{n,0}$ maximizes (A.1) subject to (3.9) and (3.10), and is the unique solution to the following first order condition:[18]

$$v_0 - p_0 = \gamma(\sigma_1^2 + \sigma_2^2)\left(I_{n,0} + q_{n,0}\right) - \gamma\sigma_2^2 z_c \left[\mathbb{E}_0[Q^*_c] - I_{n,0} - q_{n,0}\right].$$

This can be rearranged as:

$$q^c_{n,0}(p_0) = \frac{v_0 - p_0}{\gamma(\sigma_1^2 + (1 - z_c)\sigma_2^2)} - \frac{z_c\sigma_2^2}{\sigma_1^2 + (1 - z_c)\sigma_2^2}\,\mathbb{E}_0\left[\frac{Q^*_c}{2}\right] - I_{n,0}, \tag{A.2}$$

which, using (2.1) gives the equilibrium price (3.11) and quantity (3.12). The optimality of risk sharing comes from the first welfare theorem.

A.2 Proof of Lemma 1

In an analogous way to the competitive case (3.5) and (3.6), trader n's equilibrium certainty equivalent of wealth after date-1 trade is

$$\widehat{W}_{n,1} = I_n v_1 + q_{n,0}(v_1 - p_0) - \frac{\gamma\sigma_2^2}{2}(I_{n,1})^2 + \frac{N}{N-2}\frac{\gamma\sigma_2^2}{2}\left(q^*_{n,1}\right)^2 \tag{A.3}$$

Take the certainty equivalent with respect to ϵ_1 first (which is possible since Q and ϵ_1 are independent):

$$\widehat{W}_{n,0}|Q = I_n v_0 + q_{n,0}(v_0 - p_0) - \frac{\gamma(\sigma_1^2 + \sigma_2^2)}{2}(I_{n,1})^2 + \alpha\frac{\gamma\sigma_2^2}{2}\left(\frac{\bar{\gamma}}{\gamma}\,\mathbb{E}_0\left[\frac{Q}{N}\right] + \bar{I}_0 - I_{n,1}\right)^2,$$

[18] The problem is strictly concave, as $z_c < 1$. Moreover, since in (3.5), the expression for date-1 surplus involves $q^c_{n,1} = \frac{v_1 - p^c_1}{\gamma\sigma_2^2} - I_{n,0} - q_{n,0}$, the price-taking assumption involves $\partial q^c_{n,1}/\partial q_{n,0} = -1$, thus the first order condition.

where $\bar{\gamma} = \frac{N-1}{N-2}\gamma$. Then use the following lemma to compute the date-0 certainty equivalent of wealth with respect to Q.

Lemma 5. *Let $X \sim \mathcal{N}(\mu, \Sigma)$ a normal vector of dimension p ($|\Sigma| > 0$), and A a symmetric matrix. Suppose $I + 2\gamma A\Sigma$ is positive definite, then*

$$\mathbb{E}[\exp(-\gamma X'AX)] = \frac{1}{\sqrt{|I + 2\gamma A\Sigma|}} \exp\left\{-\gamma\mu'(I + 2\gamma A\Sigma)^{-1}A\mu\right\}.$$

Proof.

$$\mathbb{E}[\exp(-\gamma X'AX)] = \int_{\mathbb{R}^p} \frac{1}{\sqrt{2\pi|\Sigma|}} \exp\left\{-\gamma x'Ax - \frac{1}{2}(x-\mu)'\Sigma^{-1}(x-\mu)\right\} dx$$

where $dx \equiv dx_1 dx_2 ... dx_p$. One first computes

$$\begin{aligned} Q(x) &= -\gamma x'Ax - \frac{1}{2}(x-\mu)'\Sigma^{-1}(x-\mu) \\ &= -\frac{1}{2}(x-\mu)'(\Sigma^{-1} + 2\gamma A)(x-\mu) - 2\gamma\mu'A(x-\mu) - \gamma\mu'A\mu \end{aligned}$$

Suppose that $(\Sigma^{-1} + 2\gamma A)$ is the inverse of a covariance matrix, then the formula gives the moment generating function of a normal variable with covariance matrix $\left[(I + 2\gamma A\Sigma)\Sigma^{-1}\right]^{-1} = \Sigma(I + 2\gamma A\Sigma)^{-1}$.

$$\begin{aligned} \mathbb{E}\left[e^{-\gamma X'AX}\right] &= \frac{e^{-\gamma\mu'A\mu}}{\sqrt{|I + 2\gamma A\Sigma|}} \\ &\quad \times \int_{\mathbb{R}^p} \frac{1}{\sqrt{2\pi|\Sigma||I + 2\gamma A\Sigma|^{-1}}} e^{-2\gamma\mu'A(x-\mu)} e^{-\frac{1}{2}(x-\mu)'[\Sigma(I+2\gamma A\Sigma)^{-1}]^{-1}(x-\mu)} dx \\ &= \frac{1}{\sqrt{|I + 2\gamma A\Sigma|}} \exp\left\{\gamma\mu'A\Sigma(I + 2\gamma A\Sigma)^{-1}A\mu - \gamma\mu'A\mu\right\} \\ &= \frac{1}{\sqrt{|I + 2\gamma A\Sigma|}} \exp\left\{-\gamma\mu'(I + 2\gamma A\Sigma)^{-1}A\mu\right\} \end{aligned}$$

■

The certainty equivalent of wealth at date 0 is therefore:

$$\widehat{W}_{n,0} = I_n v_0 + q_{n,0}(v_0 - p_0) - \frac{\gamma(\sigma_1^2 + \sigma_2^2)}{2} (I_{n,1})^2 + \frac{1}{2}\frac{N}{N-2}\gamma\sigma_2^2 z \left(\mathbb{E}_0 \left[q_{n,1}^*\right]\right)^2 \quad (A.4)$$

$$= I_n v_0 + q_{n,0}(v_0 - p_0) - \frac{\gamma(\sigma_1^2 + \sigma_2^2)}{2} (I_{n,1})^2 + \frac{\alpha}{2}\gamma\sigma_2^2 z \left(\frac{N-1}{N-2}\mathbb{E}_0\left[\frac{Q}{N}\right] + \bar{I}_1^e - I_{n,1}\right)^2 \quad (A.5)$$

where $z = (1 + \bar{\gamma}^2\sigma_2^2\sigma_q^2)^{-1}$, and $\bar{I}_1^e = \frac{1}{N}\sum_m (I_m + q_{m,0}^e)$. This proves the lemma.

A.3 Proof of Proposition 3

A.4 Proof of Propositions 4 and 5

Date-0 wealth and equilibrium definition. (4.6) becomes, with $x_n = 0$,

$$\widehat{W}_{n,0}(q_{n,0}, 0) = V_n + S_{n,0}(q_{n,0}) + \widehat{S}_{n,1}(q_{n,0}). \quad (A.6)$$

I look for an equilibrium as in Definition 2 but constraining $x_n = 0$. Maximization of (A.6) with respect to $q_{n,0}$, using price impact $\lambda_{n,0} = \frac{\gamma\Sigma_{11}}{N-2}$ for trader n, implies solving the following first order condition ($\Sigma_{11} > 0$, so $\widehat{W}_{n,0}(q_{n,0}, 0)$ is a strictly concave function of $q_{n,0}$):

$$v_0 - \frac{N-2}{N-1}w_n - p_0 = \frac{N-1}{N-2}\gamma\Sigma_{11}q_{n,0} + \gamma\Sigma_{11}I_n.$$

, so that one can solve for the date-0 equilibrium similarly to date 1. In particular $\lambda_{n,0} = \frac{\gamma\Sigma_{11}}{N-2}$, so that trader n's equilibrium demand schedule is, expliciting w_n^1:

$$q_{n,0}^{\times}(p_0) = \frac{N-2}{N-1}\left[\frac{v_0 - p_0}{\gamma\Sigma_{11}} - I_n - \frac{1}{\gamma\Sigma_{11}}\frac{N-2}{N-1}w_n\right]. \quad (A.7)$$

Equilibrium price and quantities Plugging (A.7) into the market clearing condition (2.1), one gets

$$\frac{v_0 - p_0^{\times}}{\gamma\Sigma_{11}} = \bar{I}_0 + \frac{N-2}{N-1}\frac{z\sigma_2^2}{\Sigma_{11}}\left(\frac{N-1}{N-2}\mathbb{E}_0\left[\frac{Q}{N}\right] + \frac{N-1}{N}\bar{I}_1^e\right).$$

By date-1 market clearing and $q^{\times}_{n,0} = q^{e}_{n,0}$, $\bar{I}^e_1 = \bar{I}_0 + \frac{1}{N}\sum_n q^{\times}_{n,0} = \bar{I}_0$. Thus one has

$$v_0 - p_0^{\times} = \gamma\Sigma_{11}\bar{I}_0 + \frac{N-2}{N}\gamma\sigma_2^2 z\bar{I}_0 + \gamma\sigma_2^2 z\,\mathbb{E}_0\left[\frac{Q}{N}\right]. \tag{A.8}$$

Rearranging with (4.9), (A.8) leads to (5.7). Plugging (A.8) into (A.7) leads to (5.4).

Now impose equilibrium condition (4.12) into (5.4), which can be rearranged as

$$(1+\kappa_{\times})\, q^{\times}_{n,0} = \frac{N-2}{N-1}(\bar{I}_0 - I_n) + \kappa_{\times}(\bar{I}_0 - I_n),$$

So that, denoting $\psi_{\times} = \frac{\frac{N-2}{N-1}+\kappa_{\times}}{1+\kappa_{\times}}$,

$$q^{*}_{b,0} = \psi_{\times}(\bar{I}_0 - I_n), \qquad A(\sigma_q^2) = \frac{1}{N-2}\,\frac{\sigma_1^2 + \left(1 - \frac{N-1}{N}z\right)\sigma_2^2}{\sigma_1^2 + \left(1 - \frac{N-2}{N-1}z\right)\sigma_2^2} \tag{A.9}$$

The properties of $A(\sigma_q^2)$ are derived in lemma 6 in appendix A.4.1.

The date-1 quantity is straightforwardly derived from (A.9) and (4.4).

A.4.1 Properties of the rate of trade delay $A(\sigma_1^2, \sigma_q^2)$

Define, for $x = \sigma_1^2/\sigma_2^2$ and $z = \frac{1}{1+\alpha\bar{\gamma}^2\sigma_2^2\sigma_q^2} \in [0,1]$, the ratio

$$\widetilde{A}(x,z) = \frac{1}{N-2}\,\frac{x+1-\frac{N-2}{N}z}{x+1-\frac{N-2}{N-1}z}$$

so that $A(\sigma_1^2, \sigma_q^2) = \widetilde{A}(x,z)$.

Lemma 6. *Then whatever the finite parameters $N \geq 3$, $\sigma_1^2 \geq 0$ and $\sigma_2^2 > 0$:*

1. *$\widetilde{A}(x,z)$ is strictly increasing in z so that $A(\sigma_1^2, \sigma_q^2)$ strictly decreases in σ_q^2.*
2. *$\widetilde{A}(x,z)$ is strictly decreasing in x so that $A(\sigma_1^2, \sigma_q^2)$ strictly decreases in σ_1^2.*
3. *$1 < (N-2)A(\sigma_1^2, \sigma_q^2) < \frac{4}{3}$*

4. Therefore

$$\begin{aligned}
\frac{1}{N-2} < A(\sigma_1^2, \sigma_q^2) < \frac{2(N-1)}{N(N-2)} \leq \frac{4}{3} \\
\frac{3}{7} \leq \frac{N(N-2)}{N^2-2} < \frac{1}{1+A(\sigma_1^2, \sigma_q^2)} < \frac{N-2}{N-1} \\
\frac{1}{N-1} < \frac{A(\sigma_1^2, \sigma_q^2)}{1+A(\sigma_1^2, \sigma_q^2)} \leq \frac{2N-2}{N^2-2} \leq \frac{4}{7}
\end{aligned}$$

Proof. For 1., compute the derivatives

$$\begin{aligned}
\frac{\partial \widetilde{A}}{\partial z} &= \frac{\sigma_2^2}{N(N-1)} \frac{\sigma_1^2 + \sigma_2^2}{\left(\sigma_1^2 + \left(1 - \frac{N-2}{N-1} z\right) \sigma_2^2\right)^2} > 0 \\
\frac{\partial \widetilde{A}}{\partial x} &= -\frac{1}{N(N-1)} \frac{1}{\left(x + 1 - \frac{N-2}{N-1} z\right)^2} < 0
\end{aligned}$$

For 2., the first inequality is easily derived from $z \geq 0$; the case $z = 0$ corresponds to $\sigma_q^2 \to \infty$. For the second inequality, given that $\widetilde{A}(\cdot)$ is increasing,

$$(N-2)\widetilde{A}(z) \leq (N-2)\widetilde{A}(1) = \frac{\frac{\sigma_1^2}{\sigma_2^2} + 1 - \frac{N-2}{N}}{\frac{\sigma_1^2}{\sigma_2^2} + 1 - \frac{N-2}{N-1}} \leq \frac{1 - \frac{N-2}{N}}{1 - \frac{N-2}{N-1}} = 2\left(1 - \frac{1}{N}\right)$$

where the last inequality follows from the fact that the ratio $\widetilde{A}(1)$ is decreasing in the ratio σ_1^2/σ_2^2. Given that $N \geq 3$, one finally gets the desired inequality

$$(N-2)A(z) \leq \frac{4}{3}.$$

For 3., applying the mappings $x \mapsto 1/(1+x)$ and $x \mapsto x/(1+x)$ to inequalities derived in 2. (all members in these inequalities are greater than -1 so the first mapping reverses ordering, the second preserves it), one gets the desired inequalities. The last inequality is found by applying $N = 3$. ∎

A.5 Equilibrium with futures contracts

A.5.1 Certainty equivalent of wealth

After date-1 trade and with futures payoff, the certainty equivalent of wealth is

$$
\begin{aligned}
\widehat{W}_{n,1} &= I_n v_1 + q_{n,0}(v_1 - p_0) - \frac{\gamma\sigma_2^2}{2}(I_{n,1})^2 + \frac{\alpha}{2}\gamma\sigma_2^2\left(\frac{N-1}{N-2}Q^* - I_{n,1}\right)^2 \\
&\quad + x_n\left(v_0 + \epsilon_1 - \bar{\gamma}\sigma_2^2 Q^* - f_0\right) \\
&= I_n v_0 + q_{n,0}(v_0 - p_0) + x_n\left(v_0 - f_0\right) + (I_{n,1} + x_n)\epsilon_1 \\
&\quad + \frac{\alpha}{2}\gamma\sigma_2^2\left(\frac{N-1}{N-2}Q^* - I_{n,1} - \frac{x_n}{\alpha}\right)^2 - \frac{\gamma\sigma_2^2}{2}\left((1-\alpha)(I_{n,1})^2 + \alpha\left(I_{n,1} + \frac{x_n}{\alpha}\right)^2\right)
\end{aligned}
$$

where $Q^* = Q/N + \frac{N-2}{N-1}(\bar{I}_0 - I_n)$. Taking the certainty equivalent of wealth with respect to ϵ_1 and Q, one gets

$$
\begin{aligned}
\widehat{W}_{n,0} &= I_{n,0} v_0 + q_{n,0}(v_0 - p_0) + x_k n\left(v_0 - f_0\right) + \frac{\alpha}{2}\gamma\sigma_2^2 z\left(\frac{N-1}{N-2}Q^* - I_{n,1} - \frac{x_n}{\alpha}\right)^2 \\
&\quad - \frac{\gamma}{2}\left(\sigma_1^2(I_{n,1} + x_n)^2 + \sigma_2^2(1-\alpha)(I_{n,1})^2 + \sigma_2^2\alpha\left(I_{n,1} + \frac{x_n}{\alpha}\right)^2\right)
\end{aligned}
\tag{A.10}
$$

Developing and rearranging to separate terms in $I_{n,1}^2$ and terms in x_n^2 leads to expression (4.6).

A.5.2 Concavity: Proof of lemmas 2 and ??

For $\widehat{W}_{n,0}$ to be strictly concave, the first diagonal coefficient of Σ has to be positive, which is easily checked, and the determinant of Σ has to be positive:

$$
\begin{aligned}
|\Sigma| &= \left(\sigma_1^2 + \left(1 - \frac{N-2}{N}z\right)\sigma_2^2\right)\left(\sigma_1^2 + \frac{1-z}{\alpha}\sigma_2^2\right) - \left(\sigma_1^2 + \left(1 - \frac{N-1}{N}z\right)\sigma_2^2\right)^2 \\
&= \left(\frac{1}{\alpha} - 1\right)\sigma_2^2\left\{(1-z)\sigma_1^2 + \left(1 - 2\frac{N-1}{N}z\right)\sigma_2^2\right\}
\end{aligned}
\tag{A.11}
$$

Given that $\alpha < 1$, the determinant of Σ is positive as long as

$$
z < \frac{\sigma_1^2 + \sigma_2^2}{\sigma_1^2 + 2\frac{N-1}{N}\sigma_2^2} \equiv \bar{z}(\sigma_1^2) < 1 \tag{A.12}
$$

Thus for low σ_Q^2 (i.e. $z > \bar{z}(\sigma_1^2)$), the determinant is negative and $\widehat{W}_{n,0}$ is unbounded.

Trading strategies giving unbounded profit. Suppose σ_q^2 is small enough so that $|\Sigma| < 0$; given that the first diagonal coefficient of Σ is positive, this implies that the graph of the two-variable mapping $(q_{n,0}, x_n) \mapsto \widehat{W}_{n,0}$ is a hyperbolic paraboloid: there are directions $(q_{n,0}, x_n)$ that increase $\widehat{W}_{n,0}$, and others that decrease it. As for some constant, a_1 and a_2,

$$\widehat{W}_{n,0} = cst + a_1 q_{n,0} + a_2 x_n - \underbrace{(q_{n,0}, x_n)\Sigma(q_{n,0}, x_n)'}_{Q(q_{n,0}, x_n)}$$

it is sufficient to exhibit directions for which $(q_{n,0}, x_n)\Sigma(q_{n,0}, x_n)' < 0$. Set $q_{n,0} = -ax_n$, where the real number a defines the direction to find. For $x \neq 0$:

$$Q(-ax, x)/x^2 = a^2\Sigma_{11} + 2a\Sigma_{21} + \Sigma_{22}$$

This polynomial in a has roots (and can be negative for some a) if and only if its discriminant, equal to $-4|\Sigma|$, is positive. In this case, the roots are $a_\pm = (\Sigma_{21} \pm \sqrt{-|\Sigma|})/\Sigma_{11}$. For $a = a_0 = \Sigma_{21}/\Sigma_{11}$, $Q(-ax, x) < 0$. Thus it is possible to reach infinite profit by trading infinite quantities provided that $q_{k,0} = -a_0 x_k$. Moreover, $a_0 \in (0, 1)$: the strategy involves trading more futures than the underlying, in opposite directions. QED.

A.5.3 Proof of Lemma 3

We first prove the following statement regarding $|\Lambda_n + \gamma\Sigma|$.

Lemma 7. *The determinant $|\Lambda_n + \gamma\Sigma|$ is the marginal increase in c_q when trader n purchases a unit of underlying asset and shorts the amount of futures that keeps c_x constant. Equivalently, $|\Lambda_n + \gamma\Sigma|$ is the marginal increase in c_x, the marginal cost of trading futures, when trader n buys one unit of futures, and sells the sells the amount of underlying asset that keeps c_q constant.*

Proof. The determinant of the matrix $\Lambda_n + \gamma\Sigma$ is the determinant two vectors $\nabla c_q \equiv (\partial c_q/\partial q_{n,0}, \partial c_q/\partial x_n)$ and $\nabla c_x \equiv (\partial c_x/\partial q_{n,0}, \partial c_x/\partial x_n)$. It is thus well known to be the oriented area of the parallelogram spanned by these two vectors in the plane $(q_{n,0}, x_n)$: it is the orthogonal projection of ∇c_x on a line orthogonal to ∇c_q, i.e., the algebraic amount by which c_x increases when one moves along a vector orthogonal to ∇c_x, in a direction that increases x_n. The equivalent statement consider the orthogonal projection of ∇c_q on a vector orthogonal to ∇c_x. ■

Then, with a proof similar to the first part of the previous lemma, replacing c_x with the vector $\left(\frac{N-2}{N-1}w_n, w_n\right)'$, we can prove the following

Lemma 8. $\frac{N-2}{N-1}w_n(\Lambda_{n,22} + \gamma\Sigma_{22}) - w_n(\Lambda_{n,12} + \gamma\Sigma_{12})$ *is the marginal increase in the sensitivity of trader n's wealth to the impact of date-0 spot trade on date-1 spot price, when trader n purchases a unit of underlying asset and shorts the futures in the amount that keeps* c_x *constant.*

$w_n\Lambda_{n,11} + \gamma\Sigma_{11} - \frac{N-2}{N-1}w_n(\Lambda_{n,21} + \gamma\Sigma_{21})$ *is the marginal increase in the sensitivity of trader n's wealth to the impact of date-0 spot trade on date-1 spot price, when trader n purchases a unit of futures and sells the amount of underlying asset that keeps* c_q *constant.*

And lemma 3 naturally follows.

A.5.4 Proof of proposition 3

The first order conditions pin down the maximum of $\widehat{W}_{n,0}$. These conditions are, in matrix form:

$$\begin{pmatrix} v_0 - p_0 \\ v_0 - f_0 \end{pmatrix} - w_n \begin{pmatrix} \frac{N-2}{N-1} \\ 1 \end{pmatrix} = (\Lambda_n + \gamma\Sigma) \begin{pmatrix} q_{n,0} \\ x_n \end{pmatrix} + \gamma\Sigma \begin{pmatrix} I_n \\ 0 \end{pmatrix}.$$

Proposition 1 in Malamud and Rostek (2017) gives the equilibrium price impact: $\Lambda_n = \frac{\gamma}{N-2}\Sigma$. Therefore the first order conditions become

$$\begin{pmatrix} v_0 - p_0 \\ v_0 - f_0 \end{pmatrix} - w_n \begin{pmatrix} \frac{N-2}{N-1} \\ 1 \end{pmatrix} = \frac{N-1}{N-2}\gamma\Sigma \begin{pmatrix} q_{n,0} \\ x_n \end{pmatrix} + \gamma\Sigma \begin{pmatrix} I_n \\ 0 \end{pmatrix}.$$

These can be inverted, because Σ is positive definite, implying that $|\Sigma| > 0$, and the optimal demand schedules are

$$\begin{pmatrix} q^*_{n,0}(p_0, f_0) \\ x_n(p_0, f_0) \end{pmatrix} = \frac{N-2}{N-1} \left\{ \frac{1}{\gamma}\Sigma^{-1} \left[\begin{pmatrix} v_0 - p_0 \\ v_0 - f_0 \end{pmatrix} - w_n \begin{pmatrix} \frac{N-2}{N-1} \\ 1 \end{pmatrix} \right] - \begin{pmatrix} I_n \\ 0 \end{pmatrix} \right\}.$$

Plugging these demand schedules into the market clearing conditions (2.1) and (2.2):

$$\begin{pmatrix} v_0 - p^*_0 \\ v_0 - f^*_0 \end{pmatrix} - \frac{1}{N}\sum_{n=1}^{N} w_n \begin{pmatrix} \frac{N-2}{N-1} \\ 1 \end{pmatrix} = \gamma\Sigma \begin{pmatrix} \bar{I}_0 \\ 0 \end{pmatrix}.$$

And one has, from the definition of w_n,

$$\begin{aligned}
\frac{1}{N}\sum_{n=1}^{N} w_n &= \gamma\sigma_2^2 z\left(\frac{1}{N}\sum_{n=1}^{N}\sum_{m=1,m\neq n}^{N}\frac{I_{n,1}^e}{N}+\frac{N-1}{N-2}\mathbb{E}_0\left[\frac{Q}{N}\right]\right)\\
&= \gamma\sigma_2^2 z\left(\frac{N-2}{N-1}\frac{N-1}{N}\sum_{n=1}^{N}\frac{I_{n,1}^e}{N}+\mathbb{E}_0\left[\frac{Q}{N}\right]\right) = \gamma\sigma_2^2 z\left(\frac{N-2}{N}\bar{I}_1^e+\mathbb{E}_0\left[\frac{Q}{N}\right]\right)\\
&= \gamma\sigma_2^2 z\left(\frac{N-2}{N}\bar{I}_0+\mathbb{E}_0\left[\frac{Q}{N}\right]\right)
\end{aligned}$$

The last line follows from applying equilibrium condition (4.12) and date-0 market clearing (2.1), which implies $\bar{I}_1^e = \bar{I}_0$. This and (4.9) imply (4.13) and (4.14).

A.5.5 Proof of Proposition 6

Plugging equilibrium risk premia into trader n's demand schedule gives

$$\begin{pmatrix} q_{n,0}^* \\ x_n^* \end{pmatrix} = \frac{N-2}{N-1}(\gamma\Sigma)^{-1}\left[\gamma\Sigma\begin{pmatrix}\bar{I}_0 - I_n \\ 0\end{pmatrix} + \left(\sum_{m=1}^{N}\frac{w_m}{N}-w_n\right)\begin{pmatrix}\frac{N-2}{N-1}\\ 1\end{pmatrix}\right], \quad \text{(A.13)}$$

with

$$\sum_{m=1}^{N}\frac{w_m}{N}-w_n = \gamma\sigma_2^2 z\frac{N-1}{N}\bar{I}_1^e - \gamma\sigma_2^2 z\left(\bar{I}_1^e-\frac{I_{n,1}}{N}\right) = \frac{\gamma\sigma_2^2 z}{N}(I_{n,1}-\bar{I}_1^e)$$

Determination of κ_0 and ψ_0. One has, using (4.9),

$$|\Sigma|\kappa_0 = -z\sigma_2^2\left\{\frac{\sigma_1^2}{N-1}+\frac{\sigma_2^2}{N}\right\}.$$

Thus

$$1-\frac{\kappa_0}{N} = 1+\frac{1}{N}\frac{\alpha}{1-\alpha}\times\frac{z\left(\frac{1}{N-1}\sigma_1^2+\sigma_2^2/N\right)}{(1-z)\sigma_1^2+\left(1-2\frac{N-1}{N}z\right)\sigma_2^2}.$$

Recognizing $\frac{1}{N}\frac{\alpha}{1-\alpha} = \frac{1}{N}(N-1)^2\times\frac{N(N-2)}{(N-1)^2} = N-2$, one gets

$$1-\frac{\kappa_0}{N} = 1+\frac{(N-2)z\left(\frac{1}{N-1}\sigma_1^2+\sigma_2^2/N\right)}{(1-z)\sigma_1^2+\left(1-2\frac{N-1}{N}z\right)\sigma_2^2} = \frac{\left(1-\frac{z}{N-1}\right)\sigma_1^2+(1-z)\sigma_2^2}{(1-z)\sigma_1^2+\left(1-2\frac{N-1}{N}z\right)\sigma_2^2}.$$

Thus

$$A_f \equiv \left(1 - \frac{\kappa_0}{N}\right)^{-1} = \frac{1}{N-2} \times \frac{(1-z)\frac{\sigma_1^2}{\sigma_2^2} + 1 - 2\frac{N-1}{N}z}{\left[1 - \frac{z}{N-1}\right]\frac{\sigma_1^2}{\sigma_2^2} + 1 - z}. \tag{A.14}$$

$$\psi_0 \equiv \frac{1}{1+A_f} = \frac{N-2}{N-1} \frac{(1-z)\sigma_1^2 + (1 - z(N-1)/N)\,\sigma_2^2}{\left(1 - \frac{2N-3}{(N-1)^2}z\right)\sigma_1^2 + \left(1 - \left(\frac{N-2}{N-1} + \frac{2}{N}\right)z\right)\sigma_2^2} \tag{A.15}$$

$$1 - \psi_0 = \frac{1}{N-1} \frac{(1 - z/(N-1))\sigma_1^2 + (1-z)\,\sigma_2^2}{\left(1 - \frac{2N-3}{(N-1)^2}z\right)\sigma_1^2 + \left(1 - \left(\frac{N-2}{N-1} + \frac{2}{N}\right)z\right)\sigma_2^2} \tag{A.16}$$

Determination of h_0. One has, from (A.13),

$$\begin{aligned}|\Sigma| h_0 &= -\frac{\sigma_2^2 z}{N}\left[\sigma_1^2 + \left[1 - \frac{N-2}{N}z\right]\sigma_2^2 - \frac{N-2}{N-1}\left[\sigma_1^2 + \left[1 - \frac{N-1}{N}z\right]\sigma_2^2\right]\right] \\ &= -\frac{\sigma_2^2 z}{N}\frac{\sigma_1^2 + \sigma_2^2}{N-1}\end{aligned}$$

From (A.11), one gets, with $1/\alpha - 1 = 1/(N(N-2))$,

$$h_0 = -\frac{N-2}{N-1}\frac{(\sigma_1^2 + \sigma_2^2)z}{(1-z)\sigma_1^2 + \left(1 - 2\frac{N-1}{N}z\right)\sigma_2^2}. \tag{A.17}$$

The denominator is proportional to $|\Sigma|$, thus positive: so $h_0 < 0$.

A.5.6 Properties of A_f and h_0

A_f decreases with z. Setting $x = \sigma_1^2/\sigma_2^2$ to ease notation:

$$\begin{aligned}\frac{\partial A_f}{\partial z} &= \frac{1}{N-2}\frac{-\left(x + 2\frac{N-1}{N}\right)\left(x + 1 - z\left(\frac{x}{N-1} + 1\right)\right) + \left(\frac{x}{N-1} + 1\right)\left(x + 1 - z\left(x + 2\frac{N-1}{N}\right)\right)}{\left(x + 1 - z\left(\frac{x}{N-1} + 1\right)\right)^2} \\ &= -\frac{1}{N-2}\frac{(x+1)\left(\frac{N-2}{N-1}x + 1 - \frac{2}{N}\right)}{\left(\left(1 - \frac{z}{N-1}\right)x + 1 - z\right)^2} = -\frac{1}{N-1}\frac{(x+1)(x+1-1/N)}{\left(\left(1 - \frac{z}{N-1}\right)x + 1 - z\right)^2}\end{aligned}$$

which is negative as $N \geq 3$ and $x \geq 0$.

The partial derivative is also decreasing in z, so that A_f is concave in z.

A_f **decreases with** $x = \sigma_1^2/\sigma_2^2$**.**

$$\frac{\partial A_f}{\partial x} = \frac{1}{N-2} \frac{(1-z)^2 - \left(1 - 2\frac{N-1}{N} z\right)\left(1 - \frac{z}{N-1}\right)}{\left(\left(1 - \frac{z}{N-1}\right)x + 1 - z\right)^2} = \frac{z}{N-2} \frac{\frac{N-2}{N} z + \frac{1}{N-1} - \frac{2}{N}}{\left(\left(1 - \frac{z}{N-1}\right)x + 1 - z\right)^2}$$
$$= \frac{z}{N} \frac{z - \frac{1}{N-1}}{\left(\left(1 - \frac{z}{N-1}\right)x + 1 - z\right)^2}$$

Thus for $z > \frac{1}{N-1}$, the expression above increases is negative, while for $z < \frac{1}{N-1}$, it is positive. Now A_f is not defined for $z < \bar{z}(\sigma_1^2)$, where $\bar{z}(\sigma_1^2)$ is defined in the proof of Lemma 2. It is easy to show that $\bar{z}(\sigma_1^2)$ is minimal for $\sigma_1^2 = 0$, so that

$$\bar{z}(\sigma_1^2) \geq \frac{1}{2} \frac{N}{N-1} > \frac{1}{N-1}$$

since $N \geq 3$. Thus $\partial A_f / \partial x < 0$ and A_f decreases with σ_1^2.

Bounds of A_f**.** From the variations in z, knowing that $A_f = 0$ for $z = \bar{z}(\sigma_1^2)$ and $A_f = A = \frac{1}{N-2}$ for $z = 0$, one has

$$0 \leq A_f \leq \frac{1}{N-2}$$

h_0 **decreases with** z**.** The numerator of $-h_0$ trivially increases with z, the numerator decreases with z, so $-h_0$ increases with z.

h_0 **increases with** $x = \sigma_1^2/\sigma_2^2$**.**

$$\frac{\partial h_0}{\partial \sigma_1^2/\sigma_2^2} = -\frac{\frac{N-2}{N-1}\left(1 - 2\frac{N-1}{N}\right) z^2}{\left((1-z)x + 1 - 2\frac{N-1}{N} z\right)^2} = \frac{\frac{(N-2)^2}{N(N-1)} z^2}{\left((1-z)x + 1 - 2\frac{N-1}{N} z\right)^2} > 0.$$

h_0 **spans the interval** $(-\infty, 0]$**.** It is easy to see from A.17 that h_0 decreases with z, thus increases with σ_q^2. When $z = 0$, $h_0 = 0$, and as z converges from above to the value that makes the denominator (proportional to $|\Sigma|$ with a positive proportionality constant) approach zero, h_0 diverges to $-\infty$.

A.5.7 Proof of Proposition 8 and other properties of x_n^*

Proof of Proposition 8. Trader k's future position is, using (A.14) and (A.17):

$$x_n^* = -\frac{(N-2)^2}{(N-1)^3} \frac{(\sigma_1^2+\sigma_2^2)z}{\left(1-\frac{2N-3}{(N-1)^2}z\right)\sigma_1^2 + \left(1-\left(\frac{N-2}{N-1}+\frac{2}{N}\right)z\right)\sigma_2^2}(\bar{I}_0 - I_n) \qquad \text{(A.18)}$$

As z increases, the numerator of $-x_n^*/(\bar{I}_0 - I_n)$ increases, and the denominator decreases, so that $|x_n^*|$ decreases with σ_Q^2: this proves the first part of the proposition. It is also easy to see that is goes to zero as z goes to zero.

The position $x_n^*/(\bar{I}_0 - I_n)$ increases with σ_1^2. One has

$$\begin{aligned}
\frac{\partial}{\partial x}\widetilde{h}_0(1-\psi_0) &= \frac{\partial}{\partial x}\left(-\frac{1}{1+A_f}\left(\frac{N-2}{N-1}\right)^2 \frac{(x+1)z}{\left(1-\frac{z}{N-1}\right)x+1-z}\right) \\
&= -z\psi_0\left(\frac{N-2}{N-1}\right)^2 \\
&\quad \times \left\{\frac{-\frac{N-2}{N-1}z}{\left(\left(1-\frac{z}{N-1}\right)x+1-z\right)^2} + \frac{x+1}{\left(\left(1-\frac{z}{N-1}\right)x+1-z\right)^2}\frac{1}{1+A_f}\frac{\partial A_f}{\partial x}\right\}
\end{aligned}$$

Given that $\frac{\partial A_f}{\partial x} < 0$ (from Section A.5.6), the term in brackets is negative, so that $\frac{\partial}{\partial x}\widetilde{h}_0(1-\psi_0) > 0$. Since $x_n^*/(\bar{I}_0 - I_n) < 0$, $|x_n^*|$ decreases with σ_1^2.

Bounds of $x_n^*/(\bar{I}_0 - I_n)$. One knows that $x_n^* < 0$ and 0 is its limit as σ_q^2 goes to infinity, thus is its upper bound. I now look for the lower bound. Given that $x_n^*/(\bar{I}_0 - I_n)$ decreases with z, it is greater than the value its expression takes for $z = \bar{z}(\sigma_1^2)$, i.e. given that $A_f = 0$ for $z = \bar{z}$:

$$\begin{aligned}
\frac{x_n^*}{\bar{I}_0 - I_n} &\geq -\left(\frac{N-2}{N-1}\right)^2 \frac{(\sigma_1^2+\sigma_2^2)\frac{\sigma_1^2+\sigma_2^2}{\sigma_1^2+2\frac{N-1}{N}\sigma_2^2}}{\left(1-\frac{1}{N-1}\frac{\sigma_1^2+\sigma_2^2}{\sigma_1^2+2\frac{N-1}{N}\sigma_2^2}\right)\sigma_1^2 + \left(1-\frac{\sigma_1^2+\sigma_2^2}{\sigma_1^2+2\frac{N-1}{N}\sigma_2^2}\right)\sigma_2^2} \\
&= -\frac{N-2}{N-1} \times \frac{\sigma_1^2+\sigma_2^2}{\sigma_1^2+\frac{N-1}{N}\sigma_2^2} \equiv \frac{x_k^*(\bar{z}(\sigma_1^2))}{\bar{I}_0 - I_n}
\end{aligned}$$

It is easy to see that $\frac{x_n^*(\bar{z}(\sigma_1^2)}{\bar{I}_0 - I_n}$ decreases with σ_1^2/σ_2^2, so that the infimum of the futures position as a fraction of $\bar{I}_0 - I_n$ is:

$$\inf_{\sigma_1^2/\sigma_2^2, z} \frac{x_n^*}{\bar{I}_0 - I_n} = -\frac{N(N-2)}{(N-1)^2} = -\alpha \tag{A.19}$$

A.6 Traders' welfare

A.6.1 Welfares with and without futures

Without futures. Plugging equilibrium prices and quantities in the certainty equivalent of wealth (A.6) leads to

$$\widehat{W}_{n,0}^{\times} = V_n + S_{n,0}(\psi_\times) + \widehat{S}_{n,1}(\psi_\times),$$

where $S_{n,0}(\psi)$ and $\widehat{S}_{n,1}(\psi)$ are the equilibrium shares of date 0 and date 1 surpluses that accrue to trader n:

$$S_{n,0}(\psi) = \frac{\gamma(\sigma_1^2 + \sigma_2^2)}{2} \left(1 - (1-\psi)^2\right) (q_k^c)^2 + \gamma \sigma_2^2 z \psi q_k^c \, \mathbb{E}_0 \left[\frac{Q}{N}\right]$$
$$\widehat{S}_{n,1}(\psi) = \frac{\alpha}{2} \gamma \sigma_2^2 z \, (1-\psi)^2 \, (q_k^c)^2 + \gamma \sigma_2^2 z \frac{N}{N-1} (1-\psi) q_k^c \, \mathbb{E}_0 \left[\frac{Q}{N}\right] + \frac{\gamma \sigma_2^2 z}{2} \frac{N}{N-2} \left(\mathbb{E}_0 \left[\frac{Q}{N}\right]\right)^2$$

and denoting $S_n(\psi) \equiv S_{n,0}(\psi) + \widehat{S}_{n,1}(\psi)$, one has

$$S_n(\psi) = \frac{\gamma(\sigma_1^2 + (1-\alpha z)\sigma_2^2)}{2} \left(1 - (1-\psi)^2\right) \left(\bar{I}_0 - I_n\right)^2 + \gamma \sigma_2^2 z \frac{N-\psi}{N-1} (\bar{I}_0 - I_n) \, \mathbb{E}_0 \left[\frac{Q}{N}\right]$$
$$+ \frac{\gamma \sigma_2^2 z}{2} \frac{N}{N-2} \left(\mathbb{E}_0 \left[\frac{Q}{N}\right]\right)^2. \tag{A.20}$$

With futures. Similarly, plugging relevant equilibrium prices and quantities into date 0 certainty equivalent of wealth (4.6) yields:

$$\widehat{W}_{n,0} = V_n + S_n(\psi_0) - \gamma(\sigma_1^2 + (1-z)\sigma_2^2)(\bar{I}_0 - I_n - q_{n,0}^*) x_n - \frac{\gamma}{2} \Sigma_{22} (x_n^*)^2$$

where to ease notation I denoted $\widetilde{h}_0 = \frac{N-2}{N-1} h_0$.

A.6.2 Proof of theorem 1

Case $\mathbb{E}_0[Q] = 0$.

$$\widehat{W}^*_{n,0} - \widehat{W}^\times_{k,0} = -\gamma\sigma_2^2(\bar{I}_0 - I_n)^2 (1-\psi_0)^2 (x+1-\alpha z)/2$$
$$\times \underbrace{\left(1 - \left(\frac{1-\psi_\times}{1-\psi_0}\right)^2 - 2\frac{x+1-z}{x+1-\alpha z}\widetilde{h}_0 + \frac{x+\frac{1-z}{\alpha}}{x+1-\alpha z}\widetilde{h}_0^2\right)}_{\Phi(\widetilde{h}_0)}.$$

I have to show that $\Phi(\widetilde{h}_0) > 0$. Consider the roots of Φ: the discriminant of Φ is

$$\Delta = 4\frac{x+\frac{1-z}{\alpha}}{x+1-\alpha z}\left[\frac{(x+1-z)^2}{(x+1-\alpha z)(x+\frac{1-z}{\alpha})} - 1 + \left(\frac{1-\psi_\times}{1-\psi_0}\right)^2\right],$$

which is always positive because $\psi_\times < \psi_0$, so that $\left(\frac{1-\psi_\times}{1-\psi_0}\right)^2 > 1$. Given that $\tilde{h}_0 < 0$, I need to show that $\widetilde{h}_0$ is lower than the smallest root of Φ_f, which is

$$h^0_- = \frac{x+1-z}{x+\frac{1-z}{\alpha}}\left\{1 - \sqrt{1 + \left(\frac{x+1-\alpha z}{x+1-z}\right)^2\left(\left(\frac{1-\psi_\times}{1-\psi_0}\right)^2 - 1\right)}\right\}$$

Therefore, given the expression of h^0_- and that of $\widetilde{h}_0$, one has $\widetilde{h}_0 < h^0_-$ if and only if

$$\frac{x+\frac{1-z}{\alpha}}{x+1-z}\tilde{h}_0(1-\psi_0) - (1-\psi_0) < -\sqrt{(1-\psi_0)^2(1-k^2) + (1-\psi_\times)^2 k^2} \quad \text{(A.21)}$$

where $k = \frac{x+1-\alpha z}{x+1-z}$. In what follows I prove that inequality (A.21) holds in 3 steps. First, I show that the left-hand side increases with z. Second, I show that the right-hand side decreases with z. The first and second step imply that if the inequality holds for the maximum value of z, which is $\bar{z}$, then it holds for all values of z.

First step. Regarding the left-hand side of (A.21), one has

$$\frac{\partial}{\partial z}\left[\frac{x+\frac{1-z}{\alpha}}{x+1-z}\tilde{h}_0(1-\psi_0)\right] = \frac{\partial}{\partial z}\left[\frac{x+\frac{1-z}{\alpha}}{x+1-z}\right]\tilde{h}_0(1-\psi_0) + \frac{x+\frac{1-z}{\alpha}}{x+1-z}\frac{\partial}{\partial z}\left[\tilde{h}_0(1-\psi_0)\right]$$

It is easy to show that the derivative in the first term is negative, while we know from proposition 6 that $\widetilde{h}_0(1-\psi_0)$ is negative, so that the first term is positive. The second term is also positive, see Section A.5.6.

Second step. Denote Ψ the term inside the square root of the right-hand side: the RHS decreases with z if and only if Ψ increases with z. One has

$$\frac{\partial \Psi}{\partial z} = \left[(1-\psi_\times)^2 - (1-\psi_0)^2\right] \frac{\partial k^2}{\partial z} + \frac{2A_f}{(1+A_f)^3} \frac{\partial A_f}{\partial z}(1-k^2) + \frac{2A}{(1+A)^3} \frac{\partial A}{\partial z} k^2$$

For the first term, it is easy to show that k increases with z, so that $\partial k^2/\partial z > 0$, while one knows from Propositions 5 and 6 that $\psi_\times < \psi_0$, which proves that the first factor of the first term is positive. So the first term is positive. For the second term, it is easy to show that $k > 1$, while one has shown in Lemma 6 that $\partial A/\partial z > 0$ and in Section A.5.6 that $\partial A_f/\partial z < 0$. This shows the positivity of $\partial \Psi/\partial z$. QED.

Third step. Given that $\frac{x+1-\alpha z}{x+1-z} > 1$, a sufficient condition for inequality (A.21) to work is

$$\frac{x+1-\alpha z}{x+1-z} \frac{1-\psi_\times}{1-\psi_0} - \left(\frac{N-2}{N-1}\right)^2 \frac{x+\frac{1-z}{\alpha}}{x+1-z} \frac{(x+1)z}{(1-z)x+1-2\frac{N-1}{N}z} < 1$$

which, after rearranging, leads to

$$\left(\frac{x+1-z}{x+\frac{1-z}{\alpha}} + \frac{\left(\frac{N-2}{N-1}\right)^2 (x+1)z}{(1-z)x+1-2\frac{N-1}{N}z} \right) (1-\psi_0) > \frac{x+1-\alpha z}{x+1-z}(1-\psi_\times) \tag{A.22}$$

I now prove that this inequality holds for $z = \bar{z}(x)$, where $\bar{z}(x)$ is defined in (A.12). First notice that, as $1-\psi_0 = 0$ for $z = \bar{z}$ (but not the product $h_0(1-\psi_0)$), the inequality reduces to

$$\left(\frac{N-2}{N-1}\right)^2 \frac{(x+1)\bar{z}}{(1-\bar{z})x+1-2\frac{N-1}{N}\bar{z}}(1-\psi_0(\bar{z})) > \frac{x+1-\alpha\bar{z}}{x+1-\bar{z}}(1-\psi_0(\bar{z}))$$

$$\Leftrightarrow \quad -\bar{z} = \frac{N-2}{N} \frac{1}{x+2\frac{N-1}{N}} \qquad \Leftrightarrow \qquad 1-\alpha\bar{z} = \frac{(1-\alpha)x+2\frac{N-1}{N}}{x+2\frac{N-1}{N}}$$

$$\Leftrightarrow \qquad -\frac{\bar{z}}{N-1} = \frac{\frac{N-2}{N-1}x+2-\frac{2}{N}+\frac{1}{N-1}}{x+2\frac{N-1}{N}}$$

so that, plugging this into the inequality above leads to

$$(x+1)^2\left(x+\frac{N-1}{N}\right)^2 > \left(x^2+\left(2\frac{N-1}{N}\right.\right. \left.\left. +1-\alpha\right)x+1-\frac{2}{N}+\frac{1}{(N-1)^2}\right)$$
$$\times\left(x+1-\frac{2}{3N}\right)\frac{x+1-\frac{N-2}{N(N-1)}}{x+1-\frac{N-2}{N(N-1)^2}}$$

It is straightforward to show that the left-hand side decreases with N, and not difficult to check that the right-hand side increases with N: the first and second factors are obvious, while the derivative of the third factor with respect to N is

$$\frac{\partial}{\partial N}\left[\frac{x+1-\frac{N-2}{N(N-1)}}{x+1-\frac{N-2}{N(N-1)^2}}\right]=\left(\frac{x+1}{(N-1)^2}-\frac{1}{N^2}\right)\frac{\left(\frac{N-2}{N-1}(x+1)-\frac{1}{N}\right)^4}{\left(\frac{N-1}{N-1}(x+1)+1-1/N\right)^2}$$

which is positive as $x\geq 0$. Thus the above inequality holds for all N if it holds for $N\to\infty$, i.e., taking the limit, if $(x+1)^4\geq(x+1)^3$: this holds as long as $x>0$. Thus for $z=\bar{z}$, $h_-^f>\widetilde{h}_0$.

It remains to prove the strict inequality for $x=0$. In this case $\bar{z}=\frac{1}{2}\frac{N}{N-1}$ and

$$A_f=\frac{1}{N-2}\frac{1-2\frac{N-1}{N}z}{1-z}, \qquad\Rightarrow\qquad \frac{A_f}{1+A_f}=\frac{1}{N-1}\frac{1-2\frac{N-1}{N}z}{1-\frac{N-2/N}{N-1}z},$$
$$A=\frac{1}{N-2}\frac{1-\frac{1}{2}\frac{N-2}{N-1}}{1-\frac{\alpha}{2}} \qquad\Rightarrow\qquad \frac{A}{1+A}=\frac{1}{N-1}\frac{1-\frac{1}{2}\frac{N-2}{N-1}}{1-\frac{\alpha}{2}\left(\frac{N-2}{N-1}+\frac{1}{N}\right)},$$
$$\widetilde{h}_0=-\left(\frac{N-2}{N-1}\right)^2\frac{z}{1-2\frac{N-1}{N}z}, \qquad\Rightarrow\qquad \frac{\widetilde{h}_0A_f}{1+A_f}=-\left(\frac{N-2}{N-1}\right)^2\frac{1}{N-1}\frac{z}{1-\frac{N-2/N}{N-1}z},$$

while $k=(1-\alpha z)/(1-z)$. (A.21) becomes after rearranging:

$$-\frac{1-z}{1-\frac{N-2/N}{N-1}z}<-\sqrt{\left(\frac{1-2\frac{N-1}{N}z}{1-\frac{N-2/N}{N-1}z}\right)^2(1-k^2)+\left(\frac{1-\frac{N-2}{N}z}{1-\frac{N-2}{N-1}\left(\frac{N-2}{N-1}+\frac{1}{N}\right)z}\right)^2k^2}$$

For $z = \bar{z}$, this becomes

$$
\begin{aligned}
\frac{1 - \frac{N}{2(N-1)}}{1 - \frac{1}{2}\frac{N^2-2}{(N-1)^2}} &> \frac{1 - \frac{N-2}{N}\bar{z}}{1 - \frac{N-2}{N-1}\left(\frac{N-2}{N-1} + \frac{1}{N}\right)\bar{z}} k = \frac{1 - \frac{1}{2}\frac{N-2}{N-1}}{1 - \frac{\alpha}{2}\left(\frac{N-2}{N-1} + \frac{1}{N}\right)} \frac{1 - \alpha\frac{N}{2(N-1)}}{1 - \frac{N}{2(N-1)}} \\
\frac{(N-1)(N-2)}{(N-2)^2} &> \frac{N}{2N - 2 - \frac{N-2}{N-1} - N\left(\frac{N-2}{N-1}\right)^2} \frac{(2-\alpha)N - 2}{N-2} \\
\frac{N-1}{N} &> \frac{(1-\alpha/2)N - 1}{\left(1 - \frac{1}{2}\left(\frac{N-2}{N-1}\right)^2\right)N - 1 - \frac{1}{2}\frac{N-2}{N-1}},
\end{aligned}
$$

and it can be shown that the above inequality holds for all $N \geq 3$. QED.

Case $\mathbb{E}_0[Q] \neq 0$. $\mathbb{E}_0[Q]$ only affects spot trading surplus $S_n(\psi)$, and one has

$$
\begin{aligned}
S_n(\psi_0) - S_n(\psi_\times) = &\frac{\gamma(\sigma_1^2 + (1-\alpha z)\sigma_2^2)}{2}\left((1-\psi_\times)^2 - (1-\psi_0)^2\right)\left(\bar{I}_0 - I_n\right)^2 \\
&+ \gamma\sigma_2^2 z \frac{\psi_\times - \psi_0}{N-1} q_n^c \, \mathbb{E}_0\left[\frac{Q}{N}\right],
\end{aligned}
$$

which is linear in $\mathbb{E}_0[Q]$. Given that $\psi_\times < \psi_0$, the result is immediate.

Rational participation: $V_n < \widehat{W}_{n,0}^*$. From (**??**), the total trading surplus is

$$
\begin{aligned}
\frac{\widehat{W}_{n,0}^* - V_n}{\gamma(\bar{I}_0 - I_n)^2} = &\left[\frac{1}{2}\left(1 - ((1-\psi_0))^2\right) + \widetilde{h}_0(1-\psi_0)\left(1 - \psi_0 - \widetilde{h}_0(1-\psi_0)\right)\right]\sigma_1^2 \\
&+ \left[\frac{1}{2}\left(1 - (1-\alpha z)(1-\psi_0)^2\right) + \widetilde{h}_0(1-\psi_0)\left(\alpha(1-\psi_0) - \widetilde{h}_0(1-\psi_0)\right)\right]\sigma_2^2
\end{aligned}
$$

It its apparent that the coefficient in σ_1^2 is smaller than the coefficient in σ_2^2, since $\alpha \in (0,1)$ and $\widetilde{h}_0 < 0$. I show that the coefficient in σ_1^2, which I denote K_1, is positive: the positivity of total trading surplus thus follows. From the $A_f \leq 1/(N-2)$ and

$$
K_1 \geq \underbrace{\frac{1}{2}\left(1 - \frac{1}{(N-1)^2}\right)}_{\alpha} - \alpha\left(\frac{1}{N-1} + \frac{\alpha}{2}\frac{N-2}{N-1}\right) = \alpha\left(\frac{1}{2} - \frac{1}{N-1} + \frac{\alpha}{2}\frac{N-2}{N-1}\right)
$$

Given that $N \geq 3$, $1/(N-1) \geq 1/2$ and the RHS is positive. QED.

A.7 Financial traders

A.7.1 Proof of Proposition 10

Lemma 9. *The system* (6.2) *is equivalent to*

$$\begin{cases} \Lambda & = \frac{1}{N-2+K\frac{\lambda_{22}+\gamma\left(\sigma_1^2+\frac{1-z}{\alpha}\sigma_2^2\right)}{\mu+\gamma\sigma^2}} \left\{ \gamma\Sigma + \frac{K}{N-1}\frac{|\Lambda+\gamma\Sigma|}{\mu+\gamma\sigma^2} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \right\} \\ \mu & = \frac{\lambda_{22}+\gamma\left(\sigma_1^2+\frac{1-z}{\alpha}\sigma_2^2\right)}{N+(K-1)\frac{\lambda_{22}+\gamma\left(\sigma_1^2+\frac{1-z}{\alpha}\sigma_2^2\right)}{\mu+\gamma\sigma^2}} \end{cases} \tag{A.23}$$

Proof. Denote

$$B_{N,K} = N\left(\Lambda+\gamma\Sigma\right)^{-1} + \frac{K}{\mu+\gamma\sigma^2}\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix},$$

so that

$$\begin{cases} \Lambda & = B_{N-1,K}^{-1} \\ \mu & = [B_{N,K-1}^{-1}]_{1,1} \end{cases}$$

We now compute $B_{N,K}^{-1}$. First notice that

$$B_{N,K} = \frac{N}{|\Lambda+\gamma\Sigma|}\begin{pmatrix} \lambda_{22}+\gamma\left(\sigma_1^2+\frac{1-z}{\alpha}\sigma_2^2\right) & -\left[\lambda_{12}+\gamma\left(\sigma_1^2+\left(1-\frac{N-1}{N}z\right)\sigma_2^2\right)\right] \\ -\left[\lambda_{21}+\gamma\left(\sigma_1^2+\left(1-\frac{N-1}{N}z\right)\sigma_2^2\right)\right] & \lambda_{11}+\gamma\left(\sigma_1^2+\delta\sigma_2^2\right) \end{pmatrix} + \frac{K}{\mu+\gamma\sigma^2}\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

It is easy to compute

$$\begin{aligned} |B_{N,K}| &= \frac{N^2}{|\Lambda+\gamma\Sigma|^2}|\Lambda+\gamma\Sigma| + \frac{K}{\mu+\gamma\sigma^2}\frac{N}{|\Lambda+\gamma\Sigma|}\left(\lambda_{22}+\gamma\left(\sigma_1^2+\frac{1-z}{\alpha}\sigma_2^2\right)\right) \\ &= \frac{N^2}{|\Lambda+\gamma\Sigma|}\left\{1+\frac{K}{N}\frac{\lambda_{22}+\gamma\left(\sigma_1^2+\frac{1-z}{\alpha}\sigma_2^2\right)}{\mu+\gamma\sigma^2}\right\} > 0. \end{aligned}$$

Therefore

$$
\begin{aligned}
B_{N,K}^{-1} &= \frac{1}{|B_{N,K}|}\left\{\frac{N}{|\Lambda+\gamma\Sigma|}(\Lambda+\gamma\Sigma)+\frac{K}{\mu+\gamma\sigma^2}\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}\right\}\\
&= \frac{1}{N\left\{1+\frac{K}{N}\frac{\lambda_{22}+\gamma\left(\sigma_1^2+\frac{1-z}{\alpha}\sigma_2^2\right)}{\mu+\gamma\sigma^2}\right\}}\left(\Lambda+\gamma\Sigma+\frac{K}{N}\frac{|\Lambda+\gamma\Sigma|}{\mu+\gamma\sigma^2}\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}\right)
\end{aligned}
$$

Therefore the equation in Λ in the system is equivalent to:

$$
\begin{aligned}
\Lambda = &\left(1-\frac{1}{(N-1)\left(1+\frac{K}{N-1}\frac{\lambda_{22}+\gamma\left(\sigma_1^2+\frac{1-z}{\alpha}\sigma_2^2\right)}{\mu+\gamma\sigma^2}\right)}\right)^{-1}\frac{1}{(N-1)\left\{1+\frac{K}{N-1}\frac{\lambda_{22}+\gamma\left(\sigma_1^2+\frac{1-z}{\alpha}\sigma_2^2\right)}{\mu+\gamma\sigma^2}\right\}}\\
&\times\left\{\gamma\Sigma+\frac{K}{N-1}\frac{|\Lambda+\gamma\Sigma|}{\mu+\gamma\sigma^2}\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}\right\}\\
= &\frac{1}{N-2+K\frac{\lambda_{22}+\gamma\left(\sigma_1^2+\frac{1-z}{\alpha}\sigma_2^2\right)}{\mu+\gamma\sigma^2}}\left\{\gamma\Sigma+\frac{K}{N-1}\frac{|\Lambda+\gamma\Sigma|}{\mu+\gamma\sigma^2}\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}\right\}.
\end{aligned}
$$

Then the equation in μ in the system is equivalent to:

$$
\mu = \frac{\lambda_{22}+\gamma\left(\sigma_1^2+\frac{1-z}{\alpha}\sigma_2^2\right)}{N\left\{1+\frac{K-1}{N}\frac{\lambda_{22}+\gamma\left(\sigma_1^2+\frac{1-z}{\alpha}\sigma_2^2\right)}{\mu+\gamma\sigma^2}\right\}}
$$

■

One can solve (A.23) for λ_{22} and μ separately from the rest of the system, and rearranging the corresponding equations, one has:

$$
\begin{aligned}
\mu^2+\left(\frac{K-2}{N}(\lambda_{22}+\gamma\Sigma_{22})+\gamma\sigma^2\right)\mu-\gamma\sigma^2\frac{\lambda_{22}+\gamma\Sigma_{22}}{N} &= 0\\
\lambda_{22}^2+\left(\frac{N-2}{K}(\mu+\gamma\sigma^2)+\gamma\Sigma_{22}\right)\lambda_{22}-\gamma\Sigma_{22}\frac{\mu+\gamma\sigma^2}{K} &= 0
\end{aligned}
$$

Each equation in the system has two solutions when $\mu > -\gamma\sigma^2$ and $\lambda_{22} > -\gamma\Sigma_{22}$ (if these conditions are violated, certainty equivalents of wealth are no longer concave); one then being positive and the other being negative. Moreover, the original form of the system implies that $\lambda_{22}, \mu > 0$. Therefore, we look at the positive solution to each equation.

Denoting $y = \frac{\Sigma_{22}}{\sigma^2}$, $a = \mu/(\gamma\sigma^2)$ and $b = \lambda_{22}/(\gamma\Sigma_{22}) = (N+\widetilde{K}-2)^{-1}$, this system is equivalent to

$$\begin{cases} a^2 + \left(1 + (K-2)\frac{(1+b)y}{N}\right) a - \frac{(1+b)y}{N} & = 0 \\ b^2 + \left(1 + (N-2)\frac{1+a}{Ky}\right) b - \frac{1+a}{Ky} & = 0. \end{cases} \tag{A.24}$$

This system depends only on y, N and K. When $y = 1$, it can be checked that $a = b = \frac{1}{N+K-2}$. Otherwise, the positive solution are $a_+(b)$ and $b_+(a)$ and the solution to the system satisfies

$$\begin{cases} a & = a_+(b) \\ b & = b_+(a) \end{cases}.$$

Variations of a, b with y. We use the implicit function theorem to compute $(\partial a/\partial y, \partial b/\partial y)$ from the system (A.24). Denote

$$\Phi((a,b),y) = \begin{pmatrix} \Phi_1((a,b),y) \\ \Phi_2((a,b),y) \end{pmatrix} = \begin{pmatrix} a^2 + \left(1 + (K-2)\frac{(1+b)y}{N}\right) a - \frac{(1+b)y}{N} \\ b^2 + \left(1 + (N-2)\frac{1+a}{Ky}\right) b - \frac{1+a}{Ky} \end{pmatrix}$$

Then denoting $D_{a,b}\Phi$ the Jacobian matrix of Φ with respect to (a,b), one has

$$D_{a,b}\Phi = \begin{pmatrix} 2a + 1 + (K-2)\frac{(1+b)y}{N} & -(1-(K-2)a)\frac{y}{N} \\ -(1-(N-2)b)\frac{1}{Ky} & 2b + 1 + (N-2)\frac{1+a}{Ky} \end{pmatrix}$$

$$
\begin{aligned}
|D_{a,b}\Phi| &= \left(2a+1+(K-2)\frac{(1+b)y}{N}\right)\left(2b+1+(N-2)\frac{1+a}{Ky}\right) \\
&\quad -((N-2)b-1)\frac{1}{Ky}((K-2)a-1)\frac{y}{N} \\
&= (2a+1)(2b+1)+\frac{K-2}{N}\frac{N-2}{K}(1+a)(1+b)+(2a+1)(K-2)\frac{(1+b)y}{N} \\
&\quad +(2b+1)(N-2)\frac{1+a}{Ky}-\frac{((N-2)b-1)((K-2)a-1)}{NK} \\
&= (2a+1)(2b+1)+(2a+1)(K-2)\frac{(1+b)y}{N}+(2b+1)(N-2)\frac{1+a}{Ky} \\
&\quad +\frac{(K-2)(N-2)(1+a+b+ab)-((N-2)(K-2)ab-(N-2)b-(K-2)a+1)}{NK} \\
&= (2a+1)(2b+1)+(2a+1)(K-2)\frac{(1+b)y}{N}+(2b+1)(N-2)\frac{1+a}{Ky} \\
&\quad +\frac{(K-2)(N-1)a+(K-1)(N-2)b+(K-2)(N-2)-1}{NK}
\end{aligned}
$$

Given that a, b are strictly positive, for $K \geq 2$, the determinant is positive (the $-1/(NK) > -1$ is compensated by a $+1$ term from the first term). For $K = 1$, the last two terms in the numerator of the last term becomes $-(N-1)$ (still divided by N), which is again compensated with the 1 from the first term. Thus $|D_{a,b}\Phi| > 0$.

Therefore, given

$$
\begin{pmatrix} \frac{\partial \Phi_1}{\partial y} \\ \frac{\partial \Phi_2}{\partial y} \end{pmatrix} = \begin{pmatrix} \frac{1+b}{N}\left((K-2)a-1\right) \\ -\frac{1+a}{Ky^2}\left((N-2)b-1\right) \end{pmatrix}
$$

one gets

$$
\begin{aligned}
\frac{\partial a}{\partial y} &= \frac{-1}{|D_{a,b}\Phi|}\left(\left(2b+1+(N-2)\frac{1+a}{Ky}\right)\frac{1+b}{N}\left((K-2)a-1\right)\right. \\
&\qquad\qquad \left. +((K-2)a-1)\frac{y}{N}\frac{1+a}{Ky^2}\left((N-2)b-1\right)\right) \\
&= \frac{1-(K-2)a}{N|D_{a,b}\Phi|}\left(\left(2b+1+(N-2)\frac{1+a}{Ky}\right)(1+b)+\frac{1+a}{Ky}\left((N-2)b-1\right)\right) \\
&= \frac{1-(K-2)a}{N|D_{a,b}\Phi|}\left((2b+1)(1+b)+\frac{1+a}{Ky}\left((N-2)(1+b)+((N-2)b-1)\right)\right) \\
&= \frac{1-(K-2)a}{N|D_{a,b}\Phi|}\left((2b+1)(1+b)+\frac{1+a}{Ky}\left((N-3)+2(N-2)b\right)\right)
\end{aligned}
$$

Given $a, b > 0$ and $N \geq 3$, the term inside the parenthesis is positive. Therefore the sign of $\partial a/\partial y$ is that of $1-(K-2)a$. For $y = 1$, $a = (N+K-2)^{-1}$, and thus

$1-(K-2)a>0$ so that $\partial a/\partial y>0$. Thus as y decreases from 1, a decreases and thus $\partial a/\partial y$ remains positive. This implies that

$$\forall y\in(0,1),\qquad a<\frac{1}{N+K-2}\quad\text{and}\quad\frac{\partial a}{\partial y}>0.$$

Regarding b,

$$\begin{aligned}\frac{\partial b}{\partial y}&=\frac{-1}{|D_{a,b}\Phi|}\Bigg\{-\left(2a+1+(K-2)\frac{(1+b)y}{N}\right)\frac{1+a}{Ky^2}\left((N-2)b-1\right)\\&\qquad-\frac{b(N-2)-1}{Ky}\frac{1+b}{N}(a(K-2)-1)\Bigg\}\\&=\frac{-(1-(N-2)b)}{K|D_{a,b}\Phi|}\left\{\left(2a+1+(K-2)\frac{(1+b)y}{N}\right)\frac{1+a}{y^2}+\frac{1}{y}\frac{1+b}{N}(a(K-2)-1)\right\}\\&=\frac{-(1-(N-2)b)}{Ky|D_{a,b}\Phi|}\left\{(2a+1)\frac{1+a}{y}+(1+b)\frac{(K-3)+2(K-2)a}{N}\right\}\end{aligned}$$

Similarly to a, the sign of db/dy is the opposite to that of $1-(N-2)b$, which is positive for $y=1$ since then $b=\frac{1}{N+K-2}$: thus for $y=1$, $\partial b/\partial y<0$, so that b locally increases as y decreases from 1, which keeps $\partial b/\partial y$ negative as y further decreases. Thus

$$\forall y\in(0,1),\qquad b>\frac{1}{N+K-2}\quad\text{and}\quad\frac{\partial b}{\partial y}<0$$

This also implies that for $y\in(0,1)$,

$$b\equiv\frac{1}{N+\widetilde{K}-2}>\frac{1}{N+K-2}\quad\Rightarrow\quad\frac{1+b}{1+a}y<1.$$

And indeed,

$$\begin{aligned}y=\frac{\gamma\Sigma_{22}}{\gamma\operatorname{Var}_0(p_1^*)}&=\frac{\sigma_1^2+\frac{1}{\alpha}\frac{1}{1+\frac{N}{N-2}\gamma^2\sigma_2^2\sigma_q^2}\frac{N}{N-2}\gamma^2\sigma_2^4\sigma_q^2}{\sigma_1^2+\left(\frac{N-1}{N-2}\right)^2\gamma^2\sigma_2^4\sigma_q^2}=\frac{\sigma_1^2+\frac{1}{1+\frac{N}{N-2}\gamma^2\sigma_2^2\sigma_q^2}\left(\frac{N-1}{N-2}\right)^2\gamma^2\sigma_2^4\sigma_q^2}{\sigma_1^2+\left(\frac{N-1}{N-2}\right)^2\gamma^2\sigma_2^4\sigma_q^2}\\&\leq 1.\end{aligned}$$

Variations of a, b and $\widetilde{K}$ with K. One has

$$\begin{pmatrix}\frac{\partial\Phi_1}{\partial K}\\\frac{\partial\Phi_2}{\partial K}\end{pmatrix}=\begin{pmatrix}\frac{(1+b)y}{N}a\\\frac{1+a}{K^2y}(1-(N-2)b)\end{pmatrix},$$

so that

$$\frac{\partial a}{\partial K} = \frac{-1}{|D_{a,b}\Phi|}\left(\left(2b+1+(N-2)\frac{1+a}{Ky}\right)\frac{(1+b)y}{N}a + (1-(K-2)a)\frac{y}{N}(1-(N-2)b)\frac{1+a}{K^2y}\right)$$
$$= \frac{-1}{|D_{a,b}\Phi|}\left(\left(2b+1+(N-2)\frac{1+a}{Ky}\right)\frac{(1+b)y}{N}a + (1-(K-2)a)(1-(N-2)b)\frac{1+a}{K^2N}\right)$$

For $y \leq 1$, one has $1-(K-2)a > 0$ and $1-(N-2)b > 0$, so that for $y \leq 1$, $\frac{\partial a}{\partial K} < 0$. Similarly, for b:

$$\frac{\partial b}{\partial K} = \frac{-1}{|D_{a,b}\Phi|}\left\{\left(2a+1+(K-2)\frac{(1+b)y}{N}\right)\frac{1+a}{K^2y}(1-(N-2)b) + \frac{1-(N-2)b}{Ky}\frac{(1+b)y}{N}a\right\}$$
$$= \frac{-(1-(N-2)b)}{Ky|D_{a,b}\Phi|}\left\{\left(2a+1+(K-2)\frac{(1+b)y}{N}\right)\frac{1+a}{Ky} + \frac{(1+b)y}{N}a\right\} < 0.$$

And since $b = \frac{1}{N+\widetilde{K}-2}$, one has $\frac{\partial \widetilde{K}}{\partial K} = -\frac{1}{b^2}\frac{\partial b}{\partial K} > 0$. Another question is **whether $\widetilde{K}$ increases without bound as K increases**: this is equivalent to asking if b goes to zero as K goes to infinity. Since both a and b decrease with K while they remain positive, the ratio $\frac{1+b}{1+a}$ converges to a positive constant; as y is independent from K, a, b, $\widetilde{K}/K$ converges to a positive constant, so that $\lim_{K\to\infty}\widetilde{K} = +\infty$.

Determination of $\lambda_{21}, \lambda_{12}$. From (A.23) and results on λ_{22} and μ, one has

$$\lambda_{21} = \lambda_{12} = \frac{\gamma\Sigma_{21}}{N+\widetilde{K}-2} = b\gamma\Sigma_{21}. \tag{A.25}$$

Determination of λ_{11}. One first computes:

$$|\Lambda + \gamma\Sigma| = (\lambda_{11}+\gamma\Sigma_{11})(1+b)\gamma\Sigma_{22} - ((1+b)\gamma\Sigma_{21})^2$$

Therefore λ_{11} satisfies, from (A.23),

$$\lambda_{11} = b\left(\gamma\Sigma_{11} + \frac{\widetilde{K}}{N-1}\lambda_{11} + \frac{K}{N-1}\left[\frac{(1+b)\gamma\Sigma_{11}\gamma\Sigma_{22}}{(1+a)\gamma\sigma^2} - \frac{1}{(1+a)\gamma\sigma^2}((1+b)\gamma\Sigma_{21})^2\right]\right)$$
$$= b\left(\gamma\Sigma_{11} + \frac{\widetilde{K}}{N-1}\lambda_{11} + \gamma\Sigma_{11}\frac{\widetilde{K}}{N-1}\left[1-(1+b)\frac{(\Sigma_{21})^2}{\Sigma_{11}\Sigma_{22}}\right]\right)$$

where

$$\widetilde{K} = \frac{1+b}{1+a}y, \qquad \Sigma_{11} = \sigma_1^2 + \delta\sigma_2^2$$

which is equivalent to

$$\left(1 - \frac{\frac{\widetilde{K}}{N-1}}{N+\widetilde{K}-2}\right)\lambda_{11} = b\gamma\Sigma_{11}\left(1 + \frac{\widetilde{K}}{N-1}\left[1-(1+b)\frac{(\Sigma_{21})^2}{\Sigma_{11}\Sigma_{22}}\right]\right)$$

so that

$$\begin{aligned}
\lambda_{11} &= b\gamma\Sigma_{11}\left(1 + \frac{\frac{\widetilde{K}}{N-1}}{N+\frac{N-2}{N-1}\widetilde{K}-2}\right)\left(1 + \frac{\widetilde{K}}{N-1}\left[1-(1+b)\frac{(\Sigma_{21})^2}{\Sigma_{11}\Sigma_{22}}\right]\right) \\
&= b\gamma\Sigma_{11}\left(1 + \frac{\widetilde{K}}{N-1}\left[1 + \frac{1}{N+\frac{N-2}{N-1}\widetilde{K}-2} - (1+b)\frac{(\Sigma_{21})^2}{\Sigma_{11}\Sigma_{22}}\right] + \left(\frac{\widetilde{K}}{N-1}\right)^2 \frac{1-(1+b)\frac{(\Sigma_{21})^2}{\Sigma_{11}\Sigma_{22}}}{N+\frac{N-2}{N-1}\widetilde{K}-2}\right) \\
&= b\gamma\Sigma_{11} + \gamma\Delta
\end{aligned}$$

where

$$\begin{aligned}
\Delta &= \frac{b\Sigma_{11}\widetilde{K}}{N-1}\left[\frac{1}{N+\frac{N-2}{N-1}\widetilde{K}-2} + \left(1 + \frac{\frac{\widetilde{K}}{N-1}}{N+\frac{N-2}{N-1}\widetilde{K}-2}\right)\left(1-(1+b)\frac{(\Sigma_{21})^2}{\Sigma_{11}\Sigma_{22}}\right)\right] \\
&= \frac{b\Sigma_{11}\widetilde{K}}{N-1}\left[\frac{N+\widetilde{K}-1}{N+\frac{N-2}{N-1}\widetilde{K}-2} - \frac{N+\widetilde{K}-2}{N+\frac{N-2}{N-1}\widetilde{K}-2}\frac{N+\widetilde{K}-1}{N+\widetilde{K}-2}\frac{(\Sigma_{21})^2}{\Sigma_{11}\Sigma_{22}}\right] \\
&= \frac{(1+b)\frac{\widetilde{K}}{N-1}}{N+\frac{N-2}{N-1}\widetilde{K}-2}\frac{|\Sigma|}{\Sigma_{22}} = \frac{(1+b)\frac{\widetilde{K}}{N-1}}{(N-2)\left(1+\frac{\widetilde{K}}{N-1}\right)}\frac{|\Sigma|}{\Sigma_{22}} = \frac{\frac{N+\widetilde{K}-1}{N+\widetilde{K}-2}\frac{\widetilde{K}}{N-1}}{(N-2)\frac{N+\widetilde{K}-1}{N-1}}\frac{|\Sigma|}{\Sigma_{22}} \\
&= \frac{\widetilde{K}}{(N-2)(N+\widetilde{K}-2)}\frac{|\Sigma|}{\Sigma_{22}} > 0
\end{aligned} \tag{A.26}$$

We now show that λ_{11} is lower than $\lambda_0 \equiv \frac{\gamma\Sigma_{11}}{N-2}$ and decreases with K:

$$\lambda_{11} = \frac{1+\frac{\widetilde{K}}{N-2}\frac{|\Sigma|}{\Sigma_{11}\Sigma_{22}}}{N+\widetilde{K}-2}\Sigma_{11} = \frac{1}{N-2}\frac{N+\widetilde{K}\frac{|\Sigma|}{\Sigma_{11}\Sigma_{22}}-2}{N+\widetilde{K}-2}\Sigma_{11} \;\leq \frac{1}{N-2}\Sigma_{11}$$

The last inequality holds since, given $\Sigma_{11}, \Sigma_{22}, \Sigma_{21} > 0$, one has $\frac{|\Sigma|}{\Sigma_{11}\Sigma_{22}} = 1 - \frac{(\Sigma_{21})^2}{\Sigma_{11}\Sigma_{22}} <$

1, so that $\lambda_{11} < \lambda_0$. More generally,

$$\frac{\partial \lambda_{11}}{\partial K} = \frac{1}{N-2}\frac{|\Sigma|}{\Sigma_{22}}\frac{\widetilde{K}\left(\frac{|\Sigma|}{\Sigma_{11}\Sigma_{22}} - 1\right)}{(N+\widetilde{K}-2)^2}\frac{\partial \widetilde{K}}{\partial K} < 0.$$

A.7.2 Proof of Proposition 11

Plugging (5.8) and (6.5) into market clearing conditions (2.1) and (2.2) and multiplying by $\Lambda_{n,K} + \gamma\Sigma$, one gets

$$\frac{K}{N}\frac{\mathbb{E}_0[p_1^*] - f_0^*}{(1+a)\gamma \operatorname{Var}_0(p_1^*)}(\Lambda_{n,K} + \gamma\Sigma)\begin{pmatrix} 0 \\ 1 \end{pmatrix} + \begin{pmatrix} v_0 - p_0^* \\ v_0 - f_0^* \end{pmatrix} = \gamma\Sigma \begin{pmatrix} \bar{I}_0 \\ 0 \end{pmatrix} + \sum_n \frac{w_n}{N}\begin{pmatrix} \frac{N-2}{N-1} \\ 1 \end{pmatrix}$$

With $\Lambda_{n,K} + \gamma\Sigma = (1+b)\gamma\Sigma + \gamma \begin{pmatrix} \Delta & 0 \\ 0 & 0 \end{pmatrix}$, this is equivalent to

$$\frac{K}{N}\frac{\mathbb{E}_0[p_1^*] - f_0^*}{(1+a)\gamma \operatorname{Var}_0(p_1^*)}\begin{pmatrix} (1+b)\gamma\Sigma_{12} \\ (1+b)\gamma\Sigma_{22} \end{pmatrix} + \begin{pmatrix} v_0 - p_0^* \\ v_0 - f_0^* \end{pmatrix} = \gamma(\sigma_1^2 + \sigma_2^2)\bar{I}_0 \begin{pmatrix} 1 \\ 1 \end{pmatrix} + z\gamma\sigma_2^2\, \mathbb{E}_0\left[\frac{Q}{N}\right]\begin{pmatrix} 1 \\ \frac{N-1}{N-2} \end{pmatrix}. \tag{A.27}$$

The first equation shows that the spot price is increased by an amount proportional to the expected profit of the futures contract, because of financial traders. Solving for futures price first, one gets

$$\frac{\widetilde{K}}{N}(\mathbb{E}_0[p_1^*] - f_0^*) + v_0 - f_0^* = \gamma(\sigma_1^2 + \sigma_2^2)\bar{I}_0 + \frac{N-1}{N-2}z\gamma\sigma_2^2\, \mathbb{E}_0\left[\frac{Q}{N}\right].$$

$$\begin{aligned}\left(1 + \frac{\widetilde{K}}{N}\right) f_0^* = \left(1 + \frac{\widetilde{K}}{N}\right) v_0 - \gamma\left(\sigma_1^2 + \left(1 + \frac{\widetilde{K}}{N}\right)\sigma_2^2\right)\bar{I}_0 \\ - \left(z + \frac{\widetilde{K}}{N}\right)\frac{N-1}{N-2}\gamma\sigma_2^2\, \mathbb{E}_0\left[\frac{Q}{N}\right],\end{aligned}$$

so that

$$f_0^* = v_0 - \gamma\left[\frac{N\sigma_1^2}{N+\widetilde{K}} + \sigma_2^2\right]\bar{I}_0 - \left[\frac{zN}{N+\widetilde{K}} + \frac{\widetilde{K}}{N+\widetilde{K}}\right]\frac{N-1}{N-2}\gamma\sigma_2^2\, \mathbb{E}_0\left[\frac{Q}{N}\right], \tag{A.28}$$

which is (6.7). Then one can compute the expected profit of the futures contract, easily computed to give

$$\mathbb{E}_0[p_1^*] - f_0^* = \gamma \frac{N}{N+\widetilde{K}} \left(\sigma_1^2 \bar{I}_0 - (1-z)\frac{N-1}{N-2}\sigma_2^2 \, \mathbb{E}_0 \left[\frac{Q}{N} \right] \right), \tag{A.29}$$

which is (6.8), which in turn is used in (A.27) to get (with $\widehat{K} = \Sigma_{21}/\Sigma_{22}\widetilde{K}$):

$$p_0^* = v_0 - \gamma \left[\frac{N+\widetilde{K}-\widehat{K}}{N+\widetilde{K}} \sigma_1^2 + \sigma_2^2 \right] \bar{I}_0 - \left[z + \frac{(1-z)\widehat{K}\frac{N-1}{N-2}}{N+\widetilde{K}} \right] \gamma \sigma_2^2 \, \mathbb{E}_0 \left[\frac{Q}{N} \right]. \tag{A.30}$$

A.7.3 Proof of Lemma 4

One has to prove that the variance of $p_1^* - p_0^*(K)$ decreases with K. From (4.5) and (6.9), denoting $\widehat{K} = \Sigma_{21}/\Sigma_{22}$ to ease notation,

$$\begin{aligned} p_1^* - p_0^*(K) &= \epsilon_1 + \gamma(s_K - \sigma_2^2)\bar{I}_0 \\ &\quad + \gamma\sigma_2^2 \left[\left[(1-z)\left(\frac{\widehat{K}}{N+\widetilde{K}} - 1 \right) - \frac{1}{N-2} \right] \mathbb{E}_0 \left[\frac{Q}{N} \right] + \frac{N-1}{N-2} \left(\frac{Q}{N} - \mathbb{E}_0 \left[\frac{Q}{N} \right] \right) \right] \end{aligned}$$

so that, given that $\mathbb{E}_0[Q]$ and $Q - \mathbb{E}_0[Q]$ are independent,

$$\mathrm{Var}\left(p_1^* - p_0^*(K)\right) = \sigma_1^2 + \gamma^2\sigma_2^4 \left[\left[(1-z)\left(\frac{\widehat{K}}{N+\widetilde{K}} - 1 \right) - \frac{1}{N-2} \right]^2 \frac{\sigma_{EQ}^2}{N^2} + \left(\frac{N-1}{N-2} \right)^2 \frac{\sigma_Q^2}{N^2} \right]$$

Since $\Sigma_{21} < \Sigma_{22}$, the coefficient in front of $\mathrm{Var}\left(\mathbb{E}_0\left[\frac{Q}{N}\right]\right)$ is the square of a negative number; as K increases, this number is less negative, to that its square decreases: therefore the total variance decreases.

A.7.4 Proof of Proposition 12

Physical traders quantities. Plugging (6.6) and (6.7) into (5.8), together with (6.3), we get:

$$\begin{aligned}
\begin{pmatrix} q^*_{n,0} \\ x^*_n \end{pmatrix} &= (\Lambda_{n,K} + \gamma\Sigma)^{-1} \Bigg[\gamma\Sigma \begin{pmatrix} \bar{I}_0 - I_n \\ 0 \end{pmatrix} + \left(\sum_m \frac{w_m}{N} - w_n \right) \begin{pmatrix} \frac{N-2}{N-1} \\ 1 \end{pmatrix} \\
&\qquad - \frac{K}{N} \frac{E_0[p^*_1] - f^*_0}{(1+a)\gamma \operatorname{Var}_0(p^*_1)} (\Lambda_{n,K} + \gamma\Sigma) \begin{pmatrix} 0 \\ 1 \end{pmatrix} \Bigg] \\
&= (\Lambda_{n,K} + \gamma\Sigma)^{-1} \left[\gamma\Sigma \begin{pmatrix} \bar{I}_0 - I_n \\ 0 \end{pmatrix} - \frac{\gamma\sigma_2^2 z}{N} \left(\bar{I}^e_1 - I_{n,1} \right) \begin{pmatrix} \frac{N-2}{N-1} \\ 1 \end{pmatrix} \right] \\
&\qquad - \frac{K}{N} \frac{E_0[p^*_1] - f^*_0}{(1+a)\gamma \operatorname{Var}_0(p^*_1)} \begin{pmatrix} 0 \\ 1 \end{pmatrix}
\end{aligned}$$

And one has:

$$\begin{aligned}
|\Lambda_{n,K} + \gamma\Sigma| &= (1+b)\gamma^2 \begin{vmatrix} (1+b)\Sigma_{11} + \Delta & \Sigma_{21} \\ (1+b)\Sigma_{21} & \Sigma_{22} \end{vmatrix} = (1+b)\gamma^2 \left((1+b)|\Sigma| + \Delta\Sigma_{22} \right) \\
&= (1+b)\gamma^2 \frac{N-1}{N-2} |\Sigma|, \\
(\Lambda_{n,K} + \gamma\Sigma)^{-1} &= \frac{\gamma}{|\Lambda + \gamma\Sigma_f|} \begin{pmatrix} (1+b)\Sigma_{22} & -(1+b)\Sigma_{12} \\ -(1+b)\Sigma_{21} & (1+b)\Sigma_{11} + \Delta \end{pmatrix} \\
&= \frac{N-2}{N-1} \frac{1}{\gamma} \left(\Sigma^{-1} + \frac{\Delta}{(1+b)|\Sigma|} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right)
\end{aligned}$$

Therefore

$$\begin{aligned}
\begin{pmatrix} q^*_{n,0} \\ x^*_n \end{pmatrix} &= \frac{N-2}{N-1} \left[\left(Id_2 + \frac{\Delta}{(1+b)|\Sigma|} \begin{pmatrix} 0 & 0 \\ \Sigma_{21} & \Sigma_{22} \end{pmatrix} \right) \begin{pmatrix} \bar{I}_0 - I_n \\ 0 \end{pmatrix} + \left(\bar{I}^e_1 - I_{n,1} \right) \begin{pmatrix} \kappa_0 \\ h_K \end{pmatrix} \right] - \frac{K}{N} y^*_k \\
&= \frac{N-2}{N-1} \left[\begin{pmatrix} \bar{I}_0 - I_n \\ \frac{\Delta\Sigma_{21}}{(1+b)|\Sigma|} (\bar{I}_0 - I_n) \end{pmatrix} + \left(\bar{I}^e_1 - I_{n,1} \right) \begin{pmatrix} \kappa_0 \\ h_K \end{pmatrix} \right] - \frac{K}{N} y^*_k, \qquad \text{(A.31)}
\end{aligned}$$

where

$$h_K = -\frac{z\sigma_2^2}{N} \frac{\Sigma_{11} + \frac{\Delta}{1+b} - \frac{N-2}{N-1}\Sigma_{21}}{|\Sigma|}. \qquad \text{(A.32)}$$

A.7.5 Proof of Proposition 13

From (A.31), (A.32), (A.26) and (A.18), one has

$$H_K = \frac{1}{N-1}\left[\frac{b\widetilde{K}}{1+b}\frac{\sigma_1^2 + \left[1 - \frac{N-\psi_0}{N}z\right]\sigma_2^2}{\sigma_1^2 + \frac{1-z}{\alpha}\sigma_2^2} - \frac{\left(\frac{N-2}{N-1}\right)^2(\sigma_1^2+\sigma_2^2)z}{\left[1 - \frac{2N-3}{(N-1)^2}z\right]\sigma_1^2 + \left[1 - \left[\frac{N-2}{N-1} + \frac{2}{N}\right]z\right]\sigma_2^2}\right]$$

H_K increases with $\widetilde{K}$, since its dependence on $\widetilde{K}$ only goes through $b\widetilde{K}/(1+b) = (N+\widetilde{K}-1)^{-1}\widetilde{K}$, which increases with $\widetilde{K}$. $b\widetilde{K}$ also converges to 1 as $\widetilde{K}$ goes to infinity. Thus we study H_∞, the limit of H_K as $\widetilde{K}$ diverges to $+\infty$:

$$H_\infty = \frac{1}{N-1}\left[\frac{\sigma_1^2 + \left[1 - \frac{N-\psi_0}{N}z\right]\sigma_2^2}{\sigma_1^2 + \frac{1-z}{\alpha}\sigma_2^2} - \frac{\left(\frac{N-2}{N-1}\right)^2(\sigma_1^2+\sigma_2^2)z}{\left[1 - \frac{2N-3}{(N-1)^2}z\right]\sigma_1^2 + \left[1 - \left[\frac{N-2}{N-1} + \frac{2}{N}\right]z\right]\sigma_2^2}\right].$$

The proof is in two steps. First, I show that for $z = \bar{z}(\sigma_1^2)$, $H_\infty = 0$. Second, I show that H_∞ decreases with z, showing that for $z < \bar{z}$, $H_\infty > 0$.

$H_\infty = 0$ for $z = \bar{z}(\sigma_1^2)$. I evaluate H_∞ at $z = \bar{z} = \frac{x+1}{x+2\frac{N-1}{N}}$ where $x = \sigma_1^2/\sigma_2^2$. Multiplying the numerator and the denominator of both terms by $x + 2\frac{N-1}{N}$ and rearranging, one gets:

$$\begin{aligned} H_\infty &= \frac{x\left(x + 2\frac{N-1}{N}\right) + \frac{x}{N} + \frac{N-1}{N}}{x\left(x+2\frac{N-1}{N}\right) + \frac{(N-1)^2}{N(N-2)}\frac{N-2}{N}} - \frac{\left(\frac{N-2}{N-1}\right)^2(x+1)^2}{x\left(\left(\frac{N-2}{N-1}\right)^2(x+1) + 1 - \frac{2}{N}\right) - \frac{N-2}{N(N-1)}x + \frac{(N-2)^2}{N(N-1)}} \\ &= \frac{(x+1)\left(x+\frac{N-1}{N}\right)}{\left(x+\frac{N-1}{N}\right)^2} - \frac{\left(\frac{N-2}{N-1}\right)^2(x+1)^2}{\left(\frac{N-2}{N-1}\right)^2\left(x+\frac{N-1}{N}\right)(x+1)} = 0. \end{aligned}$$

H_∞ **decreases with** z**.** One computes:

$$
\begin{aligned}
\frac{\partial H_\infty}{\partial z} &= \frac{\sigma_2^4}{N-1}\Bigg[\frac{\frac{1}{N}\frac{\partial \psi_0}{\partial z}\Sigma_{22} + \left(\frac{1}{\alpha} - \frac{N-\psi_0}{N}\right)x + \frac{\psi_0}{N\alpha}}{(\Sigma_{22})^2} \\
&\qquad - \left(\frac{\frac{N-2}{N-1}(1+x)}{\left[1 - \frac{2N-3}{(N-1)^2}z\right]x + 1 - \left[\frac{N-2}{N-1} + \frac{2}{N}\right]z}\right)^2\Bigg] \\
&= \frac{\sigma_2^4}{N-1}\Bigg[\frac{\frac{\frac{1}{N(N-1)}(x+1)(x+1-1/N)}{\left(\left(1-\frac{z}{N-1}\right)x+1-z+\frac{(1-z)x+1-2\frac{N-1}{N}z}{N-2}\right)^2}\Sigma_{22} + \frac{1}{N}\left(\psi_0 + \frac{1}{N-2}\right)x + \frac{\psi_0}{N\alpha}}{(\Sigma_{22})^2} \\
&\qquad - \left(\frac{\frac{N-2}{N-1}(1+x)}{\left[1 - \frac{2N-3}{(N-1)^2}z\right]\sigma_1^2 + \left[1 - \left[\frac{N-2}{N-1} + \frac{2}{N}\right]z\right]\sigma_2^2}\right)^2\Bigg] \\
&= \frac{\sigma_2^4}{N-1}\Bigg[\frac{\frac{\frac{1}{N(N-1)}(x+1)(x+1-1/N)}{\left(\left(\frac{N-1}{N-2}-\frac{2N-3}{(N-1)(N-2)}z\right)x+\frac{N-1}{N-2}-\left(1+2\frac{N-1}{N(N-2)}\right)z\right)^2}\Sigma_{22} + \frac{1}{N}\left(\psi_0 + \frac{1}{N-2}\right)x + \frac{\psi_0}{N\alpha}}{(\Sigma_{22})^2} \\
&\qquad - \left(\frac{\frac{N-2}{N-1}(1+x)}{\left[1 - \frac{2N-3}{(N-1)^2}z\right]\sigma_1^2 + \left[1 - \left[\frac{N-2}{N-1} + \frac{2}{N}\right]z\right]\sigma_2^2}\right)^2\Bigg] \\
&= \frac{\sigma_2^4}{N-1}\Bigg[\frac{\frac{(N-2)^2}{N(N-1)^3}(x+1)(x+1-1/N)}{\left(\left(1-\frac{2N-3}{(N-1)^2}z\right)x+1-\left(\frac{N-2}{N-1}+\frac{2}{N}\right)z\right)^2\Sigma_{22}} + \frac{\frac{1}{N}\left(\psi_0 + \frac{1}{N-2}\right)x + \frac{\psi_0}{N\alpha}}{(\Sigma_{22})^2} \\
&\qquad - \left(\frac{\frac{N-2}{N-1}(1+x)}{\left[1 - \frac{2N-3}{(N-1)^2}z\right]\sigma_1^2 + \left[1 - \left[\frac{N-2}{N-1} + \frac{2}{N}\right]z\right]\sigma_2^2}\right)^2\Bigg] \\
&= \frac{\sigma_2^4}{N-1}\Bigg[\frac{\frac{1}{N}\left(\psi_0 + \frac{1}{N-2}\right)x + \frac{\psi_0}{N\alpha}}{(\Sigma_{22})^2} \\
&\qquad - \frac{\left(\frac{N-2}{N-1}\right)^2(1+x)}{\left(\left[1 - \frac{2N-3}{(N-1)^2}z\right]\sigma_1^2 + \left[1 - \left[\frac{N-2}{N-1} + \frac{2}{N}\right]z\right]\sigma_2^2\right)^2}\left[x+1-\frac{\frac{1}{N(N-1)}(x+1-1/N)}{x+\frac{1-z}{\alpha}}\right]\Bigg] \\
&= \frac{\sigma_2^4}{N-1}\Bigg[\frac{\frac{1}{N}\left(\psi_0 + \frac{1}{N-2}\right)x + \frac{\psi_0}{N\alpha}}{(\Sigma_{22})^2} \\
&\qquad - \frac{\left(\frac{N-2}{N-1}\right)^2(1+x)}{\left(\left[1 - \frac{2N-3}{(N-1)^2}z\right]\sigma_1^2 + \left[1 - \left[\frac{N-2}{N-1} + \frac{2}{N}\right]z\right]\sigma_2^2\right)^2}\Bigg[\frac{(x+1)\left(x+\frac{1-z}{\alpha}-\frac{1}{N(N-1)}\right)+\frac{1}{N^2(N-}}{x+\frac{1-z}{\alpha}}
\end{aligned}
$$

where the second line uses $1/\alpha - 1 = \frac{1}{N(N-2)}$.

$$\frac{\partial H_\infty}{\partial z} = \frac{\sigma_2^4}{(N-1)\Sigma_{22}} \left[\frac{\frac{1}{N}\left(\psi_0 + \frac{1}{N-2}\right) x + \frac{\psi_0}{N\alpha}}{x + (1-z)/\alpha} \right.$$
$$\left. - \frac{\left(\frac{N-2}{N-1}\right)^2 (1+x) \left[(x+1)\left(x + \frac{1-z}{\alpha} - \frac{1}{N(N-1)}\right) + \frac{1}{N^2(N-1)}\right]}{\left(\left[1 - \frac{2N-3}{(N-1)^2} z\right]\sigma_1^2 + \left[1 - \left[\frac{N-2}{N-1} + \frac{2}{N}\right] z\right] \sigma_2^2\right)^2} \right]$$

It is easy to check that

$$\sigma_1^2 + \frac{1-z}{\alpha}\sigma_2^2 > \left[1 - \frac{2N-3}{(N-1)^2} z\right] \sigma_1^2 + \left[1 - \left[\frac{N-2}{N-1} + \frac{2}{N}\right] z\right] \sigma_2^2$$

so that

$$\begin{aligned}
\frac{\partial H_\infty}{\partial z} &< \frac{\sigma_2^4}{(N-1)\Sigma_{22}^2} \left[\frac{\frac{1}{N-2} + \psi_0}{N} \frac{\sigma_1^2}{\sigma_2^2} + \frac{\psi_0}{N\alpha} - \left(\frac{N-2}{N-1}\left(1 + \frac{\sigma_1^2}{\sigma_2^2}\right)\right)^2 \right] \\
&= \frac{\sigma_2^4}{(N-1)\Sigma_{22}^2} \left[-\left(\frac{N-2}{N-1}\frac{\sigma_1^2}{\sigma_2^2}\right)^2 + \left[\frac{\frac{1}{N-2} + \psi_0}{N} - 2\left(\frac{N-2}{N-1}\right)^2\right] \frac{\sigma_1^2}{\sigma_2^2} + \frac{\psi_0}{N\alpha} - \left(\frac{N-2}{N-1}\right)^2 \right] \\
&< \frac{\sigma_2^4}{(N-1)\Sigma_{22}^2} \left[-\left(\frac{N-2}{N-1}\frac{\sigma_1^2}{\sigma_2^2}\right)^2 + \left[\frac{N-1}{N(N-2)} - 2\left(\frac{N-2}{N-1}\right)^2\right] \frac{\sigma_1^2}{\sigma_2^2} \right. \\
&\qquad \left. + \frac{(N-1)^2}{N^2(N-2)} - \left(\frac{N-2}{N-1}\right)^2 \right]
\end{aligned}$$

where the second inequality follows from $\psi_0 \leq 1$. For $N \geq 4$, the second and third terms inside the brackets are negative, so that in the end $\partial H_\infty/\partial z < 0$. Evaluating $\partial H_\infty/\partial z$ for $N = 3$, it is not difficult to prove that it is also negative.

A.7.6 Proof of Theorem 2

First statement. One studies $\Delta_\times \widehat{W}_n \equiv \widehat{W}^*_{n,0}(K) - \widehat{W}^\times_{n,0}$, which, from (4.6) and equilibrium quantities, is, denoting $\pi \equiv (\mathbb{E}_0[p_1^*] - f_0^*(K)) \times \widetilde{K}/N$:

$$\begin{aligned}
\Delta_\times \widehat{W}_n &= S_n(K) - S_n(\psi_\times) - \frac{\gamma\Sigma_{22}}{2}\left(x_n^*(K)\right)^2 + \left(\widehat{p}_1 - f_0^*(K) - \gamma(\sigma_1^2 + (1-z)\sigma_2^2) I_{n,1}^*\right) x_n^*(K) \\
&= S_n(0) - \left(q_{n,0}^* \frac{\Sigma_{21}}{\Sigma_{22}} + x_n^*(K)\right)\pi - S_n^\times \\
&\quad + \gamma\left[(\sigma_1^2 + (1-z)\sigma_2^2)(\bar{I}_1 - I_{n,1}^*) - \Sigma_{22}/2 x_n^*(K)\right] x_n^*(K)
\end{aligned}$$

One expands $S_n(K)$ from (A.20) and equilibrium price (6.6), $x_n^*(K)$ from (6.13) and $q_{n,0}^*$ from (6.11). Finally, $\bar{I}_1 - I_{n,1}^* = (1-\psi_0)(\bar{I}_0 - I_n)$. Rearranging, one gets:

$$\begin{aligned}
\Delta_\times \widehat{W}_n = \gamma \Bigg[& \frac{\sigma_1^2 + (1-\alpha z)\sigma_2^2}{2} \left((1-\psi_\times)^2 - (1-\psi_0)^2\right) \\
& + H_K \left((\sigma_1^2 + (1-z)\sigma_2^2)(1-\psi_0) - \frac{\Sigma_{22}}{2} H_K \right) \Bigg] (\bar{I}_0 - I_n)^2 \\
& - \left[H_K + \frac{\sigma_1^2 + \left(1 - \frac{N-\psi_0}{N} z\right)\sigma_2^2}{\Sigma_{22}} \frac{1}{1+b} + \psi_0 \frac{\Sigma_{21}}{\Sigma_{22}} \right] (\bar{I}_0 - I_n)\pi \\
& + \frac{\psi_\times - \psi_0}{N-1} \gamma \sigma_2^2 \, \mathbb{E}_0 \left[\frac{Q}{N} \right] (\bar{I}_0 - I_n) + \frac{\pi^2}{2(1+b)\gamma \Sigma_{22}}.
\end{aligned}$$

This is a quadratic function of $\bar{I}_0 - I_n$. One now shows that the coefficient of $(\bar{I}_0 - I_n)^2$ is positive: this implies that for each π, $\Delta_\times \widehat{W}_n$ is positive for large $(\bar{I}_0 - I_n)^2$. Given that $\psi_\times < \psi_0$, the first term of this coefficient is positive. Given that $H_K > 0$, it is sufficient to show that the following quantity is positive:

$$(\sigma_1^2 + (1-z)\sigma_2^2)(1-\psi_0) - \frac{\Sigma_{22}}{2} H_K = (1 - \psi_0 - H_K/2)\sigma_1^2 + (1 - \psi_0 - H_K/(2\alpha))\,\sigma_2^2$$

Since $\alpha < 1$, the RHS is positive iff $1 - \psi_0 - H_K/(2\alpha) > 0$. Since $1 - \psi_0$ decreases with z and H_K increases with z (see Proposition 13), this quantity is minimal for $z = \bar{z}(\sigma_1^2)$. As $H_K(\bar{z}) = 0$ (proof of Proposition 13) and $\psi_0(\bar{z}) = 1$ (see (A.14) in the proof of Proposition 6 and the definition of $\bar{z}$ in the proof of Lemma 2), one has $1 - \psi_0 - H_K/(2\alpha) \geq 0$. QED.

Second statement. One now studies $\Delta \widehat{W}_n \equiv \widehat{W}_{n,0}^*(K) - \widehat{W}_{n,0}^*(0)$:

$$\begin{aligned}
\Delta \widehat{W}_n = & \underbrace{S_n(K) - S_n(0)}_{-q_{n,0}^* \frac{\widehat{K}}{N}(\mathbb{E}_0[p_1^*] - f_0^*(K))} - \frac{\gamma \Sigma_{22}}{2} \left((x_n^*(K))^2 - (x_n^*(0))^2 \right) \\
& + \left(\widehat{p}_1 - f_0^*(K) - \gamma(\sigma_1^2 + (1-z)\sigma_2^2) I_{n,1}^* \right) x_n^*(K) \\
& - \underbrace{\left(\widehat{p}_1 - f_0^*(0) - \gamma(\sigma_1^2 + (1-z)\sigma_2^2) I_{n,1}^* \right)}_{\gamma(\sigma_1^2 + (1-z)\sigma_2^2)(\bar{I}_1 - I_{n,1}^*)} x_n^*(0).
\end{aligned}$$

I first show that $\Delta \widehat{W}_n$ can be expressed as a quadratic form in $\pi \equiv (\mathbb{E}_0[p_1^*] - f_0^*(K)) \times \widetilde{K}/N$ and $(\bar{I}_0 - I_n)$. Then I show that $\Delta \widehat{W}_n$ exhibits positive and negative values for each value of $(\mathbb{E}_0[p_1^*] - f_0^*(K)) \times \widetilde{K}/N$, depending on $\bar{I}_0 - I_n$.

Lemma 10.

$$\Delta\widehat{W}_n = \left[1 - \left(\frac{b}{1+b}\right)^2\right]\frac{\pi^2}{\Sigma_{22}} - \left[\frac{bH_K}{1+b} + \frac{\sigma_1^2 + (1-(N-\psi_0)/Nz)\,\sigma_2^2}{(1+b)\Sigma_{22}}\right]\pi(\bar{I}_0 - I_n)$$
$$+ \gamma\left[(\sigma_1^2 + (1-z)\sigma_2^2)(1-\psi_0) - \frac{\Sigma_{22}\left[h_0(1-\psi_0)\frac{N-2}{N-1} + H_K\right]}{2}\right] a_K(\bar{I}_0 - I_n)^2.$$

Proof. Given that

$$x_n^*(K) = x_n^*(0) + a_K(\bar{I}_0 - I_n) - \frac{K}{N}y_k^*,$$

and equilibrium prices (6.6) and (6.7), one has

$$\begin{aligned}&\left(\widehat{p}_1 - f_0^*(K) - \gamma(\sigma_1^2 + (1-z)\sigma_2^2)I_{n,1}^*\right)x_n^*(K)\\ &\quad= \left[\gamma(\sigma_1^2 + (1-z)\sigma_2^2)(\bar{I}_1 - I_{n,1}^*) - \pi\right](x_n^*(0) + a_K(\bar{I}_0 - I_n) - Ky_k^*/N).\end{aligned}$$

One can decompose the welfare impact of having K financial traders as

$$\begin{aligned}\Delta\widehat{W}_n &= -\left[q_{n,0}^*\frac{\Sigma_{21}}{\Sigma_{22}} + x_n^*(0)\right]\pi + \left(\gamma(\sigma_1^2 + (1-z)\sigma_2^2)(\bar{I}_1 - I_{n,1}^*) - \pi\right)\left[a_K(\bar{I}_0 - I_n) - Ky_k^*/N\right]\\ &\quad - \gamma\Sigma_{22}/2\left(2x_n^*(0) + a_K(\bar{I}_0 - I_n) - Ky_k^*/N\right)\left(a_K(\bar{I}_0 - I_n) - Ky_k^*/N\right)\\ &= -\left(H_K(\bar{I}_0 - I_n) + \Sigma_{21}/\Sigma_{22}q_{n,0}^*\right)\pi\\ &\quad - \gamma\Sigma_{22}/2\left(x_n^*(0) + H_K(\bar{I}_0 - I_n) - Ky_k^*/N\right)\left(a_K(\bar{I}_0 - I_n) - Ky_k^*/N\right)\\ &\quad + \gamma(\sigma_1^2 + (1-z)\sigma_2^2)(\bar{I}_1 - I_{n,1}^*)(a_K(\bar{I}_0 - I_n) - Ky_k^*/N) + K/N\pi y_k^*\end{aligned}$$

One has, from (6.10), $Ky_k^*/N = \frac{\pi}{(1+b)\gamma\Sigma_{22}}$ and $H_K = x_n^*(0)(\bar{I}_0 - I_n)^{-1} + a_K$. Moreover, $a_K = \frac{b\tilde{K}}{N-1}\frac{\Sigma_{21}}{\Sigma_{22}}$. Rearranging,

$$\begin{aligned}\Delta\widehat{W}_n &= -\left[\left(1 - \frac{1}{1+b}\right)H_K + \frac{\Sigma_{21}}{\Sigma_{22}}\psi_0 + \frac{\sigma_1^2 + (1-z)\sigma_2^2}{(1+b)\Sigma_{22}}\right](\bar{I}_0 - I_n)\pi\\ &\quad + \left[\gamma(\sigma_1^2 + (1-z)\sigma_2^2)(1-\psi_0) - \frac{\gamma\Sigma_{22}}{2}(h_0 + H_K)\right]a_K(\bar{I}_0 - I_n)^2\\ &\quad + \pi Ky_k^*/N - \gamma\Sigma_{22}/2\left(Ky_k^*/N\right)^2\end{aligned}$$

Expanding $\Sigma_{21} = \sigma_1^2 + (1-(N-1)/Nz)\sigma_2^2)$, replacing y_k^* by its expression as a function of π and rearranging, one gets the desired result. ■

I now study the properties of $\Delta\widehat{W}_n$. The coefficient in π^2 is clearly positive. The coefficient in $(\bar{I}_0 - I_n)^2$ is positive as well given $a_K > 0$ while the second factor is positive as shown in the proof of the first statement. I now study whether, given π, the inventory $\underline{I}$ such that $\bar{I}_0 - \underline{I}$ minimizes ΔW_n is negative. Given the positivity of both coefficients in π^2 and $(\bar{I}_0 - I_n)^2$, this quantity is such that

$$\bar{I}_0 - \underline{I} = \frac{\frac{bH_K}{1+b} + \frac{\sigma_1^2 + (1-(N-\psi_0)/Nz)\sigma_2^2}{(1+b)\Sigma_{22}}}{2(\gamma(\sigma_1^2 + (1-z)\sigma_2^2)(1-\psi_0) - \Sigma_{22}\left(h_0(1-\psi_0)\frac{N-2}{N-1} + H_K\right)/2)}\pi$$

And the welfare difference taken $\underline{\Delta \widetilde{W}}$ at $\bar{I}_0 - \underline{I}$ given π verifies

$$\frac{\underline{\Delta \widehat{W}}\Sigma_{22}}{\pi^2} = 1 - \left(\frac{b}{1+b}\right)^2 - \frac{\frac{1}{2}\Sigma_{22}\left(1 - \frac{a_K}{2}\right)\left[\frac{b}{1+b}H_K + \frac{1}{1+b}\left(\sigma_1^2 + \left(1 - \frac{N-\psi_0}{N}z\right)\sigma_2^2\right)/\Sigma_{22}\right]^2}{\gamma\left((\sigma_1^2 + (1-z)\sigma_2^2)(1-\psi_0) - \Sigma_{22}[h_0(1-\psi_0)\frac{N-2}{N-1} + H_K]/2\right)}$$

For $K \to \infty$, this becomes, since $b/(1+b) = (N + \tilde{K} - 1)^{-1}$,

$$\frac{\underline{\Delta \widehat{W}}^\infty \Sigma_{22}}{\pi^2} = 1 - \frac{\frac{1}{2}\Sigma_{22}\left(1 - \frac{1}{2(N-1)}\frac{\Sigma_{21}}{\Sigma_{22}}\right)\left[\left(\sigma_1^2 + \left(1 - \frac{N-\psi_0}{N}z\right)\sigma_2^2\right)/\Sigma_{22}\right]^2}{\gamma\left((\sigma_1^2 + (1-z)\sigma_2^2)(1-\psi_0) - \Sigma_{22}[h_0(1-\psi_0)\frac{N-2}{N-1} + H_\infty]/2\right)}.$$

For $z = \bar{z}(\sigma_1^2)$, since $H_\infty(\bar{z}) = 0$ (see proof of Proposition 13) and $\psi_0(\bar{z}) = 1$, thus:

$$\frac{\underline{\Delta \widehat{W}}^\infty(\bar{z})\Sigma_{22}}{\pi^2} = 1 - \frac{\left(1 - \frac{1}{2(N-1)}\frac{\Sigma_{21}}{\Sigma_{22}}\right)\left[\frac{\Sigma_{21}}{\Sigma_{22}}\right]^2}{-\gamma h_0(1-\psi_0)\frac{N-2}{N-1}}.$$

It is not difficult to show that for $z = \bar{z}$, denoting $x = \sigma_1^2/\sigma_2^2, \frac{\Sigma_{21}}{\Sigma_{22}} = \frac{x^2 - \frac{2N-1}{N}x + \frac{N-1}{N}}{\left(x + \frac{N-1}{N}\right)^2}$, which equals $(N-1)/N$ for $x = 0$. And for $x = 0$, the proof of Proposition 6 shows that $h_0(1-\psi_0)\frac{N-2}{N-1} = -\alpha$. Therefore

$$\sigma_1^2 = 0, z = \bar{z}(0) \quad \Rightarrow \quad \frac{\underline{\Delta \widehat{W}}^\infty(\bar{z})\Sigma_{22}}{\pi^2} = 1 - \left(1 - \frac{1}{2}\frac{N}{(N-1)^2}\right)\frac{N}{N-2} < 0 \; \forall N.$$

QED.

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B A tax/subsidy scheme to restore a more efficient equilibrium

As illustrated by equation (4.5), date-1 price can be influenced by strategic trading, which is also the case without futures; the expression of date-0 wealth (4.6) shows how traders wish to do this to influence futures payoff. Adding the following tax/subsidy scheme to traders' wealth (4.6) exactly cancels the incentive to influence futures payoff:

$$\tau_{xq} = \frac{\gamma \sigma_2^2}{N} \times x_k \times q_{k,0}$$

This amount is negative and corresponds to a tax if date-0 transactions in the spot and futures markets are in opposite directions; it is positive and corresponds to a subsidy if transactions are in the same direction. In the appendix, I also examine the case with a similar tax/subsidy scheme proportional to x_k: it is simply a transfer between buyers and sellers and does not change the equilibrium allocation.

This scheme may not be optimal, but it removes the incentive to influence futures payoff and induces an equilibrium where futures make traders better off.

Theorem 3. 1. *The certainty equivalent of wealth with the tax/subsidy scheme is strictly concave for all parameters.*

2. *Sellers of the underlying asset sell futures to buyers of the underlying asset:*

$$q_{k,0}^{\tau} = \frac{1}{1 + A_\tau}\, q_k^c \tag{B.1}$$

$$q_{k,1}^{\tau} = \frac{N-2}{N-1} \times \frac{A_\tau}{1 + A_\tau}\, q_k^c + \frac{Q}{N} \tag{B.2}$$

$$x_k^{\tau} = h_\tau \left(q_{k,1}^{\tau} - \frac{Q}{N} \right) \tag{B.3}$$

where $A_\tau > 0$ and $h_\tau \in (0, 1/2)$.

3. *Traders defer more trades to date 1 than without futures*

$$|q_{k,0}^{\tau}| < |q_{k,0}^{n}| \qquad \textit{and} \qquad |q_{k,1}^{\tau}| > |q_{k,1}^{n}|.$$

4. The tax/subsidy scheme does not affect prices:

$$p_0^\tau = p_0^f, \qquad f_0^\tau = f_0^*.$$

5. Traders' equilibrium utilities are greater than without futures:

$$\widehat{W}_{k,0}^\tau > \widehat{W}_{k,0}^n.$$

This proposition is proved in appendix B.1. Such a tax scheme therefore induces a better equilibrium within the model.

In the real world, one may fear that such a tax/subsidy hits arbitrageurs, who simultaneously trade in opposite directions in spot and futures markets, and in principle provide socially useful links between fragmented markets. An analysis of the impact of arbitrage is beyond the scope of this paper.

B.1 Proof of theorem 3

I study the equilibrium when there is the following tax scheme on joint transactions for an individual trader of class k:

$$\tau_k = \underbrace{-\gamma\sigma_2^2\left(\bar{I}_1^e - \frac{q_{k,0}}{N}\right)}_{\tau_x} x_k - \underbrace{\frac{\gamma\sigma_2^2}{N}}_{\tau_{xq}} q_{k,0} x_k \tag{B.4}$$

Specification (B.4) implies that futures payoff net of the tax is now

$$p_1^* - f_0 - \tau_k = \epsilon_1 - \gamma\sigma_2^2 \frac{Q}{N},$$

so that the term $\gamma\sigma_2^2\bar{I}_1^e$ is removed from futures payoff. The first term in (B.4) does not depend on $q_{k,0}$ as indicated by (5.1) and is a tax for futures sellers, and a subsidy for futures buyers. With this tax scheme, the certainty equivalent of trader k's wealth is $\widehat{W}_{k,0}^\tau = \widehat{W}_{k,0}^f - \tau$, where $\widehat{W}_{k,0}^f$ is set in (4.6).

B.1.1 Concavity of $\widehat{W}_{k,0}^f - \tau_k$.

Differentiating $\widehat{W}_{k,0}^f - \tau_k$ with respect to $q_{k,0}$ and $x_{k,0}$ for trader k and setting the derivatives to zero gives an expression formally identical to (**??**), except that Σ and

K_f are respectively replaced with

$$\Sigma_\tau = \Sigma - \begin{pmatrix} 0 & \sigma_2^2/N \\ \sigma_2^2/N & 0 \end{pmatrix}, \quad K_\tau = K_f + \begin{pmatrix} 0 & \sigma_2^2/N \\ \sigma_2^2 & 0 \end{pmatrix}.$$

Now I compute $(\Sigma_\tau)^{-1}$. The determinant of Σ_τ is

$$|\Sigma_\tau| = \frac{2}{N}\sigma_2^2 \left\{ \left[1 + \frac{1-z}{2(N-2)} \right] \sigma_1^2 + \frac{1-z}{\alpha}\sigma_2^2 \right\} \tag{B.5}$$

which is always positive since $z \leq 1$: this proves point 1 of theorem 3.

B.1.2 Equilibrium prices (point 4) and quantities (point 2).

Optimal demand schedules solve the first order conditions. Equilibrium risk premia are similar to (**??**), replacing with Σ_τ and K_τ. This gives the equilibrium prices:

$$p_0^\tau = v_0 - \gamma(\sigma_1^2 + \sigma_2^2)\bar{I}_0 - \gamma\sigma_2^2 z\, \mathbb{E}_0\left[Q/N\right] \tag{B.6}$$

$$f_0^\tau - \tau_x = -\gamma\sigma_1^2\bar{I}_0 - \frac{N-1}{N-2}\gamma\sigma_2^2 z\, \mathbb{E}_0\left[Q/N\right] \tag{B.7}$$

Similarly to the case without taxes: denote κ_1^τ and κ_2^τ the quantities, to be computed later, such that

$$q_{b,0}^\tau = \frac{1}{1+A_\tau}\frac{S}{N}\left(I_{s,0} - I_{b,0}\right) \qquad \text{with} \qquad A_\tau = \frac{1}{N-2}\left(1 - \frac{\kappa_1^\tau}{N}\right)^{-1} \tag{B.8}$$

$$x_b^\tau = -\frac{N-2}{N-1} \times \frac{1+\kappa_2^\tau}{N} \times \frac{A_\tau}{1+A_\tau} \times \frac{S}{N}\left(I_{s,0} - I_{b,0}\right) \tag{B.9}$$

Computation of κ_1^τ and $q_{b,0}^\tau$. One has

$$|\Sigma_\tau|\kappa_1^\tau = \sigma_2^2 \left\{ \left(1 - \frac{z}{N-1}\right)\sigma_1^2 + \frac{N-1}{N}(1-z)\sigma_2^2 \right\}$$

Therefore

$$1-\frac{\kappa_1^\tau}{N}=\frac{N-1}{N-2}\frac{\left(1-\frac{z}{(N-1)^2}\right)\sigma_1^2+(1-z)\sigma_2^2}{\left[2+\frac{1-z}{N-2}\right]\sigma_1^2+\frac{2}{\alpha}(1-z)\sigma_2^2}$$

$$A_\tau(\sigma_q^2)=\frac{1}{N-1}\times\frac{\left(2+\frac{1-z}{N-2}\right)\sigma_1^2+\frac{2}{\alpha}(1-z)\sigma_2^2}{\left(1-\frac{z}{(N-1)^2}\right)\sigma_1^2+(1-z)\sigma_2^2} \tag{B.10}$$

Together with (B.8), this gives equilibrium date-0 quantity traded $q_{b,0}^\tau$.

When both $\sigma_1^2=0$ and $z=1$, A_τ is undefined; it is easy to see that when σ_1^2 converges to zero and z converges to 1, whatever the order in taking the limits, A_τ converges to $2\frac{N-1}{N(N-2)}$. So I define this value for A_τ when $\sigma_1^2=0$ and $z=1$.

Variations of A_τ Denote $x=\sigma_1^2/\sigma_2^2$. Then

$$\frac{\partial A_\tau}{\partial x}=\frac{2}{N-1}\frac{\left(\frac{1}{2}-\frac{1}{N}\right)(1-z)\left(\frac{1}{N-2}+z\right)}{\left(\left(1-\frac{z}{(N-1)^2}\right)\sigma_1^2+(1-z)\sigma_2^2\right)^2}>0$$

given that $N\geq 3$, so that $1/2>1/N$. Differentiating with respect to z:

$$\frac{\partial A_\tau}{\partial z}=\frac{2}{N-1}\frac{x^2}{(...)^2}\left\{1-\frac{N(N-3/2)}{(N-1)^2}\right\}$$

Given that for all N, $\frac{N(N-3/2)}{(N-1)^2}>1$, one deduces that $\frac{\partial A_\tau}{\partial z}<0$.

Computation of x_k^τ. The second coefficient κ_2^τ of $\Sigma_\tau^{-1}K_\tau$ is such that:

$$|\Sigma_\tau|\kappa_2^\tau=-\sigma_2^2\left\{\frac{1}{N}\left[2+\frac{(N-1)^2}{N-2}-z\right]\sigma_1^2+\left[1+\frac{2}{N\alpha}\right](1-z)\sigma_2^2\right\}$$

Therefore

$$1+\kappa_2^\tau=-\frac{N}{2}\frac{\left[1-\frac{1}{N}\frac{N-3}{N-2}z\right]\sigma_1^2+(1-z)\sigma_2^2}{\left[1+\frac{1-z}{2(N-2)}\right]\sigma_1^2+\frac{1-z}{\alpha}\sigma_2^2}$$

As $0<z\leq 1$, one has $1+\kappa_2^\tau<0$. Finally set $h_\tau=-(1+\kappa_2^\tau)/N$ to get the result.

Variations and bounds of h_τ. One can show, denoting $x = \sigma_1^2/\sigma_2^2$, that:

$$\frac{\partial h_\tau}{\partial z} = \frac{1}{2(N-2)} \frac{x\left(\frac{1}{N}\left(1-\frac{N-3}{\alpha}\right)+x\left(\frac{1}{2}-\frac{N-3}{N}\frac{N-3/2}{N-2}\right)\right)}{\left(\left[1+\frac{1-z}{2(N-2)}\right]x+\frac{1-z}{\alpha}\right)^2}$$

For $N = 3$, it is clear that $\partial h_\tau/\partial z > 0$. For $N \geq 4$, one can check that $1-(N-3)/\alpha < 0$; while $1/2 - \frac{(N-3)(N-3/2)}{N(N-2)} < 0$ for $N \geq 6$. Thus for $N \geq 6$, without ambiguity $\partial h_y/\partial z < 0$. For $N = 4, 5$, there are x_4, x_5 such that for $x < x_N$, $\partial h_\tau/\partial z < 0$ and for $x > x_N$, $\partial h_\tau/\partial z > 0$.

Case $N \geq 4$. In this case the partial derivative of h_y with respect to z above is unambiguously negative, thus:

$$h_\tau \leq \frac{1}{2}\frac{\sigma_1^2+\sigma_2^2}{\left[1+\frac{1}{2(N-2)}\right]\sigma_1^2+\frac{1}{\alpha}\sigma_2^2} = \frac{1}{2}\frac{x+1}{\frac{N-3/2}{N-2}x+\frac{1}{\alpha}}$$

Computing the derivative of the RHS term with respect to x, it is easy to show that the RHS decreases in x, so that it is maximized for x, and thus $0 < h_\tau \leq \frac{\alpha}{2}$.

Case $N = 3$. The sign of $\partial h_\tau/\partial z$ is minus the sign of its numerator, which equals $x - 1$. Thus for $N = 3$, $\partial h_\tau/\partial z < 0$ iff $x > 1$. And inspecting h_τ, one easily sees that the numerator is smaller than the denominator, so that $h_\tau \leq \frac{1}{2}$.

B.1.3 Trading pace (Point 3.)

Here I show that for all $\sigma_1^2 \geq 0$ and all $\sigma_q^2 \geq 0$, $A_\tau(\sigma_1^2, \sigma_q^2) > A(\sigma_1^2, \sigma_q^2)$. I examine the difference $A - A_\tau$ and proceed in three steps. First, I show that $A - A_\tau$ increases with z for any σ_1^2. Second, I show that for $z = 1$, $A - A_\tau$ decreases with σ_1^2. Thus I find that the maximum of $A - A_\tau$ is attained for $\sigma_1^2 = 0$ and $z = 1 \Leftrightarrow \sigma_q^2 \to \infty$: I simply show that $A(0, \infty) = A_\tau(0, \infty)$, then the claim is proven.

First step. From lemma 6, we know that $\frac{\partial A}{\partial z} > 0$, while I compute:

$$\frac{\partial A_\tau}{\partial z} = -\frac{\sigma_1^4}{N-1}\frac{N^2+4}{(N-1)^2(N-2)} \times \left(\left(1-\frac{z}{(N-1)^2}\right)\sigma_1^2+(1-z)\sigma_2^2\right)^{-2} < 0$$

Second step. Now for $z = 1$,

$$A - A_\tau = \frac{1}{N-2}\left(\frac{\sigma_1^2+\sigma_2^2/N}{\sigma_1^2+\frac{\sigma_2^2}{N-1}} - \frac{2(N-2)}{N-1-\frac{1}{N-1}}\right)$$

Therefore

$$\frac{\partial(A - A_\tau)}{\partial\sigma_1^2/\sigma_2^2} = \frac{\partial A}{\partial\sigma_1^2/\sigma_2^2} - \frac{1}{N(N-1)}\frac{1}{\left(\sigma_1^2 + \left(1 - \frac{N-2}{N-1}z\right)\sigma_2^2\right)^2} < 0$$

which proves the second step. The third step is straightforward.

B.1.4 Welfare (point 5.)

I reexpress the welfare difference as:

$$\widehat{W}_{k,0}^{\tau} - \widehat{W}_{k,0}^{n} = -\gamma\sigma_2^2(q_k^c)^2\left\{\left(\frac{A_\tau}{1+A_\tau}\right)^2\left[\frac{x+1-\alpha z}{2} - \tilde{h}_\tau\left(1 - \frac{\tilde{h}_\tau}{2}\right)x + \frac{1-z}{\alpha}\tilde{h}_\tau\left(\alpha - \frac{\tilde{h}_\tau}{2}\right)\right.\right.$$
$$\left.\left. - \left(\frac{A}{1+A}\right)^2\frac{x+1-\alpha z}{2}\right]\right\}$$

$$\widehat{W}_{k,0}^{\tau} - \widehat{W}_{k,0}^{n} = -\gamma\sigma_2^2(q_k^c)^2\left(\frac{A_\tau}{1+A_\tau}\right)^2\frac{x+1-\alpha z}{2}$$
$$\times\underbrace{\left(1 - \left(\frac{A}{1+A}\frac{1+A_\tau}{A_\tau}\right)^2 - 2\frac{x+1-z}{x+1-\alpha z}\tilde{h}_\tau + \frac{x+\frac{1-z}{\alpha}}{x+1-\alpha z}\tilde{h}_\tau^2\right)}_{\Phi(\tilde{h}_\tau)}.$$

Hence $\widetilde{W}_{k,0}^{\tau} > \widetilde{W}_{k,0}^{n}$ iff $\Phi(\tilde{h}_\tau) < 0$. I consider Φ as a second degree polynomial in $\tilde{h}_\tau$, taking A and A_τ as given. Given that the coefficient in $\tilde{h}_\tau^2$ is positive, it is negative iff it has roots $h_- < h_+$ and if the equilibrium value of $\tilde{h}_\tau$ is in the interval $[h_-, h_+]$. I know check this. Φ has roots if and only if its discriminant

$$\Delta_{n\tau} = 4\frac{x+\frac{1-z}{\alpha}}{x+1-\alpha z}\left[\frac{(x+1-z)^2}{(x+1-\alpha z)(x+\frac{1-z}{\alpha})} - 1 + \left(\frac{A}{1+A}\frac{1+A_\tau}{A_\tau}\right)^2\right]$$

is positive. I show that the term in brackets in Δ_{ny}, which determines its sign, decreases with x: it is easy to check that

$$\frac{\partial}{\partial x}\frac{(x+1-z)^2}{(x+1-\alpha z)(x+\frac{1-z}{\alpha})} = \frac{(x+1-z)\left(x+\frac{1-z}{N(N-2)}\right)}{(x+1-\alpha z)^2(x+\frac{1-z}{\alpha})^2}(\alpha - 1) \leq 0$$

because $\alpha < 1$, and with equality iff $x = 0$ and $z = 0$. In addition, Lemma 6 shows that A decreases with x, thus $A/(1+A)$ also decreases; while A_τ increases with x, so $(1+A_\tau)/A_\tau$ also decreases with x. Thus the term in brackets decreases with x. It is easy to check that as $x \to \infty$, the first two terms in brackets cancel out, so that

$$\Delta_{n\tau} \geq 4\frac{x+\frac{1-z}{\alpha}}{x+1-\alpha z}\lim_{x\to\infty}\left(\frac{A}{1+A}\frac{1+A_\tau}{A_\tau}\right)^2$$

and given that $\lim_{x\to\infty} A = \frac{1}{N-2} > 0$ and

$$\lim_{x\to\infty} A_\tau = \frac{2+\frac{1-z}{N-2}}{1-\frac{z}{N-1}} > 0,$$

the limit of $\left(\frac{A}{1+A}\frac{1+A_\tau}{A_\tau}\right)^2$ is strictly positive. Thus $\Delta_{n\tau} > 0$, implying that Φ has two real roots. The roots of Φ are

$$h_\pm = \frac{x+1-z}{x+\frac{1-z}{\alpha}}\left\{1 \pm \sqrt{1-\left(\frac{x+1-\alpha z}{x+1-z}\right)^2\left(1-\left(\frac{A}{1+A}\frac{1+A_\tau}{A_\tau}\right)^2\right)}\right\}$$

Given that $h_\tau \leq 1/2$ (see the proof of Theorem 3), and $h_+ \geq 1$ (in particular $(x+1-z)/(x+(1-z)/\alpha) > 1$), one has $\tilde{h}_\tau < h_+$.

It remains to check that $\tilde{h}_\tau > h_-$. This is equivalent to showing that

$$\sqrt{1+\left(\frac{x+1-\alpha z}{x+1-z}\right)^2\left(\left(\frac{1+A_y^{-1}}{1+A^{-1}}\right)^2-1\right)}+\frac{1}{2}\frac{N-2}{N-1}\frac{x+\frac{1-z}{\alpha}}{x+1-z}\frac{\left(1-\frac{1}{N}\frac{N-3}{N-2}z\right)x+1-z}{\left(1+\frac{1-z}{N-2}\right)x+\frac{1-z}{\alpha}} > 1$$

and given that $\frac{x+1-\alpha z}{x+1-z} > 1$, replacing the ratio by 1 in the above inequality and rearranging, it holds if

$$\frac{1+A_\tau^{-1}}{1+A^{-1}}+\frac{1}{2}\frac{N-2}{N-1}\frac{x+\frac{1-z}{\alpha}}{x+1-z}\frac{\left(1-\frac{1}{N}\frac{N-3}{N-2}z\right)x+1-z}{\left(1+\frac{1-z}{N-2}\right)x+\frac{1-z}{\alpha}} > 1$$

The left-hand side (LHS) is decreasing in x: the ratio $\frac{1+A_\tau^{-1}}{1+A^{-1}}$ decreases with x as shown above, and it is easy to check that each non-constant factor in the second term of the decreases with x. Therefore the previous inequality holds for all $x > 0$

and $z \in [0,1)$ if it holds for the limit of the LHS as x becomes infinite, i.e.:

$$
\begin{aligned}
\frac{1+(N-1)\frac{1-\frac{z}{(N-1)^2}}{2+\frac{1-z}{N-2}}}{N-1} + \frac{1}{2}\frac{N-2}{N-1}\frac{1-\frac{1}{N}\frac{N-3}{N-2}z}{1+\frac{1-z}{2(N-2)}} &> 1 \\
\frac{2}{N-1} + \frac{1-\frac{z}{(N-1)^2}}{1+\frac{1-z}{2(N-2)}} + \frac{N-2}{N-1}\frac{1-\frac{1}{N}\frac{N-3}{N-2}z}{1+\frac{1-z}{2(N-2)}} &> 2 \\
\frac{\frac{N-1}{N-2}+1-\left(\frac{1}{(N-2)(N-1)}+\frac{N-3}{N(N-2)}\right)z}{1+\frac{1-z}{2(N-2)}} &> 2 \\
\frac{N-1}{N-2}+1-\frac{1}{N-2}\left(\frac{1}{N-1}+\frac{N-3}{N}\right)z &> 2+\frac{1}{N-2}-\frac{z}{N-2} \\
\left(\frac{1}{N-1}+\frac{N-3}{N}\right)z &< z
\end{aligned}
$$

The latter inequality always hold for $z > 0$, since $\frac{1}{N-1} + \frac{N-3}{N} = 1 - \frac{2N-3}{N(N-1)} < 1$.

From this I conclude that $\tilde{h}_\tau > h_-$, and the theorem is proven.

C Internet appendix: heterogenous risk aversions

C.1 Setting

The two types of traders are now denoted $k = a, b$ with respective risk aversions parameters γ_a, γ_b,. Type a traders start with inventory $I_{a,0}$, type b traders with $I_{b,0} < I_{a,0}$. Type a traders may or may not be sellers depending on risk aversions. There are N_a trader of type a, N_b traders of type b, with $N_a + N_b \geq 3$ to ensure existence of equilibria in linear strategies.

Two periods of trading $t = 0, 1$, one risky asset that matures at date $t = 2$.

The date-0 and date-1 market clearing conditions are

$$
\begin{aligned}
N_a q_{a,0} + N_b q_{b,0} &= 0 \\
N_a q_{a,1} + N_b q_{b,1} &= Q
\end{aligned}
$$

C.2 Competitive benchmark

C.2.1 Date 1

Traders submit demand schedules

$$q_{k,1}^c(p_1) = \frac{v_1 - p_1}{\gamma_k \sigma_2^2} - I_{k,1}$$

By market clearing:

$$\left(\frac{N_a}{\gamma_a} + \frac{N_b}{\gamma_b}\right) \frac{v_1 - p_1^c}{\sigma_2^2} = N_a I_{a,1} + N_b I_{b,1} + Q$$

which is equivalent to

$$\begin{aligned} v_1 - p_1^c &= \bar{\gamma}_c \sigma_2^2 Q^*, \\ \bar{\gamma}_c &= \left(\frac{N_a}{\gamma_a} + \frac{N_b}{\gamma_b}\right)^{-1} = \frac{\gamma_a/N_a \ \gamma_b/N_b}{\gamma_a/N_a + \gamma_b/N_b} \\ Q^* &= N_k I_{k,1} + N_{-k} I_{-k,1} + Q \end{aligned}$$

Therefore equilibrium trade are

$$\begin{aligned} q_{k,1}^c &= \frac{\bar{\gamma}_c}{\gamma_k} Q^* - I_{k,1} \\ &= \left(\frac{\bar{\gamma}_c}{\gamma_k} N_k - 1\right) I_{k,1} + \frac{\bar{\gamma}_c}{\gamma_k}(N_{-k} I_{-k,1} + Q) \\ &= \left(\frac{\gamma_{-k}/N_{-k}}{\gamma_k/N_k + \gamma_{-k}/N_{-k}} - 1\right) I_{k,1} + \frac{\gamma_{-k}/N_k}{\gamma_k/N_k + \gamma_{-k}/N_{-k}} I_{-k,1} + \frac{\bar{\gamma}_c}{\gamma_k} Q \end{aligned}$$

Denoting

$$s_k^c = \frac{\gamma_{-k}/N_{-k}}{\gamma_k/N_k + \gamma_{-k}/N_{-k}} \in (0,1)$$

(notice that $s_k^c + s_{-k}^c = 1$), one gets

$$q_{k,1}^c = s_k^c \frac{N_{-k}}{N_k} I_{-k,1} - (1 - s_k^c) I_{k,1} + \frac{s_k^c}{N_k} Q \tag{C.1}$$

Post trade, the equilibrium utility is

$$\begin{aligned}
\widehat{W}^c_{k,1} &= I_{k,0}v_1 + q_{k,0}(v_1 - p_0) - \frac{\gamma_k\sigma_2^2}{2}(I_{k,1})^2 \\
&\quad + q^c_{k,1}(v_1 - p^c_1) - \frac{\gamma_k\sigma_2^2}{2}((I_{k,1} + q^c_{k,1})^2 - (I_{k,1})^2) \\
&= I_{k,0}v_1 + q_{k,0}(v_1 - p_0) - \frac{\gamma_k\sigma_2^2}{2}(I_{k,1})^2 + \frac{\gamma_k\sigma_2^2}{2}(q^c_{k,1})^2
\end{aligned}$$

C.2.2 Date 0

The date-0 certainty equivalent of wealth is

$$\widehat{W}^c_{k,0} = I_{k,0}v_0 + q_{k,0}(v_0 - p_0) - \frac{\gamma_k(\sigma_1^2 + \sigma_2^2)}{2}(I_{k,1})^2 + \frac{\gamma_k\sigma_2^2}{2}z^c(\mathbb{E}_0[q^c_{k,1}])^2$$

where $z^c = \left(1 + \bar{\gamma}_c^2\sigma_2^2\sigma_Q^2\right)^{-1}$. I find the maximum of $\widehat{W}^c_{k,0}$:

$$\begin{aligned}
\frac{\partial \widehat{W}^c_{k,0}}{\partial q_{k,0}} &= v_0 - p_0 - \gamma_k(\sigma_1^2 + \sigma_2^2)I_{k,1} - \gamma_k\sigma_2^2 z^c\left(\mathbb{E}_0[Q^*] - I_{k,1}\right) \\
&= v_0 - p_0 - \gamma_k(\sigma_1^2 + \underbrace{(1 - z^c)}_{\delta^c}\sigma_2^2)I_{k,1} - \gamma_k\sigma_2^2 z^c\,\mathbb{E}_0[Q^*]
\end{aligned}$$

The problem is easily seen to be concave as $z < 1$. Thus the solution solves the FOC, which implies the following demand schedule:

$$q^c_{k,0}(p_0) = \frac{v_0 - p_0}{\gamma_k(\sigma_1^2 + \delta^c\sigma_2^2)} - I_{k,1} - \frac{z^c\sigma_2^2}{\sigma_1^2 + \delta_c\sigma_2^2}\,\mathbb{E}_0[Q^*] \tag{C.2}$$

Together with the date-0 market clearing condition, this implies:

$$\underbrace{\left(\frac{N_a}{\gamma_a} + \frac{N_b}{\gamma_b}\right)}_{1/\bar{\gamma}^c}\frac{v_0 - p^c_0}{\sigma_1^2 + \delta^c\sigma_2^2} = N_k I_{k,0} + N_{-k}I_{-k,0} + \frac{z^c\sigma_2^2}{\sigma_1^2 + \delta^c\sigma_2^2}E_0[Q^*]$$

so that

$$v_0 - p^c_0 = \bar{\gamma}_c(\sigma_1^2 + \sigma_2^2)\left(N_k I_{k,0} + N_{-k}I_{-k,0}\right) + z^c\bar{\gamma}_c\sigma_2^2\,\mathbb{E}_0[Q] \tag{C.3}$$

Plugging equilibrium risk premia into demand schedules yields equilibrium trades:

$$
\begin{aligned}
q^c_{k,0} &= \frac{\bar{\gamma}_c(\sigma_1^2+\sigma_2^2)}{\gamma_k(\sigma_1^2+\delta^c\sigma_2^2)}(N_k I_{k,0}+N_{-k}I_{-k,0}) - I_{k,0} - \frac{z^c\sigma_2^2}{\sigma_1^2+\delta^c\sigma_2^2}(N_k I_{k,0}+N_{-k}I_{-k,0}) \\
&\quad + \frac{z^c\sigma_2^2}{\sigma_1^2+\delta^c\sigma_2^2}\,\mathbb{E}_0[Q] - \frac{z^c\sigma_2^2}{\sigma_1^2+\delta^c\sigma_2^2}\,\mathbb{E}_0[Q] \\
&= \frac{\bar{\gamma}_c}{\gamma_k}(N_k I_{k,0}+N_{-k}I_{-k,0}) - I_{k,0}
\end{aligned}
$$

In particular, there is no trade associated with $E_0[Q]$. Equilibrium inventories are

$$
\begin{aligned}
I_{k,0}+q^c_{k,0} &= \frac{\bar{\gamma}_c}{\gamma_k}(N_k I_{k,0}+N_{-k}I_{-k,0}) && \text{(C.4)} \\
&= \frac{1}{N_k}\times\frac{\bar{\gamma}_c}{\gamma_k/N_k}\times I_0 && \text{(C.5)}
\end{aligned}
$$

where $I_0 = N_a I_{a,0} + N_b I_{b,0}$ is the market total inventory of the asset. The second formulation gives is intuitive: class a traders as a whole get a fraction

$$
s^c_a = \frac{\bar{\gamma}_c}{\gamma_a/N_a} = \frac{\gamma_b/N_b}{\gamma_a/N_a+\gamma_b/N_b} \in (0,1)
$$

of the market total inventory, and class b traders receive the complementary share. If $N_a = N_b$, traders with higher risk aversion get a lower share; γ_k/N_k is an equivalent collective risk aversion of traders of class k. [19] This share of total market inventory is split equally across traders within class a, so that an individual traders gets a fraction $1/N_a$ of the share that goes to class a traders as a whole.

with $\bar{\gamma}_c/\gamma_k = \frac{1}{N_k}\frac{\gamma_{-k}/N_{-k}}{\gamma_k/N_k+\gamma_{-k}/N_{-k}}$, one gets

$$
\begin{aligned}
q^c_{k,0} &= \frac{1}{N_k}\left(\frac{\gamma_{-k}/N_{-k}}{\gamma_k/N_k+\gamma_{-k}/N_{-k}}(N_k I_{k,0}+N_{-k}I_{-k,0}) - N_k I_{k,0}\right) \\
&= \frac{1}{N_k}\left(s^c_k N_{-k}I_{-k,0} - (1-s^c_k)N_k I_{k,0}\right) && \text{(C.6)}
\end{aligned}
$$

[19]Traders of class k collectively are not equivalent to a single trader of risk aversion γ_k/N_k, because they are competing with each other within a class. This proves important in the imperfectly competitive case.

C.3 Imperfect competition: backward resolution

C.3.1 Date 1

The resolution follows Malamud and Rostek (2017). In equilibrium, traders submit the demand schedule

$$q^*_{k,1}(p_1) = \frac{v_1 - p_1}{\gamma_k(1+\beta_{k,1})\sigma_2^2} - \frac{I_{k,1}}{1+\beta_{k,1}} \tag{C.7}$$

where $\beta_{k,1} > 0$ and is such that $\gamma_k\beta_{k,1}$ is monotone decreasing in γ_k *(what about $\gamma_k(1+\beta_{k,1})$?)*. By market clearing:

$$\underbrace{\left(\frac{N_a}{\gamma_a(1+\beta_{a,1})} + \frac{N_b}{\gamma_b(1+\beta_{b,1})}\right)}_{1/\bar{\gamma}_1} \frac{v_1 - p_1^*}{\sigma_2} = \frac{N_a I_{a,1}}{1+\beta_{a,1}} + \frac{N_b I_{b,1}}{1+\beta_{b,1}} + Q$$

Thus one has

$$\bar{\gamma}_1 = \frac{\gamma_a(1+\beta_{a,1})/N_a \times \gamma_b(1+\beta_{b,1})/N_b}{\gamma_b(1+\beta_{b,1})/N_b + \gamma_a(1+\beta_{a,1})/N_a}$$

When $\gamma_a = \gamma_b = \gamma$, the expression becomes $\bar{\gamma}_1 = \gamma/(N_a N_b(1/N_a + 1/N_b)) = \gamma/(N_a + N_b)$. Rearranging one gets:

$$v_1 - p_1^* = \bar{\gamma}_1\sigma_2^2 \left(\frac{N_a I_{a,1}}{1+\beta_{a,1}} + \frac{N_b I_{b,1}}{1+\beta_{b,1}} + Q\right) \tag{C.8}$$

Equilibrium trades. Plugging (C.8) into (4.3), one gets

$$q^*_{k,1} = \frac{\bar{\gamma}_1}{\gamma_k(1+\beta_{k,1})} \underbrace{\left(\frac{N_k I_{k,1}}{1+\beta_{k,1}} + \frac{N_{-k} I_{-k,1}}{1+\beta_{-k,1}} + Q\right)}_{Q^*} - \frac{I_{k,1}}{1+\beta_{k,1}} \tag{C.9}$$

Denoting

$$s_{k,1} = \frac{\bar{\gamma}_1}{\gamma_k(1+\beta_{k,1})/N_k} = \frac{\gamma_{-k}(1+\beta_{-k,1})/N_{-k}}{\gamma_k(1+\beta_{k,1})/N_k + \gamma_{-k}(1+\beta_{-k,1})/N_{-k}},$$

($s_{k,1} + s_{-k,1} = 1$) one gets

$$q^*_{k,1} = s_{k,1}\frac{N_{-k}}{N_k}\frac{I_{-k,1}}{1+\beta_{-k,1}} - (1 - s_{k,1})\frac{I_{k,1}}{1+\beta_{k,1}} + \frac{s_{k,1}}{N_k}Q \tag{C.10}$$

When $\gamma_a = \gamma_b = \gamma$, $s_{k,1} = (1 + N_{-k}/N_k)^{-1}$, so that

$$q^*_{k,1} = \frac{N-2}{N-1}\frac{N_{-k}}{N_k + N_{-k}}\left(I_{-k,1} - I_{k,1}\right) + \frac{Q}{N_{-k} + N_k}$$

Expression (C.10) is analogous to the competitive case, although the coefficients are distorted. Regarding equilibrium inventories:

$$I_{k,0} + q^*_{k,0} = \frac{s_{k,1}}{N_k}Q^* + \frac{\beta_{k,1}}{1+\beta_{k,1}}I_{k,1} \tag{C.11}$$

A trader of class k retains a fraction $\beta_{k,1}/(1+\beta_{k,1})$ of her initial inventory, and gets a distorted share $s_{k,1}/N_k$ of a reduced quantity Q^* put on the market at date 1.

The share $s_{k,1}$ of total quantity put on the market decreases for low risk aversion traders with respect to the competitive case, and increases for high risk aversion traders. This is because $\gamma_k\beta_{k,1}$ decreases with γ_k (examine $s_{k,1}$). Thus traders with lower risk aversion put less quantity on the market (effect on Q^*), and get less of this total quantity because they face a higher price impact (effect on $s_{k,1}$).

Utility post date-1 trade. Plugging equilibrium price and quantities into the certainty equivalent of wealth:

$$\widehat{W}_{k,1} = I_{k,0}v_1 + q_{k,0}(v_1 - p_0) - \frac{\gamma_k\sigma_2^2}{2}(I_{k,1})^2 + S_{k,1}$$

where

$$\begin{aligned} S_{k,1} &= q^*_{k,1}(v_1 - p^*_1) - \frac{\gamma_k\sigma_2^2}{2}\left((I_{k,1} + q^*_{k,1})^2 - (I_{k,1})^2\right) \\ &= q^*_{k,1}\left(v_1 - p^*_1\right) - \frac{\gamma_k\sigma_2^2}{2}\left(2I_{k,1}q^*_{k,1} + (q^*_{k,1})^2\right) \end{aligned}$$

and since, with (4.3), one has

$$v_1 - p^*_1 = \gamma_k(1+\beta_{k,1})\sigma_2^2 q^*_{k,1} + \gamma_k\sigma_2^2 I_{k,1},$$

one gets

$$\begin{aligned} S_{k,1} &= q^*_{k,1}\left(\gamma_k(1+\beta_{k,1})\sigma_2^2 q^*_{k,1} + \gamma_k\sigma_2^2 I_{k,1} - \frac{\gamma_k\sigma_2^2}{2}\left(2I_{k,1} + q^*_{k,1}\right)\right) \\ &= \frac{1+2\beta_{k,1}}{2}\gamma_k\sigma_2^2\left(q^*_{k,1}\right)^2 \end{aligned}$$

Therefore the certainty equivalent of wealth after date 1 trade is

$$\widehat{W}_{k,1} = I_{k,0}v_1 + q_{k,0}(v_1 - p_0) - \frac{\gamma_k\sigma_2^2}{2}(I_{k,1})^2 + \underbrace{\frac{1+2\beta_{k,1}}{2}\gamma_k\sigma_2^2\left(q_{k,1}^*\right)^2}_{S_{k,1}} \tag{C.12}$$

$$= I_{k,0}v_1 + q_{k,0}(v_1 - p_0) - \frac{\gamma_k\sigma_2^2}{2}(I_{k,1})^2 + \frac{\alpha_k\gamma_k\sigma_2^2}{2}\left(\frac{\bar{\gamma}_1}{\gamma_k}\left(\frac{N_k I_{k,1}}{1+\beta_{k,1}} + \frac{N_{-k}I_{-k,1}}{1+\beta_{-k,1}} + Q\right) - I_{k,1}\right)^2$$

where $\alpha_k = \frac{1+2\beta_{k,1}}{(1+\beta_{k,1})^2} = 1 - \left(\frac{\beta_{k,1}}{1+\beta_{k,1}}\right)^2$.

Comparison of date-1 surpluses for higher and lower risk aversions. Suppose $\gamma_a < \gamma_b$ and $N_a = N_b$. Then $\beta_{a,1} > \beta_{b,1}$, and

$$s_{a,1} = \frac{1}{1 + \frac{\gamma_a(1+\beta_{a,1})}{\gamma_b(1+\beta_{b,1})}}$$

Date 1 surpluses for class a and class b traders are:

$$S_{a,1} = \frac{(1+2\beta_{a,1})\gamma_a\sigma_2^2}{2}\left(s_{a,1}Q^* - \frac{I_{a,1}}{1+\beta_{a,1}}\right)^2$$
$$S_{b,1} = \frac{(1+2\beta_{b,1})\gamma_b\sigma_2^2}{2}\left((1-s_{a,1})Q^* - \frac{I_{b,1}}{1+\beta_{b,1}}\right)^2$$

and

$$\frac{S_{a,1} - S_{b,1}}{\sigma_2^2/2} = \gamma_a(1+2\beta_{a,1})\left(s_{a,1}^2(Q^*)^2 - 2s_{a,1}\frac{I_{a,1}}{1+\beta_{a,1}}Q^* + \left(\frac{I_{a,1}}{1+\beta_{a,1}}\right)^2\right)$$
$$- \gamma_b(1+2\beta_{b,1})\left((1-s_{a,1})^2(Q^*)^2 - 2(1-s_{a,1})\frac{I_{b,1}}{1+\beta_{b,1}}Q^* + \left(\frac{I_{b,1}}{1+\beta_{b,1}}\right)^2\right)$$

C.3.2 Date 0

Traders' wealth Taking the certainty equivalent of wealth with respect to ϵ_1 and Q, one gets:

$$\widehat{W}_{k,0} = I_{k,0}v_0 + q_{k,0}(v_0 - p_0) - \frac{\gamma_k(\sigma_1^2+\sigma_2^2)}{2}(I_{k,1})^2 + \frac{1+2\beta_{k,1}}{2}z_k\gamma_k\sigma_2^2\left(\mathbb{E}_0[q_{k,1}^*]\right)^2 \tag{C.13}$$

where $z_k = \left(1 + \alpha_k\bar{\gamma}_1^2\sigma_2^2\sigma_Q^2\right)^{-1}$.

In contrast to the competitive equilibrium, the factor z now depends on the class of traders. Traders with higher risk aversion have a lower $\beta_{k,1}$ (restrict their date-1 demand less) than traders with lower risk aversion, thus a higher α_k. This also means that traders with a higher risk aversion attach a higher value to a unit traded at date 1, but they also discount uncertainty over Q more. Overall, the value they attach to a unit traded at date 1 is proportional to $\frac{\alpha_k}{1+\alpha_k\bar{\gamma}_1^2\sigma_2^2\sigma_Q^2}$, which is increasing in α_k, so that traders with higher risk aversion value one unit traded at date 1 more.

Now I find the maximum of $\widehat{W}_{k,0}(q_{k,0})$. I differentiate $\widehat{W}_{k,1}$ with respect to $q_{k,0}$:

$$\begin{aligned}\frac{\partial\widehat{W}_{k,0}}{\partial q_{k,0}} &= v_0 - p_0 - \lambda_{k,0}q_{k,0} - \gamma_k(\sigma_1^2+\sigma_2^2)I_{k,1}\\ &\quad + \left(\frac{s_{k,1}}{N_k}-1\right)\alpha_k z_k\gamma_k\sigma_2^2\left(\frac{\bar{\gamma}_1}{\gamma_k}\left(\frac{N_kI^e_{k,1}}{1+\beta_{k,1}}+\frac{N_{-k}I^e_{-k,1}}{1+\beta_{-k,1}}+\mathbb{E}_0[Q]\right)-I_{k,1}\right)\\ &= v_0 - p_0 - \lambda_{k,0}q_{k,0} - \gamma_k(\sigma_1^2+\delta_k\sigma_2^2)I_{k,1}\\ &\quad - \left(1-\frac{s_{k,1}}{N_k}\right)\frac{s_{k,1}}{N_k}\frac{1+2\beta_{k,1}}{1+\beta_{k,1}}z_k\gamma_k\sigma_2^2\left(\frac{(N_k-1)I^e_{k,1}}{1+\beta_{k,1}}+\frac{N_{-k}I^e_{-k,1}}{1+\beta_{-k,1}}+\mathbb{E}_0[Q]\right)\end{aligned}$$

where

$$\delta_k = 1-\left(1-\frac{s_{k,1}}{N_k}\right)^2\alpha_k z_k \in (0,1)$$

From the fact that $\delta_k > 0$, and provided that $\lambda_{k,0} > 0$, one has the concavity of $\widehat{W}_{k,0}$ in $q_{k,0}$.

Equilibrium demand schedules and prices In equilibrium,[20]

$$\lambda_{k,0} = \beta_{k,0}\gamma_k(\sigma_1^2+\delta_k\sigma_2^2)$$

[20]The proposition by Malamud and Rostek (2017) applies if one sets $\hat{\gamma}_a = \gamma_a$ and $\hat{\gamma}_b = \gamma_b\frac{\sigma_1^2+\delta_b\sigma_2^2}{\sigma_1^2+\delta_a\sigma_2^2}$, and $\Sigma = \sigma_1^2+\delta_a\sigma_2^2$.

Equating the derivative to zero gives the following demand schedule:

$$
\begin{aligned}
q_{k,0}^{n}(p_0) &= \frac{v_0 - p_0}{\tilde{\gamma}_k(1+\beta_{k,0})} - \frac{\widetilde{I}_{k,0}}{1+\beta_{k,0}} \\
\widetilde{I}_{k,0} &= I_{k,0} + A_k\left(\frac{(N_k-1)I_{k,1}^{e}}{1+\beta_{k,1}} + \frac{N_{-k}I_{-k,1}^{e}}{1+\beta_{-k,1}} + \mathbb{E}_0[Q]\right) \\
A_k &= \frac{1+2\beta_{k,1}}{1+\beta_{k,1}}\left(1-\frac{s_{k,1}}{N_k}\right)\frac{s_{k,1}}{N_k}\frac{z_k\sigma_2^2}{\sigma_1^2+\delta_k\sigma_2^2}
\end{aligned}
\tag{C.14}
$$

where $\tilde{\gamma}_k = \gamma_k(\sigma_1^2 + \delta_k\sigma_2^2)$. $\widetilde{I}_{k,0}$ reflects a class k traders' initial inventory $I_{k,0}$ plus her willingness to trade in anticipation of date-1 market. Plugging (C.14) into the date-0 market clearing condition implies

$$
v_0 - p_0^n = \bar{\gamma}_0\left(\frac{N_k\widetilde{I}_{k,0}}{1+\beta_{k,0}} + \frac{N_{-k}\widetilde{I}_{-k,0}}{1+\beta_{-k,0}}\right) \tag{C.15}
$$

where

$$
\bar{\gamma}_0 = \left(\frac{N_k}{\tilde{\gamma}_k(1+\beta_{k,0})} + \frac{N_{-k}}{\tilde{\gamma}_{-k}(1+\beta_{-k,0})}\right)^{-1}
$$

Equilibrium trades Plugging (C.15) into (C.14), one gets the equilibrium quantity given other traders' strategies:

$$
\begin{aligned}
q_{k,0}^{n} &= \frac{1}{1+\beta_{k,0}}\left(\frac{\bar{\gamma}_0}{\tilde{\gamma}_k}\left(\frac{N_k\widetilde{I}_{k,0}}{1+\beta_{k,0}} + \frac{N_{-k}\widetilde{I}_{-k,0}}{1+\beta_{-k,0}}\right) - \widetilde{I}_{k,0}\right) \\
&= \frac{1}{1+\beta_{k,0}}\left(\left(\frac{\bar{\gamma}_0}{\tilde{\gamma}_k(1+\beta_{k,0})/N_k} - 1\right)\widetilde{I}_{k,0} + \frac{\bar{\gamma}_0}{\tilde{\gamma}_k}\frac{N_{-k}\widetilde{I}_{-k,0}}{1+\beta_{-k,0}}\right) \\
&= s_{k,0}\frac{N_{-k}}{N_k}\frac{\widetilde{I}_{-k,0}}{1+\beta_{-k,0}} - (1-s_{k,0})\frac{\widetilde{I}_{k,0}}{1+\beta_{k,0}},
\end{aligned}
$$

where I denote

$$
s_{k,0} = \frac{\tilde{\gamma}_{-k}(1+\beta_{-k,0})/N_{-k}}{\tilde{\gamma}_k(1+\beta_{k,0})/N_k + \tilde{\gamma}_{-k}(1+\beta_{-k,0})/N_{-k}} \in (0,1)
$$

the share of total "inventory" $\frac{N_k \widetilde{I}_{k,0}}{1+\beta_{k,0}} + \frac{N_{-k}\widetilde{I}_{-k,0}}{1+\beta_{-k,0}}$ that traders of class k collectively get. In equilibrium, $q^e_{k,0} = q^*_{k,0}$ and by market clearing $q^e_{-k,0} = q^*_{-k,0} = -q^*_{k,0}$. Thus

$$\begin{aligned}
\widetilde{I}_{k,0} &= I_{k,0} + A_k \left(\frac{(N_k-1)(I_{k,0}+q^n_{k,0})}{1+\beta_{k,1}} + \frac{N_{-k}\left(I_{-k,0} - \frac{N_k}{N_{-k}} q^n_{k,0}\right)}{1+\beta_{-k,1}} + \mathbb{E}_0[Q] \right) \\
&= I_{k,0} + A_k \left(\frac{(N_k-1)I_{k,0}}{1+\beta_{k,1}} + \frac{N_{-k}I_{-k,0}}{1+\beta_{-k,1}} + \mathbb{E}_0[Q] \right) + A_k \left(\frac{N_k-1}{1+\beta_{k,1}} - \frac{N_{-k}}{1+\beta_{-k,1}} \frac{N_k}{N_{-k}} \right) q^n_{k,0} \\
&= \left(1 + A_k \frac{N_k-1}{1+\beta_{k,1}}\right) I_{k,0} + A_k \frac{N_{-k}}{1+\beta_{-k,1}} I_{-k,0} + A_k\, \mathbb{E}_0[Q] + A_k \left(\frac{N_k-1}{1+\beta_{k,1}} - \frac{N_k}{1+\beta_{-k,1}} \right) q^n_{k,0}
\end{aligned}$$

Similarly

$$\begin{aligned}
\widetilde{I}_{-k,0} &= I_{-k,0} + A_{-k} \left(\frac{(N_{-k}-1)\left(I_{-k,0} - \frac{N_k}{N_{-k}} q^n_{k,0}\right)}{1+\beta_{k,1}} + \frac{N_k(I_{k,0}+q^n_{k,0})}{1+\beta_{-k,1}} + \mathbb{E}_0[Q] \right) \\
&= \left(1 + A_{-k} \frac{N_{-k}-1}{1+\beta_{-k,1}}\right) I_{-k,0} + A_{-k} \frac{N_k}{1+\beta_{k,1}} I_{k,0} + A_{-k}\, \mathbb{E}_0[Q] \\
&\quad + A_{-k} \left(\frac{N_k}{1+\beta_{k,1}} - \frac{N_{-k}-1}{1+\beta_{-k,1}} \frac{N_k}{N_{-k}} \right) q^*_{k,0}
\end{aligned}$$

Therefore

$$\begin{aligned}
q^n_{k,0} = &\frac{s_{k,0}}{1+\beta_{-k,0}} \frac{N_{-k}}{N_k} \left\{ \left(1 + A_{-k} \frac{N_{-k}-1}{1+\beta_{-k,1}}\right) I_{-k,0} + A_{-k} \frac{N_k}{1+\beta_{k,1}} I_{k,0} + A_{-k}\, \mathbb{E}_0[Q] \right\} \\
&- \frac{1-s_{k,0}}{1+\beta_{k,0}} \left\{ \left(1 + A_k \frac{N_k-1}{1+\beta_{k,1}}\right) I_{k,0} + A_k \frac{N_{-k}}{1+\beta_{-k,1}} I_{-k,0} + A_k\, \mathbb{E}_0[Q] \right\} \\
&+ \left\{ \frac{s_{k,0}}{1+\beta_{-k,0}} \frac{N_{-k}}{N_k} A_{-k} \left(\frac{N_k}{1+\beta_{k,1}} - \frac{N_{-k}-1}{1+\beta_{-k,1}} \frac{N_k}{N_{-k}} \right) \right. \\
&\qquad \left. - \frac{1-s_{k,0}}{1+\beta_{k,0}} A_k \left(\frac{N_k-1}{1+\beta_{k,1}} - \frac{N_k}{1+\beta_{-k,1}} \right) \right\} q^*_{k,0}
\end{aligned}$$

and finally

$$\begin{aligned}
q^n_{k,0} = \frac{1}{D_k} &\left\{ \frac{s_{k,0}}{1+\beta_{-k,0}} \frac{N_{-k}}{N_k} \left[\left(1 + A_{-k} \frac{N_{-k}-1}{1+\beta_{-k,1}}\right) I_{-k,0} + A_{-k} \frac{N_k}{1+\beta_{k,1}} I_{k,0} + A_{-k}\, \mathbb{E}_0[Q] \right] \right. \\
&\left. - \frac{1-s_{k,0}}{1+\beta_{k,0}} \left[\left(1 + A_k \frac{N_k-1}{1+\beta_{k,1}}\right) I_{k,0} + A_k \frac{N_{-k}}{1+\beta_{-k,1}} I_{-k,0} + A_k\, \mathbb{E}_0[Q] \right] \right\}
\end{aligned} \tag{C.16}$$

$$D_k = 1 + \frac{1-s_{k,0}}{1+\beta_{k,0}} A_k \left(\frac{N_k-1}{1+\beta_{k,1}} - \frac{N_k}{1+\beta_{-k,1}} \right) - \frac{s_{k,0}}{1+\beta_{-k,0}} A_{-k} \left(\frac{N_{-k}}{1+\beta_{k,1}} - \frac{N_{-k}-1}{1+\beta_{-k,1}} \right)$$

This expression is to be plugged into $v_0 - p_0^n$, $v_1 - p_1^n$ and $q_{k,1}^n$ to fully solve the equilibrium.

C.3.3 Traders' welfare

For a trader of class k, equilibrium welfare is

$$\widehat{W}_{k,0}^n = V_k + S_{k,0}^n + \widehat{S}_{k,1}^n$$

where

$$\begin{aligned}
V_k &= I_{k,0}v_0 - \frac{\gamma_k(\sigma_1^2 + \sigma_2^2)}{2}(I_{k,0})^2 \\
S_{k,0}^n &= q_{k,0}^n(v_0 - p_0^n) - \frac{\gamma_k(\sigma_1^2 + \sigma_2^2)}{2}\left((I_{k,0} + q_{k,0}^n)^2 - (I_{k,0})^2\right) \\
\widehat{S}_{k,1}^n &= z_k\gamma_k\sigma_2^2\left(\mathbb{E}_0[q_{k,1}^n]\right)^2
\end{aligned}$$

C.4 Equilibrium with futures - date 0

C.4.1 Traders' wealths

Lemma 11 (Traders' wealths). *The date-0 certainty equivalent of wealth for type-k trader is*

$$\begin{aligned}
\widehat{W}_{k,0}^f(q_{k,0}, x_k) = V_k + S_{k,0}(q_{k,0}) + \widehat{S}_{k,1}(q_{k,0}) + x_k(\widehat{p}_1 - f_0) \\
- \gamma_k(\sigma_1^2 + (1 - z_k)\sigma_2^2)I_{k,1}x_k - \gamma_k\left(\sigma_1^2 + \frac{1 - z_k}{\alpha_k}\sigma_2^2\right)(x_k)^2 \quad \text{(C.17)}
\end{aligned}$$

where

$$\begin{aligned}
\widehat{p}_1 &= v_0 - z_k\bar{\gamma}_1\sigma_2^2\,\mathbb{E}_0[Q^*] \\
Q^* &= \frac{N_k I_{k,1}^e}{1 + \beta_{1,1}} + \frac{N_{-k}I_{-k,1}^e}{1 + \beta_{k,1}} + Q
\end{aligned}$$

Proof. The date-1 utility post trade is, for a trader of class k:

$$
\begin{aligned}
\widehat{W}^f_{k,1} &= I_{k,0}v_1 + q_{k,0}(v_1 - p_0) - \frac{\gamma_k\sigma_2^2}{2}(I_{k,1})^2 + \frac{\alpha_k}{2}\gamma_k\sigma_2^2\left(\frac{\bar{\gamma}_1}{\gamma_k}Q^* - I_{k,1}\right)^2 \\
&\quad + x_k(v_0 + \epsilon_1 - \bar{\gamma}_1\sigma_2^2 Q^* - f_0) \\
&= I_{k,0}v_0 + q_{k,0}(v_0 - p_0) + x_k(v_0 - f_0) + (I_{k,1} + x_k)\epsilon_1 \\
&\quad + \frac{\alpha_k}{2}\gamma_k\sigma_2^2\left(\frac{\bar{\gamma}_1}{\gamma_k}Q^* - I_{k,1} - \frac{x_k}{\alpha_k}\right)^2 - \frac{\gamma_k\sigma_2^2}{2}\left((1-\alpha_k)(I_{k,1})^2 + \alpha_k\left(I_{k,1} + \frac{x_k}{\alpha}\right)^2\right)
\end{aligned}
$$

The date-0 certainty equivalent of wealth is thus

$$
\begin{aligned}
\widehat{W}^f_{k,0} &= I_{k,0}v_0 + q_{k,0}(v_0 - p_0) + x_k(v_0 - f_0) - \frac{\gamma_k\sigma_1^2}{2}(I_{k,1} + x_k)^2 \\
&\quad + \frac{\alpha_k z_k\gamma_k\sigma_2^2}{2}\left(\frac{\bar{\gamma}_1}{\gamma_k}\mathbb{E}_0[Q^*] - I_{k,1} - \frac{x_k}{\alpha_k}\right)^2 - \frac{\gamma_k\sigma_2^2}{2}\left((1-\alpha_k)(I_{k,1})^2 + \alpha_k\left(I_{k,1} + \frac{x_k}{\alpha_k}\right)^2\right) \\
&= I_{k,0}v_0 + q_{k,0}(v_0 - p_0) + x_k(v_0 - f_0) - \frac{\gamma_k\sigma_1^2}{2}\left((I_{k,1})^2 + 2x_kI_{k,1} + (x_k)^2\right) \\
&\quad + \frac{\alpha_k z_k\gamma_k\sigma_2^2}{2}\left[\left(\frac{\bar{\gamma}_1}{\gamma_k}\mathbb{E}_0[Q^*] - I_{k,1}\right)^2 - 2\left(\frac{\bar{\gamma}_1}{\gamma_k}\mathbb{E}_0[Q^*] - I_{k,1}\right)\frac{x_k}{\alpha_k} + \left(\frac{x_k}{\alpha_k}\right)^2\right] \\
&\quad - \frac{\gamma_k\sigma_2^2}{2}\left((I_{k,1})^2 + \alpha_k\left(2I_{k,1}\frac{x_k}{\alpha_k} + \left(\frac{x_k}{\alpha_k}\right)^2\right)\right) \\
&= I_{k,0}v_0 + q_{k,0}(v_0 - p_0) + x_k(v_0 - f_0) - \frac{\gamma_k(\sigma_1^2 + \sigma_2^2)}{2}(I_{k,1})^2 + \frac{\alpha_k z_k\gamma_k\sigma_2^2}{2}\left(\frac{\bar{\gamma}_1}{\gamma_k}\mathbb{E}_0[Q^*] - I_{k,1}\right)^2 \\
&\quad - \frac{\gamma_k\sigma_1^2}{2}(2I_{k,1}x_k + (x_k)^2) - \frac{\gamma_k\sigma_2^2}{2}\left(2I_{k,1}x_k + \frac{(x_k)^2}{\alpha_k}\right) \\
&\quad + \frac{z_k\gamma_k\sigma_2^2}{2}\left[-2\left(\frac{\bar{\gamma}_1}{\gamma_k}\mathbb{E}_0[Q^*] - I_{k,1}\right)x_k + \frac{(x_k)^2}{\alpha_k}\right]
\end{aligned}
$$

Finally, rearranging, one gets (C.17). ■

Notice that $\partial Q^*/\partial I_{k,1} = \frac{1}{1+\beta_{k,1}}$.

Concavity Differentiating (C.17), one gets

$$\frac{\partial \widehat{W}^f_{k,0}}{\partial q_{k,0}} = v_0 - p_0 - \lambda_{k,qq} q_{k,0} - \lambda_{k,qx} x_k - \gamma_k(\sigma_1^2 + \delta_k \sigma_2^2) I_{k,1}$$
$$- \gamma_k \left(\sigma_1^2 + \left(1 - \left(1 - \frac{s_{k,1}}{N_k}\right) z_k\right) \sigma_2^2\right) x_k - \left(1 - \frac{s_{k,1}}{N_k}\right) \alpha_k z_k \bar{\gamma}_1 \sigma_2^2 \, \mathbb{E}_0[\tilde{Q}_k]$$
$$\frac{\partial \widehat{W}^f_{k,0}}{\partial x_k} = \widehat{p}_1 - f_0 - \lambda_{k,xq} q_{k,0} - \lambda_{k,xx} x_k - \gamma_k(\sigma_1^2 + (1 - z_k)\sigma_2^2) I_{k,1} - \gamma_k \left(\sigma_1^2 + \frac{1 - z_k}{\alpha_k} \sigma_2^2\right) x_k$$

where $\widetilde{Q}_k = \frac{(N_k - 1) I_{k,1}}{1 + \beta_{k,1}} + \frac{N_{-k} I_{-k,1}}{1 + \beta_{-k,1}} + Q$. Developing $\widehat{p}_1$, one has

$$\frac{\partial \widehat{W}^f_{k,0}}{\partial x_k} = v_0 - f_0 - \lambda_{k,xq} q_{k,0} - \lambda_{k,xx} x_k - \gamma_k \left(\sigma_1^2 + \left(1 - \left(1 - \frac{s_{k,1}}{N_k}\right) z_k\right) \sigma_2^2\right) I_{k,1}$$
$$- \gamma_k \left(\sigma_1^2 + \frac{1 - z_k}{\alpha_k} \sigma_2^2\right) x_k - z_k \bar{\gamma}_1 \sigma_2^2 \, \mathbb{E}_0[\widetilde{Q}_k]$$

In matrix terms, the gradient of $\widehat{W}^f_{k,1}$ is

$$\nabla \widehat{W}^f_{k,0} = \begin{pmatrix} v_0 - p_0 \\ v_0 - f_0 \end{pmatrix} - (\Lambda_{k,0} + \gamma_k \Sigma_k) \begin{pmatrix} q_{k,0} \\ x_k \end{pmatrix} - \gamma_k \Sigma_k \begin{pmatrix} I_{k,0} \\ 0 \end{pmatrix} - z_k \bar{\gamma}_1 \sigma_2^2 \, \mathbb{E}_0[\widetilde{Q}_k] \begin{pmatrix} \alpha_k \left(1 - \frac{s_{k,1}}{N_k}\right) \\ 1 \end{pmatrix}$$

and

$$\Lambda_{k,0} = \begin{pmatrix} \lambda_{k,qq} & \lambda_{k,qx} \\ \lambda_{k,xq} & \lambda_{k,xx} \end{pmatrix}$$
$$\Sigma_k = \begin{pmatrix} \sigma_1^2 + \delta_k \sigma_2^2 & \sigma_1^2 + \left(1 - \left(1 - \frac{s_{k,1}}{N_k}\right) z_k\right) \sigma_2^2 \\ \sigma_1^2 + \left(1 - \left(1 - \frac{s_{k,1}}{N_k}\right) z_k\right) \sigma_2^2 & \sigma_1^2 + \frac{1 - z_k}{\alpha_k} \sigma_2^2 \end{pmatrix}.$$

Assuming that $\widehat{W}^f_{k,0}$ is concave, one has in equilibrium:

$$\Lambda_{k,0} = \beta_{k,0} \gamma_k \Sigma_{k,f}$$

with $\beta_{k,0} > 0$. (Proposition 1 in Malamud and Rostek (2017) shows that $\beta_{k,0}$ is the same as without futures.) Thus $\widehat{W}^f_{k,0}$ is concave if and only if $|\Sigma|$ is positive:

$$
\begin{aligned}
|\Sigma_{k,f}| &= \left(\sigma_1^2 + \left(1 - \left(1 - \frac{s_{k,1}}{N_k}\right)^2 \alpha_k z_k\right)\sigma_2^2\right)\left(\sigma_1^2 + \frac{1-z_k}{\alpha_k}\sigma_2^2\right) \\
&\qquad - \left(\sigma_1^2 + \left(1 - \left(1 - \frac{s_{k,1}}{N_k}\right) z_k\right)\sigma_2^2\right)^2 \\
&= \sigma_2^2 \left\{ \left[1 - \left(1 - \frac{s_{k,1}}{N_k}\right)^2 \alpha_k z_k + \frac{1-z_k}{\alpha_k} - 2\left(1 - \left(1 - \frac{s_{k,1}}{N_k}\right) z_k\right)\right]\sigma_1^2 \right. \\
&\qquad \left. + \left[\left(1 - \left(1 - \frac{s_{k,1}}{N_k}\right)^2 \alpha_k z_k\right)\frac{1-z_k}{\alpha_k} - \left(1 - \left(1 - \frac{s_{k,1}}{N_k}\right) z_k\right)^2\right]\sigma_2^2 \right\} \\
&= \sigma_2^2 \left\{ \left[\frac{1}{\alpha_k} - 1 + \left(2 - \left(1 - \frac{s_{k,1}}{N_k}\right)\alpha_k\right)\left(1 - \frac{s_{k,1}}{N_k}\right) z_k - \frac{z_k}{\alpha_k}\right]\sigma_1^2 \right. \\
&\qquad + \left[\frac{1}{\alpha_k}\left(1 - \left(1 + \left(1 - \frac{s_{k,1}}{N_k}\right)^2 \alpha_k\right) z_k + \left(1 - \frac{s_{k,1}}{N_k}\right)^2 \alpha_k z_k^2\right) \right. \\
&\qquad\qquad \left.\left. - \left(1 - 2\left(1 - \frac{s_{k,1}}{N_k}\right) z_k + \left(1 - \frac{s_{k,1}}{N_k}\right)^2 z_k^2\right)\right]\sigma_2^2 \right\} \\
&= \sigma_2^2 \left\{ \left[\frac{1}{\alpha_k} - 1 + \left(2 - \left(1 - \frac{s_{k,1}}{N_k}\right)\alpha_k\right)\left(1 - \frac{s_{k,1}}{N_k}\right) z_k - \frac{z_k}{\alpha_k}\right]\sigma_1^2 \right. \\
&\qquad \left. + \left[\frac{1}{\alpha_k} - 1 - \left(\frac{1}{\alpha_k} - 1 + \left(\frac{s_{k,1}}{N_k}\right)^2\right) z_k\right]\sigma_2^2 \right\}
\end{aligned}
$$

Finally:[21]

$$
\begin{aligned}
|\Sigma_{k,f}| = \left(\frac{1}{\alpha_k} - 1\right)\sigma_2^2 &\left\{ \left[1 - \frac{\left(1 - \alpha_k + \alpha_k \frac{s_{k,1}}{N_k}\right)^2}{1-\alpha_k} z_k\right]\sigma_1^2 \right. \\
&\qquad \left. + \left[1 - \left(1 + \frac{\alpha_k}{1-\alpha_k}\left(\frac{s_{k,1}}{N_k}\right)^2\right) z_k\right]\sigma_2^2 \right\}. \qquad \text{(C.18)}
\end{aligned}
$$

[21] One can check that when $\gamma_a = \gamma_b$ and setting $N = N_a + N_b$ and with $\beta_{k,0} = \beta_{k,1} = (N-2)^{-1}$ for $k = a, b$, one gets back to

$$|\Sigma_{k,f}| = \left(\frac{1}{\alpha} - 1\right)\sigma_2^2 \left\{(1-z)\sigma_1^2 + \left(1 - 2\frac{N-1}{N}\right)\sigma_2^2\right\}$$

Then $|\Sigma| > 0$ if and only if

$$\left[1 - \frac{\left(1-\alpha_k+\alpha_k\frac{s_{k,1}}{N_k}\right)^2}{1-\alpha_k} z_k\right]\frac{\sigma_1^2}{\sigma_2^2} + 1 - \left(1 + \frac{\alpha_k}{1-\alpha_k}\left(\frac{s_{k,1}}{N_k}\right)^2\right) z_k > 0$$

$$z_k < \frac{(1-\alpha_k)\left(\frac{\sigma_1^2}{\sigma_2^2}+1\right)}{\left(1-\alpha_k+\alpha_k\frac{s_{k,1}}{N_k}\right)^2\frac{\sigma_1^2}{\sigma_2^2} + 1 - \alpha_k + \alpha_k\left(\frac{s_{k,1}}{N_k}\right)^2} \equiv \bar{z}_k\left(\frac{\sigma_1^2}{\sigma_2^2}\right)$$

For $\sigma_1 = 0$, the $\bar{z}_k$ equals $\frac{1-\alpha_k}{1-\alpha_k+\alpha_k(s_{k,1}/N_k)^2} < 1$. For $\sigma_1^2/\sigma_2^2 \to \infty$, the ratio equals $\frac{1-\alpha_k}{1-\alpha_k+\alpha_k s_{k,1}/N_k} \times \frac{1}{1-\alpha_k+\alpha_k s_{k,1}/N_k}$.

Clearly, for z_k not too large at least, $|\Sigma_{k,f}| > 0$. Investigate the case for which z_k closer to 1 later (study variations of $\beta_{a,1}, \beta_{b1}$ as functions of (γ_a, γ_b).)

C.4.2 Equilibrium demand schedules, quantities and prices

Assuming $|\Sigma_{k,f}| > 0$, the equilibrium demand schedules are given by the FOC:

$$\begin{pmatrix} q_{k,0}^*(p_0, f_0) \\ x_k^*(p_0, f_0) \end{pmatrix} = \frac{1}{\gamma_k(1+\beta_{k,0})}\Sigma_k^{-1}\begin{pmatrix} v_0 - p_0 \\ v_0 - f_0 \end{pmatrix} - \frac{1}{1+\beta_{k,0}} J_k \tag{C.19}$$

where

$$J_k = \begin{pmatrix} I_{k,0} \\ 0 \end{pmatrix} + \frac{\bar{\gamma}_1}{\gamma_k} z_k\, \mathbb{E}_0[\widetilde{Q}_k]\sigma_2^2\Sigma_k^{-1}\begin{pmatrix} \alpha_k\left(1-\frac{s_{k,1}}{N_k}\right) \\ 1 \end{pmatrix} \tag{C.20}$$

With market clearing, this implies

$$\begin{pmatrix} v_0 - p_0^* \\ v_0 - f_0^* \end{pmatrix} = \overline{\Gamma\Sigma}_0\left(\frac{N_k}{1+\beta_{k,0}} J_k + \frac{N_{-k}}{1+\beta_{-k,0}} J_{-k}\right) \tag{C.21}$$

where

$$\overline{\Gamma\Sigma}_0 = \left(\frac{N_k}{\gamma_k(1+\beta_{k,0})}\Sigma_k^{-1} + \frac{N_{-k}}{\gamma_{-k}(1+\beta_{-k,0})}\Sigma_{-k}^{-1}\right)^{-1}$$

And for $\gamma_1 = \gamma_2 = \gamma$, one has $\Sigma_a = \Sigma_b = \Sigma$ and

$$\overline{\Gamma\Sigma}_0 = \left(N\frac{N-2}{N-1}\right)^{-1}\gamma\Sigma = \frac{N-1}{N-2}\frac{\gamma}{N}\Sigma$$

Equilibrium trades Plugging (C.21) into (C.19), one gets

$$\begin{pmatrix} q^*_{k,0} \\ x^*_k \end{pmatrix} = \frac{N_{-k}/N_k}{1+\beta_{-k,0}}\Delta_{k,0}J_{-k} - \frac{1}{1+\beta_{k,0}}\left(Id_2 - \Delta_{k,0}\right)J_k \tag{C.22}$$

where, denoting Id_2 the 2-dimensional identity matrix,

$$\begin{aligned}\Delta_{k,0} &= \frac{N_k}{\gamma_k(1+\beta_{k,0})}\Sigma_k^{-1}\overline{\Gamma\Sigma}_0 \\ &= \left(Id_2 + \frac{N_{-k}}{N_k}\frac{\gamma_k(1+\beta_{k,0})}{\gamma_{-k}(1+\beta_{-k,0})}\Sigma_{-k}^{-1}\Sigma_k\right)^{-1}\end{aligned}$$

J_k and J_{-k} depend on $q^*_{k,0}$, as in equilibrium, $I_{k,0}+q^e_{k,0} = I_{k,0}+q^*_{k,0}$ and $I^e_{-k,1} = I_{-k,0} - q^*_{k,0}$. Thus I compute J_k and J_{-k} more explicitly to solve for $q^*_{k,0}$.

Provided that $|\Sigma_k| > 0$, one has

$$\Sigma_k^{-1} = \frac{1}{|\Sigma_k|}\begin{pmatrix} \sigma_1^2 + \frac{1-z_k}{\alpha_k}\sigma_2^2 & -\left(\sigma_1^2 + \left(1-\left(1-\frac{s_{k,1}}{N_k}\right)z_k\right)\sigma_2^2\right) \\ -\left(\sigma_1^2 + \left(1-\left(1-\frac{s_{k,1}}{N_k}\right)z_k\right)\sigma_2^2\right) & \sigma_1^2 + \delta_k\sigma_2^2 \end{pmatrix}$$

so that

$$\Sigma_k^{-1}\begin{pmatrix} \alpha_k\left(1-\frac{s_{k,1}}{N_k}\right) \\ 1 \end{pmatrix} = \frac{1}{|\Sigma_k|}\begin{pmatrix} -\left(1-\alpha_k\left(1-\frac{s_{k,1}}{N_k}\right)\right)\sigma_1^2 - \frac{s_{k,1}}{N_k}\sigma_2^2 \\ \left(1-\alpha_k\left(1-\frac{s_{k,1}}{N_k}\right)\right)(\sigma_1^2+\sigma_2^2) \end{pmatrix}$$

Plugging this into (C.20), one gets

$$\begin{aligned} J_k &= \begin{pmatrix} I_{k,0} - A_k^f\left(\frac{(N_k-1)(I_{k,0}+q^*_{k,0})}{1+\beta_{k,0}} + \frac{N_{-k}\left(I_{-k,0}-\frac{N_k}{N_{-k}}q^*_{k,0}\right)}{1+\beta_{-k,0}} + \mathbb{E}_0[Q]\right) \\ B_k^f\left(\frac{(N_k-1)(I_{k,0}+q^*_{k,0})}{1+\beta_{k,0}} + \frac{N_{-k}\left(I_{-k,0}-\frac{N_k}{N_{-k}}q^*_{k,0}\right)}{1+\beta_{-k,0}} + \mathbb{E}_0[Q]\right) \end{pmatrix} \\ &= \underbrace{\begin{pmatrix} I_{k,0} - A_k^f\left(\frac{(N_k-1)I_{k,0}}{1+\beta_{k,0}} + \frac{N_{-k}I_{-k,0}}{1+\beta_{-k,0}} + \mathbb{E}_0[Q]\right) \\ B_k^f\left(\frac{(N_k-1)I_{k,0}}{1+\beta_{k,0}} + \frac{N_{-k}I_{-k,0}}{1+\beta_{-k,0}} + \mathbb{E}_0[Q]\right) \end{pmatrix}}_{J_k^0} + \left(\frac{N_k-1}{1+\beta_{k,0}} - \frac{N_k}{1+\beta_{-k,0}}\right)q^*_{k,0}\begin{pmatrix} -A_k^f \\ B_k^f \end{pmatrix}\end{aligned}$$

where

$$
\begin{aligned}
A_k^f &= \frac{\bar{\gamma}_1}{\gamma_k} z_k \frac{\sigma_2^2}{|\Sigma_k|} \left[\left(1 - \alpha_k \left(1 - \frac{s_{k,1}}{N_k}\right)\right) \sigma_1^2 + \frac{s_{k,1}}{N_k} \sigma_2^2 \right] \\
B_k^f &= \frac{\bar{\gamma}_1}{\gamma_k} z_k \frac{\sigma_2^2}{|\Sigma_k|} \left(1 - \alpha_k \left(1 - \frac{s_{k,1}}{N_k}\right)\right) (\sigma_1^2 + \sigma_2^2)
\end{aligned}
$$

and

$$
J_{-k} = J_{-k}^0 + \left(\frac{N_k}{1+\beta_{k,0}} - \frac{N_{-k}-1}{1+\beta_{-k,0}} \frac{N_k}{N_{-k}} \right) q_{k,0}^* \begin{pmatrix} -A_{-k}^f \\ B_{-k}^f \end{pmatrix}
$$

Therefore, (C.22) becomes:

$$
\begin{aligned}
\begin{pmatrix} q_{k,0}^* \\ x_k^* \end{pmatrix} &= \frac{N_{-k}/N_k}{1+\beta_{-k,0}} \Delta_{k,0} J_{-k}^0 - \frac{1}{1+\beta_{k,0}} \left(Id_2 - \Delta_{k,0}\right) J_k^0 \\
&\quad + q_{k,0}^* \Bigg\{ \left(\frac{N_k}{1+\beta_{k,0}} - \frac{N_{-k}-1}{1+\beta_{-k,0}} \frac{N_k}{N_{-k}} \right) \frac{N_{-k}/N_k}{1+\beta_{-k,0}} \Delta_{k,0} \begin{pmatrix} -A_{-k}^f \\ B_{-k}^f \end{pmatrix} \\
&\qquad - \left(\frac{N_k - 1}{1+\beta_{k,0}} - \frac{N_k}{1+\beta_{-k,0}} \right) \frac{1}{1+\beta_{k,0}} (Id_2 - \Delta_{k,0}) \begin{pmatrix} -A_k^f \\ B_k^f \end{pmatrix} \Bigg\} \\
&= \frac{N_{-k}/N_k}{1+\beta_{-k,0}} \Delta_{k,0} J_{-k}^0 - \frac{1}{1+\beta_{k,0}} \left(Id_2 - \Delta_{k,0}\right) J_k^0 \\
&\quad + q_{k,0}^* \Bigg\{ \left(\frac{N_{-k}}{1+\beta_{k,0}} - \frac{N_{-k}-1}{1+\beta_{-k,0}} \right) \frac{1}{1+\beta_{-k,0}} \Delta_{k,0} \begin{pmatrix} -A_{-k}^f \\ B_{-k}^f \end{pmatrix} \\
&\qquad - \left(\frac{N_k - 1}{1+\beta_{k,0}} - \frac{N_k}{1+\beta_{-k,0}} \right) \frac{1}{1+\beta_{k,0}} (Id_2 - \Delta_{k,0}) \begin{pmatrix} -A_k^f \\ B_k^f \end{pmatrix} \Bigg\}
\end{aligned}
$$

I first solve for $q_{k,0}^*$ first (equation in the first line of the vector equation above), denoting $\Delta_{k,0} = (\Delta_{k,0}^{ij})_{i,j=q,x}$:

$$
\begin{aligned}
q_{k,0}^* = \frac{1}{D_k^f} \Bigg\{ & \frac{N_{-k}/N_k}{1+\beta_{-k,0}} \left(\Delta^{qq} J_{-k,q}^0 + \Delta^{qx} J_{-k,x}^0 \right) \\
& - \frac{1}{1+\beta_{k,0}} \left((1 - \Delta^{qq}) J_{k,q}^0 - \Delta^{qx} J_{k,x}^0 \right) \Bigg\}
\end{aligned}
$$

where

$$D_k^f = 1 - \left(\frac{N_{-k}}{1+\beta_{k,0}} - \frac{N_{-k}-1}{1+\beta_{-k,0}}\right)\frac{1}{1+\beta_{-k,0}}\left(-\Delta^{qq}A_{-k}^f + \Delta^{qx}B_{-k}^f\right)$$
$$+\left(\frac{N_k-1}{1+\beta_{k,0}} - \frac{N_k}{1+\beta_{-k,0}}\right)\frac{1}{1+\beta_{k,0}}\left(-(1-\Delta^{qq})A_k^f - \Delta^{qx}B_k^f\right)$$

Then plugging $q_{k,0}^*$ into the expression for x_k^* gives

$$x_k^* = \frac{N_{-k}/N_k}{1+\beta_{-k,0}}\left(\Delta^{xq}J_{-k,q}^0 + \Delta^{xx}J_{-k,x}^0\right) - \frac{N_k/N_{-k}}{1+\beta_{k,0}}\left((1-\Delta^{xq})J_{k,q}^0 - \Delta^{xx}J_{k,x}^0\right)$$

Equilibrium prices From (C.21):

$$\begin{pmatrix} v_0 - p_0^* \\ v_0 - f_0^* \end{pmatrix} = \overline{\Gamma\Sigma}_0\left(\frac{N_k}{1+\beta_{k,0}}J_k + \frac{N_{-k}}{1+\beta_{-k,0}}J_{-k}\right)$$

with

$$J_k = \begin{pmatrix} I_{k,0} \\ 0 \end{pmatrix} + \widetilde{Q}_k^0\begin{pmatrix} -A_k^f \\ B_k^f \end{pmatrix} + \left(\frac{N_k-1}{1+\beta_{k,0}} - \frac{N_k}{1+\beta_{-k,0}}\right)q_{k,0}^*\begin{pmatrix} -A_k^f \\ B_k^f \end{pmatrix}$$
$$J_{-k} = \begin{pmatrix} I_{-k,0} \\ 0 \end{pmatrix} + \widetilde{Q}_{-k}^0\begin{pmatrix} -A_{-k}^f \\ B_{-k}^f \end{pmatrix} - \left(\frac{N_{-k}-1}{1+\beta_{-k,0}} - \frac{N_{-k}}{1+\beta_{k,0}}\right)\frac{N_k}{N_{-k}}q_{k,0}^*\begin{pmatrix} -A_k^f \\ B_k^f \end{pmatrix}$$

with

$$\widetilde{Q}_k^0 = \left(\frac{(N_k-1)I_{k,0}}{1+\beta_{k,0}} + \frac{N_{-k}I_{-k,0}}{1+\beta_{-k,0}} + \mathbb{E}_0[Q]\right)$$

C.4.3 Welfare

I simply plug equilibrium quantities into (C.17).

C.5 Numerical implementation

I test the case $N_a = N_b = 2$, for γ_a and γ_b ranging from 1 to 10, $\sigma_1^2 = 0$, $\sigma_2^2 = 1$, $\sigma_Q^2 = 3$. I do not test for lower values of γ_a, γ_b because it leads to non-concave wealth in the case with futures.

I also assume that a-traders have higher initial inventory: $I_{a,0} = 2$ and $I_{b,1} = 1$. Finally, I look at the case where $\mathbb{E}_0[Q] = 0$ and $Q = 0$.

C.5.1 Results

Checks for $\gamma_a = \gamma_b$ Here I present results assuming $N_a = N_b = 2$, $I_{a,0} = 10$ and $I_{b,0} = 0$. Later on I show welfare comparisons for Figure 1 presents level sets of equilibrium trades $q^n_{a,0}$, $q^n_{a,1}$ and of equilibrium risk premia $v_0 - p^n_0$ and $v_0 - p^n_1$, when there are no futures, as a function of γ_a and γ_b. The black diagonal line corresponds to $\gamma_a = \gamma_b$.

Figure 2 does the same, together with equilibrium trade and risk premium for futures, when there are futures.

Figure 3 shows comparison of equilibria, together with comparison of equilibrium utilities for both classes of traders.

I check that for $\gamma_a = \gamma_b \equiv \gamma$:

- $\beta_{a,1} = \beta_{b,1} = 1/(N-2)$ and $s_{a,1} = s_{b,1} = 1/2$ (not reported)
- without futures,
  - $q^n_{a,0}, q^n_{a,1} < 0$.
  - the total quantity traded by a-traders is lower than the competitive quantity:
  $$|q^n_{a,0} + q^n_{a,1}| < \frac{N_b}{N_a + N_b}(I_{b,0} - I_{a,0})$$
  - As γ increases, $|q^n_{a,0}|$ increases
- with futures,
  - $q^*_{a,0} < 0$ and $q^*_{a,1} \leq 0$; $x^*_a > 0$.
  - as γ increases, $|q^*_{a,0}|$ decreases and $|q^*_{a,1}|$ increases[22]
  - $|x^*_a|$ decreases as γ increases.
  - the date-0 spot risk premium $v_0 - p^*_0$ equals the spot risk premium without futures $v_0 - p^n_0$
  - date-1 spot risk premia are also equal with and without futures.

Comments.

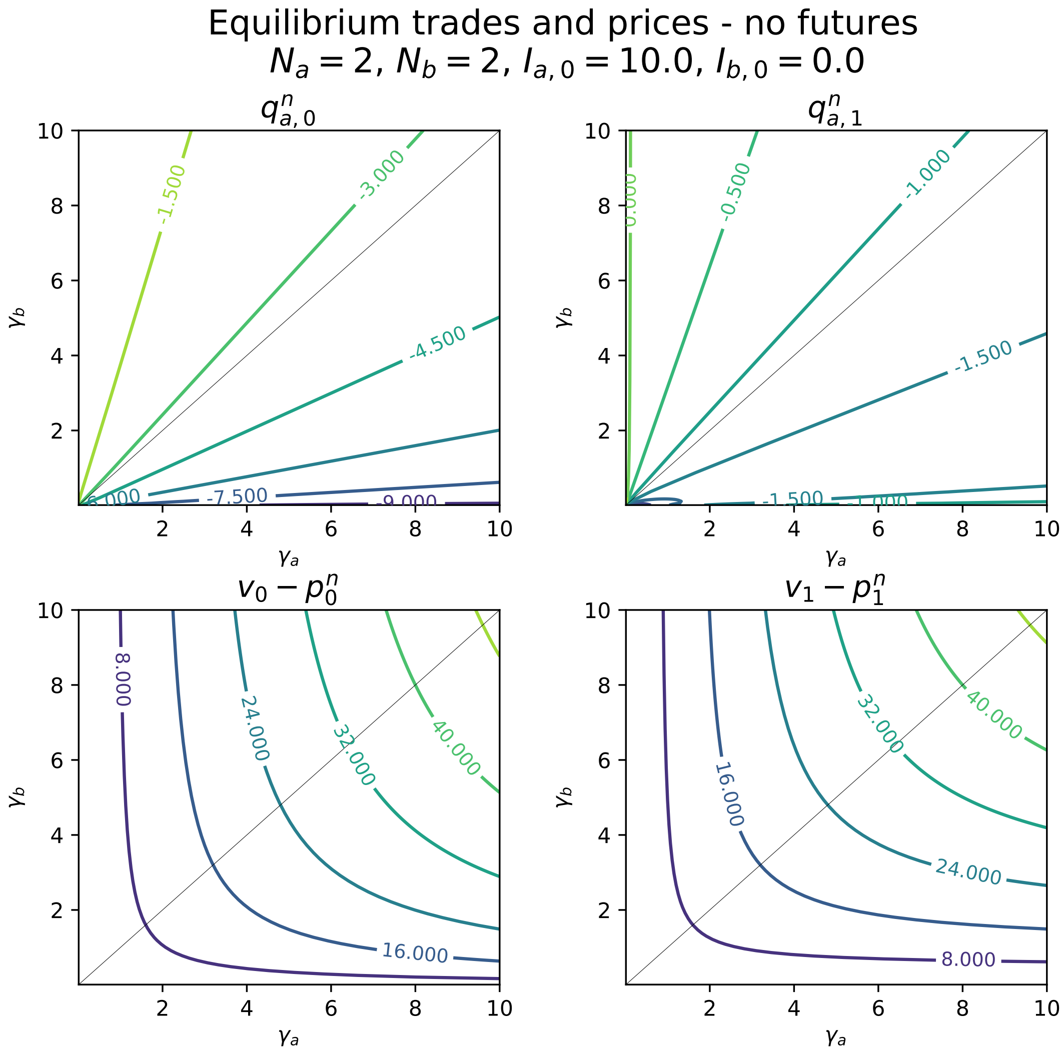
Futures trading. As indicated by the upper middle panel of figure 2, futures are traded by inventory-rich a-traders when b-traders have lower risk aversion.

[22] A_f decreases with z, thus increases with γ. As $q^*_{a,0}$ decreases with A_f, it also decreases with γ.

Hedging ratios. As shown by in upper middle panel of figure 3, a negative hedging ratio is quite general, as suggested in : only for high γ_a and very low γ_b, or low γ_a/high γ_b does the hedging ratio turn positive.

Welfare. The upper and lower right panels of figure 3 show that the welfare decrease result is not a knife-edge case: for each both classes of traders and all values of γ_a plotted here, there an interval of $\left(\underline{\gamma}_b(\gamma_a), \overline{\gamma}_b(\gamma_a)\right)$, containing $\gamma_b = \gamma_a$, for which adding futures decreases a-traders' welfare. When b-traders are less risk-averse (region below the diagonal), the welfare loss is lower, and quickly turns to a welfare gain for trader a as γ_b decreases for a fixed γ_a.

Figure 4 shows the areas in which futures hurt traders of class a and/or trader of class b for different numbers of traders ($N_a = 20, N_b = 1$ and $N_a = 20, N_b = 1$), keeping aggregate inventory $N_a I_{a,0}$ at 20. The green areas show values for which futures hurt at least one trader, the grey areas show values for which there is no equilibrium. This figure show that for these values of N_a and N_b, the welfare result is even more robust than for $N_a = N_b$.



Equilibrium trades and prices - no futures
$N_a = 2$, $N_b = 2$, $I_{a,0} = 10.0$, $I_{b,0} = 0.0$
$q^n_{a,0}$
$q^n_{a,1}$
$v_0 - p^n_0$
$v_1 - p^n_1$
γ_a
γ_b
-1.500
-3.000
-4.500
-7.500
-9.000
-0.500
-1.000
24.000
32.000
40.000
8.000
16.000

Figure 1

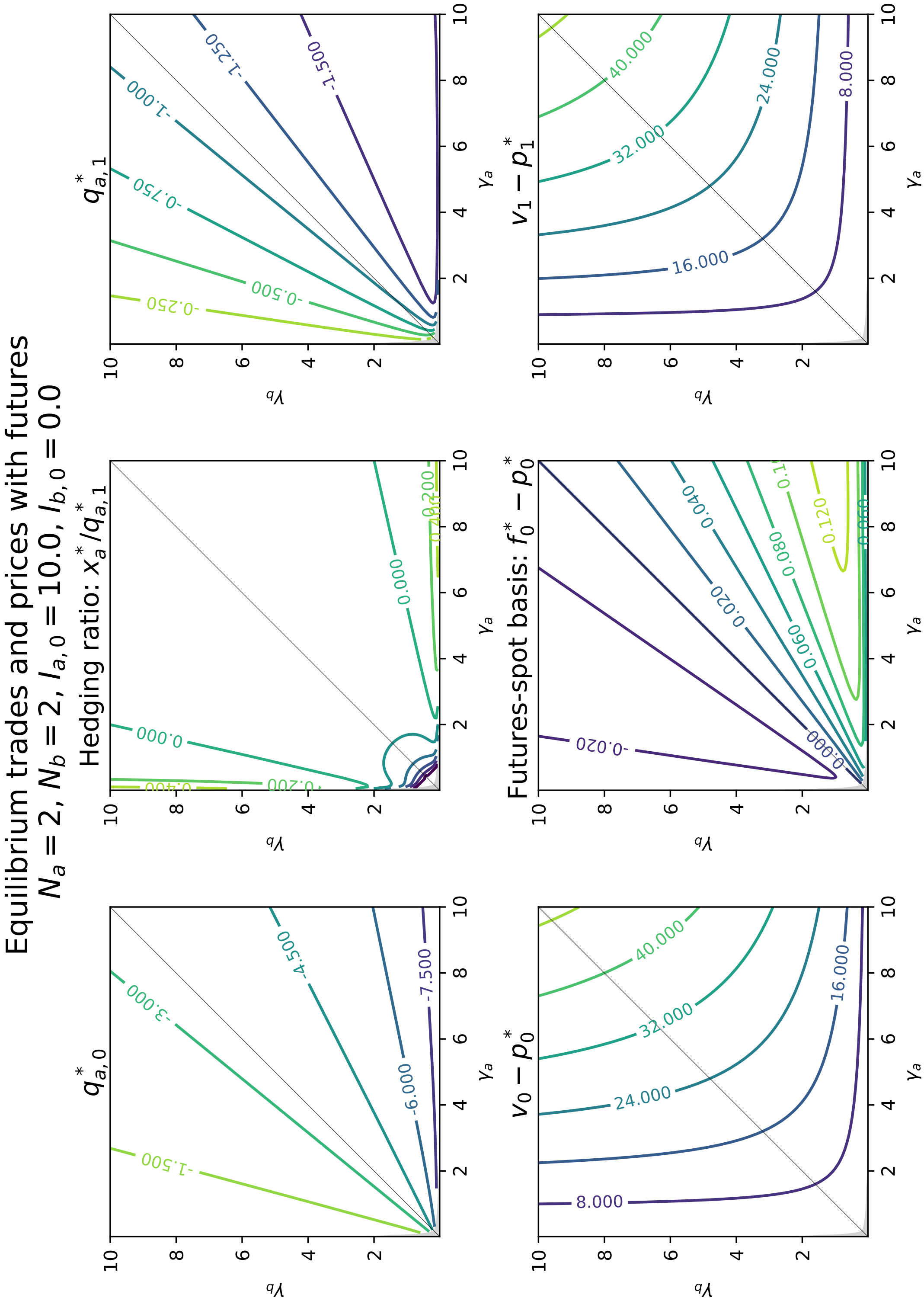


Figure 2 – Equilibrium trades and prices with futures, as a function of (γ_a, γ_b). Values of (γ_a, γ_b) for which one certainty equivalent of wealth is not concave are shaded in light grey.

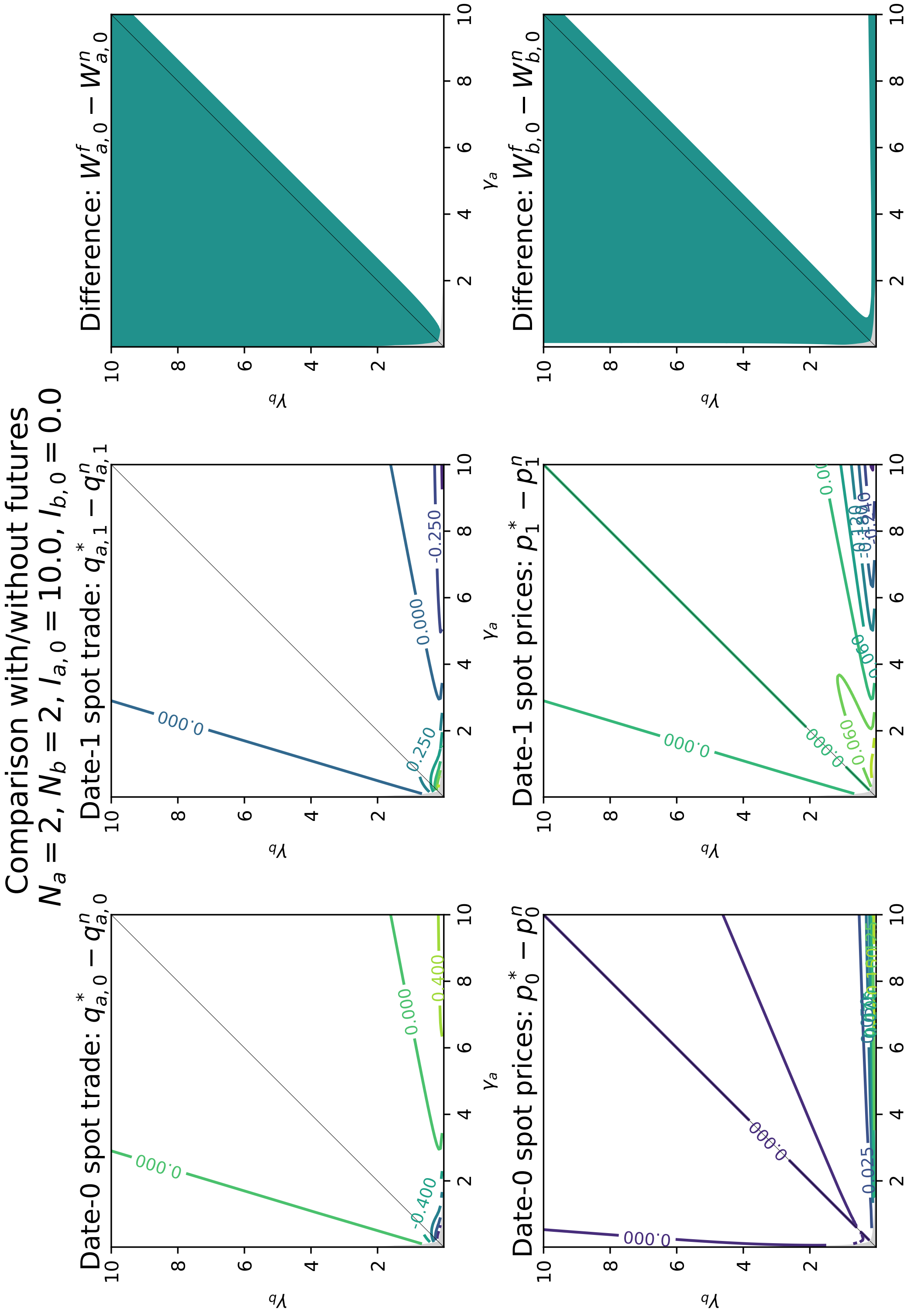


Figure 3 – Traders' equilibrium utilities with and without futures. In upper and lower right panels, values for which $\widehat{W}^f_{k,0} < \widehat{W}^n_{k,0}$ are shaded in green. Values of (γ_a, γ_b) for which one certainty equivalent of wealth is not concave are shaded in light grey.

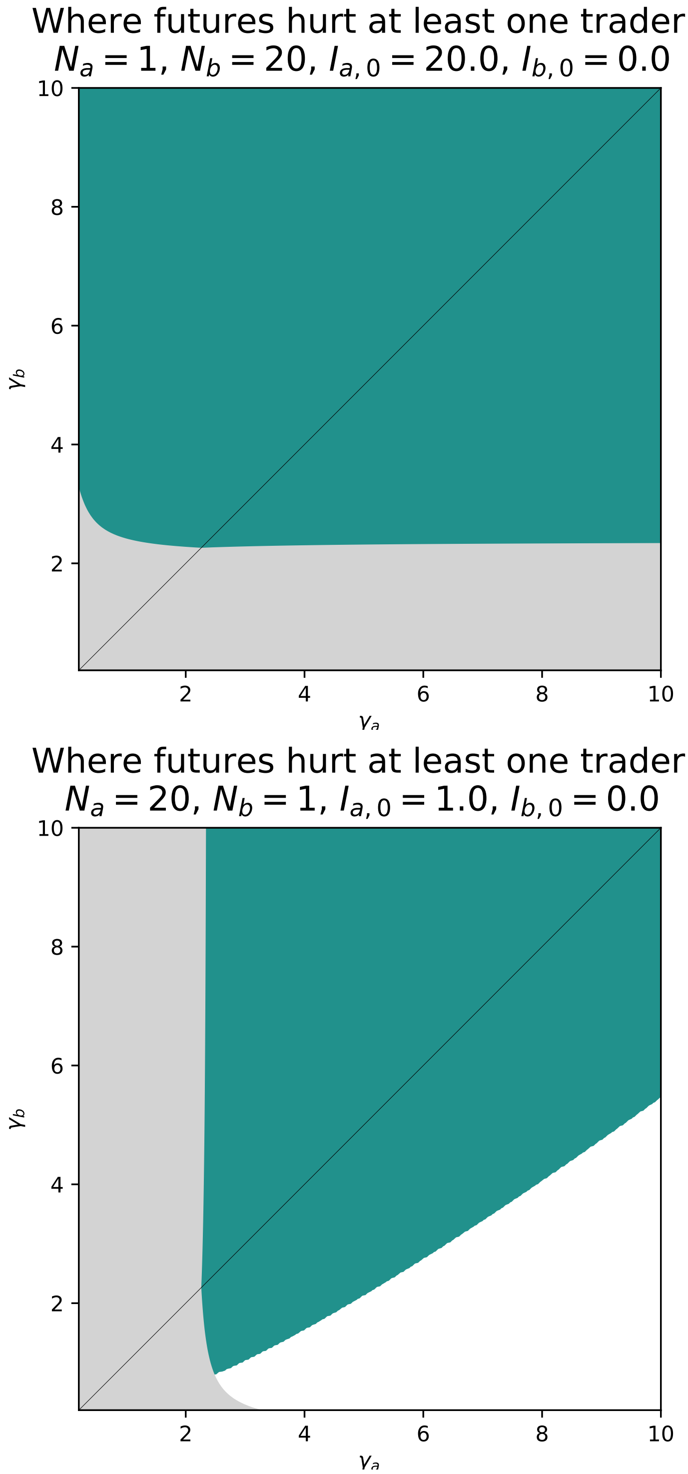


Figure 4 – Values of (γ_a, γ_b) for which $\widehat{W}^f_{a,0} < \widehat{W}^n_{a,0}$ or $\widehat{W}^f_{b,0} < \widehat{W}^n_{b,0}$ are shaded in green. Values of (γ_a, γ_b) for which one certainty equivalent of wealth is not concave are shaded in light grey.