Optimal Fee Pricing*

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Abstract

We show that the trading-fee breakdown (fee pricing) depends on the distribution of investor gains-from-trade relative to the tick size. Absent price discreteness, an increase in investor gains-from-trade increases the total fee proportionally, but the fee breakdown has no effect. With price discreteness, the fee breakdown can mitigate the loss of welfare due to difficulty in trading when gains-from-trade are small relative to the tick size. However, when gains-from-trade are large, the exchange fee breakdown plays only a small role and exchanges extract rents from investors gains-from-trade by increasing total fee. The resulting gap between welfare relative to fees set by a Social Planner can be large. Consequently, a regulator can improve welfare substantially by imposing a cap on exchange fees.

JEL classification: G10, G20, G24, D40

Keywords: Market fee pricing, make-take, limit order markets, liquidity, market microstructure

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Optimal trading fees for limit orders (make fee) and market orders (take fee) is at the top of the agenda of financial regulators around the world. This is generally called *fee pricing*. Regulators are vigilant about fee pricing for several reasons: First, following the introduction of Regulation National Market System (Reg NMS, 2007) in the US (and related regulation in Europe), fee pricing has become a strategic tool for trading platforms and exchanges to attract trading volume especially for liquid stocks (Cardella, Hao and Kalcheva, 2015 and O'Hara, 2015). In particular, negative fees, called rebates, incentivize investors to submit certain types of orders, while investors using other order types are charged positive fees. For example, Maker-Taker pricing pays investors rebates when their limit orders (making liquidity) are executed and charges fees on market orders (taking liquidity), while under Taker-Maker (also called inverted) pricing the fees and rebates are reversed. Second, even though competition among exchanges is gradually reducing the total fees charged by exchanges, the economic magnitude of fee pricing revenue for exchanges is material. For example, for the London Stock Exchange group, it totals £407m, which represents 19 percent of the Total Group income in 2019. Third, rebate-based pricing has been criticized by some practitioners as well as by Angel, Harris, and Spatt (2013), Harris (2015), and Spatt (2019) on agency, price transparency and regulatory grounds.

There are a number of question about fee pricing. What is the optimal way to set trading fees? What is the optimal breakdown between make fee and take fee? What are the determinants of the fee breakdown and of the resulting total fee? Does the fee breakdown depend on the investors' gains-from-trade and on the tick size, or, does the optimal fee pricing depend on the relative tick size (tick-to-gains-from-trade ratio)? In real market the owners of trading platforms set trading fees, whereas regulators generally set the tick size and can only impose a cap on fees.[1] How does this regulatory environment affect the optimal fee pricing?

The existing literature (Colliard and Foucault (2012)) shows that, absent a discrete tick size, there is no role for the fee breakdown as investors can neutralize any trading fees by choosing a different limit price.

[1] See Section 1 for institutional details on trading fees.

Only the total net fee matters. In contrast, when prices are instead discrete, Foucault, Kadan, and Kandel (2013), and Chao, Yao, and Ye (2018) show that the fee breakdown can mitigate the loss of welfare due to price discreteness. In particular, if investors' private valuations — which proxy for gains-from-trade — are in between prices on a discrete price grid, transactions cannot take place and hence there is a loss of welfare without rebates. However, if fee pricing has a positive role on social welfare, why do regulators generally impose a cap on fees? Is the fee breakdown set by owners of trading platforms optimal in terms of social welfare? Is the fee breakdown relevant for any stock/market characterized by different gains-from-trade?

We obtain three sets of new results compared to the existing literature. One set of results is on the optimal fee pricing in relation to the gains-from-trade that characterize the market. Another set of results motivates why regulators impose fee pricing restrictions on exchanges that are tied to the tick size. A third set of results is on the importance of the endogenous choice between different possible limit orders and between limit and market orders when setting the optimal fee breakdown.

Our paper extends the existing literature (Foucault et al. (2013) and Chao et al. (2018)) by introducing a new important feature that needs to be considered when setting the optimal fee pricing, which is the heterogeneity of ex ante gains-from-trade across markets and across stocks. As is standard in the literature (Chao et al. (2018)) we proxy the ex ante heterogeneity of gains-from-trade among market participants with the support of the distribution of the investors' private values and show that the optimal fee pricing depends on the tick size relative to the gains from trading. We therefore derive new policy implications on how to manage the heterogeneity of gains-from-trade across markets and across securities.

We show that when the gains-from-trade are small, not only fee pricing with a rebate set by an exchange is important as it resolves the loss of welfare due to the existence of frictions determined by price discreteness, but a social planner can Pareto improve on the exchange fee pricing by setting a fee breakdown that minimizes the total fees. When instead the gains-from-trade increase relative to the tick size, the fee breakdown plays no longer an important role in alleviating the frictions from price discreteness (as the relative tick size becomes irrelevant), whereas the exchange has an incentive to increase total fee and set positive fees to extract a rent from the investors' increasing gains-from-trade. Hence, the regulatory action that a social planner can take is to set a cap on fees that minimizes the exchange rent extraction and at the same time maximizes both investors' welfare and total welfare.

By considering different valuation supports (investor populations with different gains-from-trade) we show that it is the tick size relative to the valuation support (as opposed to the absolute value of the tick size) that matters for fee pricing. When the relative tick size is large (as the support of the investors' personal evaluation is small), the fee breakdown matters, whereas when the relative tick size is small, the fee breakdown plays a shrinking role as the discrete-price friction becomes less material. We show that the absolute value of the tick size is only a normalization: All else equal a small tick market is isomorphic to a corresponding larger tick market with the same relative valuation support and price grid.

An increase in investor gains-from-trade has two effects on exchange fee pricing: it tends to increase total fee as the exchange increases its rent extraction, and it changes the fee breakdown. To isolate the two effects in Section 2 we start from a continuous price model where there is no tick size and therefore there is no role for fee pricing to alleviate the welfare loss induced by the discreteness of the price grid. This model shows that total fee increases as a constant proportion of the gains-from-trade due to the exchange rent extraction. We then introduce a two-period discrete price model and show the effects of an increase in investor gains-from-trade both on total fee and on fee breakdown. With discrete prices, we show the fee breakdown matters for large relative tick sizes but as investor gains-from-trade increase and the relative tick size decreases, the owner of the trading platform has only an incentive to increase the total fee. Therefore, fee breakdown becomes irrelevant and regulators have to step in with a cap on fees to prevent exchanges to extract excessive rents from the investor gains-from-trade to the detriment of social welfare.

The two-period model raises the issue of having a monopolistic liquidity provider at the first period of the trading game which affects trading volume and therefore the exchange's objective function. In particular, with a book that opens empty, the first trader is necessarily a liquidity supplier while the second trader is necessarily a liquidity taker at the price posted by the first trader. In Section 3, we extend the existing literature by showing how the addition of a third round of trading affects fee pricing and its relation to the valuation support.[2] With a three-period model we introduce endogenous choice between limit and market orders which affects the liquidity provision of the first trader who anticipates that the second trader can eventually undercut his order instead of taking the liquidity he offers at his chosen price. We show that with endogenous choice between limit and market orders the breakdown offered by the owner of the trading platform is no longer symmetric as the exchange tries to maximize trading volume by inducing the second trader to either take the order posted by the first trader or post liquidity on the other side of the market in order to maximise the chances the third traders hits an existing limit order. We also find that the exchange exploits increased investor arrival by increasing its fees. Our results have welfare and policy implications that we discuss in the conclusions (Section 5).

1 Background information and prior research

Reg NMS established the regulatory foundation for the current architecture of US equity markets. This regulation includes an explicit limit on the cost of accessing (i.e., posting and trading on) quotes displayed by U.S. equity trading platforms. Rule 610 caps trading fees to no more than \$0.003 per share for stocks priced over \$1, and to no more that 0.3% of the quoted price for stocks priced below \$1. In addition, the Sub-Penny Rule 612 of Reg NMS prohibits exchanges, market makers, and electronic platforms from

[2] In order to illustrate the essential economics we use a model with a small number of trading period. The model could, of course, be extended to allow for additional trading periods but with additional mathematical complexity.

displaying, ranking or accepting quotes on NMS securities in sub-penny increments unless a stock is priced less than $1 per share. Thus, under Reg NMS, trading fees cannot exceed one third of the tick size.[3] The recent SEC (2022) proposal aims to significantly reduce the trading fee cap for U.S. stocks.

In Europe, MiFID II (Directive 2014/65/EU) and MiFIR (Regulation 600/2014/EU) mandates a reduction in the tick size for European stocks and thereby implicitly reduces the maximum trading fees given that the standard practice on European exchanges is to cap fees relative to the tick size.[4] MiFID II also sharpened the regulation of trading fees by requiring new incentives on market making agreements under Stress Market Conditions (RTS 8), a maximum Order-To-Trade ratio for each instrument (RTS 9), and a periodic disclosure by exchanges of the percentage of fees and rebates on total turnover (RTS 27). It also bans "cliff-edge" pricing structures in which customer-specific fees are reduced retroactively for market participants who reach a trading volume threshold (RTS 10).

Fee pricing is investigated in several theoretical papers. The starting point for work on price discreteness frictions is Colliard and Foucault (2012), which shows the breakdown between make and take fees has no effects on the cum-fee-spread (net of fees spread) in a competitive market with continuous prices. The reason is that traders can neutralize changes in fees by making offsetting changes in the pricing aggressiveness of limit orders. Subsequent research has identified two channels through which a price-discreteness friction affects fee pricing and trading: Market monitoring and limit order price choice. Foucault et al. (2013) show how price discreteness and fee pricing affect investor monitoring incentives and the order-arrival process in a coordination game matching buyers and sellers. However, investor gains-from trade are non-random and

[3]According to the S.E.C. (2018) Release No.34-82873 on Transaction Fee Pilot for NMS Stocks "For maker-taker exchanges, the amount of the taker fee is bounded by the cap imposed by Rule 610(c) on the fees the exchange can charge to access its best bid/offer for NMS stocks. This cap applies to the fees assessed on an incoming order that executes against a resting order or quote, but does not directly limit rebates paid. The Rule 610(c) cap on fees also typically indirectly limits the amount of the rebates that an exchange offers to less than $0.003 per share in order to maintain net positive transaction revenues. For taker-maker exchanges, the amount of the maker fee charged to the provider of liquidity is not bounded by the Rule 610(c) cap, but such fees typically are no more than $0.003, and the taker of liquidity earns a rebate." If the price of a protected quotation is less than $1.00, the trading fee is no more than 0.3% of the quotation price per share SEC (2009).

[4]See Article 49 of MiFID II and the following Regulatory Technical Standard 11 (RTS 11, ESMA 2017). ESMA (2015)

known, and there is no decision about posted limit prices (which are exogenously fixed in their analysis) or choice between limit and market orders. Foucault et al. (2013) show that, in single market with a discrete tick size, the make-take breakdown affects market quality. In contrast, we study a trading game in which potential buyers and sellers and an exchange decide how to split random investor gains-from-trade. In particular, investors endogenously choose the limit prices at which limit orders are posted so as to maximize their expected share of the gains-from-trade, and the exchange's fee pricing affects both the probability of transactions and the exchange's profit per trade. Panayides, Rindi, and Werner (2017) show how a change in trading fees affects market quality when two trading platforms compete for the provision of liquidity.

Our analysis is closely related to Chao et al. (2018), which models optimal fee pricing both in a single monopolistic market and also with competition between multiple markets. In terms of modeling structure, we extend their model by expanding the scope of endogenous order choice (i.e., limit price choice and choice of market or limit orders) and allowing for multiple rounds of trading. These changes lead to two sets of new insights: First, optimal fee pricing is strictly rebate-based in Chao et al. (2018), but we show that optimal fee pricing depends on the investor population in the market. As a result, optimal fee pricing is rebate-based only when investor gains-from-trade are small relative to the price tick size. When instead the ex ante gains-from-trade are large, strictly positive fee pricing by exchanges is possible. Second, we show how a regulator optimally interact with the exchange to optimize social welfare.

Angel et al. (2013) and Spatt (2019) take a different approach from the price-friction literature. They emphasize that trading fees and rebates have potential effects via the transparency of economic prices (price + fee pricing) vs quoted prices, the efficacy of regulatory protections based on quoted prices, agency issues when brokers do not pass through fees and rebates to their clients, and impeding intermarket competition. Harris (2015) points out further that negative fees allow for intra-tick trading, thus by-passing the Reg NMS trade-through rule. Li, Ye, and Zheng (2020) show how fees affect order routing decision in fragmented markets and create demand for complex order types. The different theoretical considerations are not mutually exclusive. Moreover, a complete understanding of fee pricing is likely to involve interactions between these various effects and price frictions.

A sizable empirical literature investigates different aspects of fee pricing.[5] Malinova and Park (2015) find evidence following changes in fee and rebates on the Toronto Stock Exchange (TSX) that appears to support the Colliard and Foucault (2012) irrelevance prediction provided that the TSX price tick-size is interpreted as being economically small. However, using Rule 605 data, O'Donoghue (2015) finds that changes in the split of trading fees between liquidity suppliers and demanders affect order choice and execution quality as predicted by Foucault et al. (2013). Battalio, Corwin, Jennings (2016) find that fees and rebates appear to affect broker order-routing decisions. Panayides et al. (2020) find that quoted and cum-fee spreads are affected by change in total fees on the BATS European platforms, CXE and BXE. Menkveld (2013) shows that rebate-based pricing is related to HFTs. Cardella, Hao, and Kalcheva (2015) document that Reg NMS was followed by the adoption of rebate–based fee pricing by most trading platforms in U.S. markets and by a sharp increase in HFT firm trading. O'Hara (2015) also links HFT trading activity and the increased use of rebate–based fee pricing around the world.

2 Two-Period Model

We begin our analysis with a parsimonious model with two dates, t_1 and t_2, on which investors arrive sequentially and potentially submit orders. With only two periods, the model dynamics can be solved analytically in closed-form. We use this model to develop basic insights about fee pricing and different price grids (continuous and discrete), different amounts of ex ante trading demand (ranging from low to high), and different

[5] In addition to the research discussed here, see also Bourke, DeSantis, and Porter (2019), Baldacci, Possamaï, and Rosenbaum (2019), Brauneis, Mestel, Riordan, and Theissen (2019), Skjeltorp, Sojli, and Tham (2012), He, Jarnecic, and Liu (2015), Clapham, Gomber, Lausen, and Panz (2017), Anand, Hua, and McCormick (2016), Comerton-Forde, Grégoire, and Zhong (2019), Lin, Swan, and Harris (2019) and Brolley and Malinova (2013).

fee decision-making (profit-maximizing exchange, Social Planner, and regulatory-constrained exchange). These models all include endogenous limit order choice in terms of the posted limit price. Section 3 extends this basic model to more than two periods to describe fee pricing where investors endogenously choose between limit and market orders in addition to their choice of the limit price.

At each period $t_z \in \{t_1, t_2\}$ a risk-neutral trader arrives characterized by a private valuation equal to β_{t_z} which is an i.i.d drawn from a uniform distribution, $U[\underline{\beta}, \overline{\beta}]$, where $\underline{\beta}$ and $\overline{\beta}$ are the limits of the trader valuation supports. The mean of the valuation support v is constant over time and denotes the ex ante *asset value*. Traders with more extreme β_{t_z} realizations have stronger demands to trade, whereas traders with β_{t_z} realizations close to v are more willing to supply liquidity. Therefore, the *support width* $\Delta = \overline{\beta} - \underline{\beta}$ measures the ex ante gains-from-trade and, thus, the associated ex ante demand for trade. The wider the support, $[\underline{\beta}, \overline{\beta}]$, the higher is the probability that arriving traders will have strong heterogeneous directional demands to trade, such as, e.g., long–term asset managers. The smaller the support $[\underline{\beta}, \overline{\beta}]$, the higher is the probability that arriving traders will prefer to profit as passive liquidity providers.

Investors trade using limit orders (which supply depth to the book) and market orders (which take depth from the book by hitting standing limit orders). The state of the limit order book at time t_z is a vector $L_{t_z} = [D^P_{t_z}]$, where $D^P_{t_z}$ indicates the total limit order depth at price P at time t_z. The initial limit order book L_{t_0} is assumed to be empty, and then we model how the book evolves over time. Let x_{t_z} denote a generic action (i.e, order submission) taken by an investor at a date t_z. Let X^L denote the set of possible limit buy and sell orders at all available limit prices, and let X_{t_z} denote the set of all available limit orders and possible market orders given the standing book $L_{t_{z-1}}$.

Trading and limit order book dynamics take a simple form in a two-period market: An investor arriving at time t_1 chooses between submitting a limit buy order LBP or limit sell order LSP at one of the available price levels P on a (discrete or continuous) price grid or a no trade NT. Market orders are not possible at t_1

given the empty initial book. Next, given the standing book L_{t_1}, the investor arriving at t_2 chooses between submitting a market buy order MBP_k that is immediately executed at the best offer (if there is a limit sell in the book L_{t_1}) or a market sell order MSP_k that is immediately executed at the best bid (if there is a limit buy in the book) or, instead, does not trade NT.[6] In the two-period model, limit orders are not used at t_2 since, after the final round of investor arrival, a limit orders posted at t_2 would not be executed.

The trading platform may set different trading fees $\xi(x)$ for different order types x. (We also consider fees set by a Social Planner.) An investor offering liquidity via a limit order pays a *make fee* MF. An investor taking liquidity via a market order (or via a marketable limit order) pays a *take fee* TF. Fee pricing is denoted as the set $\Xi = \{\xi(x)\}_{\forall x} = \{MF, TF\}$. Rebates are negative fees, which are a cost for the trading platform and a reward for the investor receiving them. Under a *Maker-Taker* structure, investors submitting market orders pay a take fee ($TF > 0$) to the trading platform, and investors posting limit orders receive a make rebate ($MF < 0$) whenever their limit order executes. In a *Taker-Maker* structure, the fees and rebates are reversed so that limit-order submitters pay make fees ($MF > 0$), and market-order submitters receive take rebates ($TF < 0$).

Quoted prices and the trading fees and rebates determine the net prices paid and received by investors when trading, which we call *cum-fee prices*. Let $P^{cum,MS} = P - TF$ denote the cum-fee price received from a market order to sell at the quoted price P (net of take fees paid to the exchange) , and let $P^{cum,MB} = P + TF$ be the cum-fee price paid on a market order to buy at P (net of take fees paid to the exchange). Similarly, $P^{cum,LS} = P - MF$ is the cum-fee price for a limit order to sell and $P^{cum,LB} = P + MF$ is the cum-fee price for a limit order to buy.

Our model determines optimal fees for an exchange or, alternatively, for a Social Planner in a Stackelberg game. The exchange/Social Planner is a Stackelberg leader and investors are Stackelberg followers. We

[6] Marketable limit orders that cross with the best available bid/ask on the opposite side of the standing book $L_{t_{z-1}}$ are treated as market orders in terms of both order execution and exchange fee pricing.

solve the model in two steps: Taking market fee pricing Ξ as given, we first solve for optimal investor trading strategies — i.e., the optimal responses of the Stackelberg followers — in the trading subgame by backward induction. Given this characterization of optimal investor trading, we then solve for the optimal fee pricing Ξ given an exchange's profit-maximization problem or a Social Planner's total welfare-maximization problem.

Given a standing book $L_{t_{z-1}}$ and fee pricing Ξ, the expected payoff on an order x_{t_z} for an investors arriving at time $t_z$with a private valuation β_{t_z} is:

$$\pi_{t_z}(x_{t_z}|\beta_{t_z},\Xi,L_{t_{z-1}}) = \begin{cases} [\beta_{t_z} - P(x_{t_z}) - \xi(x_{t_z})]\,Pr(\theta_{t_z}^{x_{t_z}}|\Xi,L_{t_{z-1}}) & \text{if } x_{t_z} \text{ is a buy order} \\ [P(x_{t_z}) - \beta_{t_z} - \xi(x_{t_z})]\,Pr(\theta_{t_z}^{x_{t_z}}|\Xi,L_{t_{z-1}}) & \text{if } x_{t_z} \text{ is a sell order} \\ 0 & \text{if } x_{t_z} \text{ is } NT \end{cases} \tag{1}$$

where $P(x_{t_z})$ is the posted price at which order x_{t_z} trades if it is executed and $\xi(x_{t_z}) = TF$ for market orders and MF for limit orders. $\theta_{t_z}^{x_{t_z}}$ denotes the (endogenous) set of future trading states in which an order x_{t_z} submitted at time t_z is executed, and $Pr(\theta_{t_z}^{x_{t_z}}|\Xi,L_{t_{z-1}})$ is the associated probability of execution. If x_{t_z} is a market order, then $P(x_{t_z})$ is the best standing quote on the other side of the market at time t_z, and $Pr(\theta_{t_z}^{x_{t_z}}|\Xi,L_{t_{z-1}}) = 1$, since market orders are executed immediately at the standing bid or ask (if that side of the book is non-empty). If x_{t_z} is a non-marketable limit order, then the execution price $P(x_{t_z})$ is its limit price, and the execution probability $Pr(\theta_{t_z}^{x_{t_z}}|\Xi,L_{t_{z-1}})$ is the probability (between 0 and 1) of later investors choosing to hit standing limit orders via market orders. Limit order execution probabilities depend parametrically on the valuation support S. Liquidity supply decisions at t_z and the order-execution probabilities $Pr(\theta_{t_z}^{x_{t_z}}|\Xi,L_{t_{z-1}})$ in (1) are both endogenous.

A novel feature of our analysis, relative to other fee pricing models, is the endogenous choice of posted limit prices. An investor arriving at time t_z chooses his order x_{t_z} — and, in particular, limit prices — to maximize his expected payoff from (1):

$$\max_{x_{t_z} \in X_{t_z}} \pi_{t_z}(x_{t_z} | \beta_{t_z}, \Xi, L_{t_{z-1}}) \tag{2}$$

given his private value realization β_{t_z}. A key intuition is that the investor's optimal order choice depends on a trade-off between order-execution probabilities and price improvement: More aggressive limit order prices reduce the payoff conditional on execution, but can increase the probability of execution. This means we need to consider the market order submission decision given different hypothetical limit orders at multiple hypothetical possible posted limit prices. The optimization problem in (2) is tractable because the investor expected payoff from (1) for different orders x_{t_z} are linear in the realized investor valuation β_{t_z}. We can also identify the set of a priori prices at which investors might potentially post limit orders. From Lemma 3 in Appendix A, a necessary condition for a limit order at a posted price P to be used by investors is that it has a positive execution probability in that the corresponding market-order cum-fee price is $P^{cum,MS} \leq \overline{\beta}$ for limit buys and, by symmetry, $P^{cum,MB} \geq \underline{\beta}$ for limit sells.

Order submissions in the last round of trading (at t_2 in the two-period market) take a simple form: An investor submits a market sell MSP at t_2 to hit a limit buy at a posted price P if his payoff given the cum-fee price is positive, i.e., $P^{cum,MS} - \beta_{t_2} > 0$ and symmetrically for limit sells and otherwise does not trade:[7]

$$x_{t_2} = \begin{cases} MSP & \text{if there is a limit buy at } P \text{ and } \beta_{t_1} < P^{cum,MS} \\ MBP & \text{if there is a limit sell at } P \text{ and } \beta_{t_1} > P^{cum,MB} \\ NT & \text{otherwise} \end{cases} \tag{3}$$

It follows then that the execution probability $Pr(\theta_{t_1}^{x^{LBP}} | \Xi, L_{t_0})$ of a limit buy LBP posted at P at t_1 is the

[7]We extended our previous notation so that, for example, $x_{t_2}^{MSP}$ and MSP_{t_2} are used interchangeably for a market sell order at P at t_2. When possible, we simplify the notation to make it consistent with the notation used in the figures.

order-submission probability of a market sell, MSP, at t_2 given the cum-fee price $P^{cum,MS} = P - TF$:[8]

$$
\begin{aligned}
Pr(\theta_{t_1}^{x^{LBP}}|\Xi, L_{t_0}) &= Pr(x_{t_2}^{MSP}|\Xi, L_{t_1}) = Pr(\beta_{t_2}|\beta_{t_2} < P - TF) \\
&= \max\left\{0, \min\left\{\frac{P - TF - \underline{\beta}}{\Delta}, 1\right\}\right\} = \max\left\{0, \min\left\{\frac{(P - v) + 0.5\Delta - TF}{\Delta}, 1\right\}\right\}
\end{aligned} \tag{4}
$$

where the last equality is because the investor valuation β_{t_2} is drawn from $U[\underline{\beta}, \bar{\beta}]$ with support width Δ.[9] In particular, this probability is well-defined (i.e., ≤ 1) for all priori possible limit prices from Lemma 3. By symmetry, the execution probability $Pr(\theta_{t_1}^{x^{LSP}}|\Xi, L_{t_0})$ of a limit sell, LSP posted at t_1 is the order-submission probability of a market buy MBP at t_2 given the cum-fee market-buy price $P^{cum,MB} = P + TF$:

$$
\begin{aligned}
Pr(\theta_{t_1}^{x^{LSP}}|\Xi, L_{t_0}) &= Pr(x_{t_2}^{MBP}|\Xi, L_{t_1}) = Pr(\beta_{t_2}|\beta_{t_2} > P + TF) \\
&= \max\left\{0, \min\left\{\frac{\bar{\beta} - P - TF}{\Delta}, 1\right\}\right\} = \max\left\{0, \min\left\{\frac{(v - P) + 0.5\Delta - TF}{\Delta}, 1\right\}\right\}
\end{aligned} \tag{5}
$$

Next, consider the initial time t_1 in the two-period market. The limit order book opens empty, and so the investor arriving at t_1 chooses between submitting a limit order and submitting no order (NT). Lemma 4 in Appendix A shows that an investor with $\beta_{t_1} > v$ is a potential buyer who only submits a limit buy or a NT. This investor optimally posts a limit buy LBP_k at a price P_k if two conditions hold: First, the expected payoff $\pi_{t_z}(x_{t_1}^{LBP_k}|\beta_{t_z}, \Xi, L_{t_0})$ from an order LBP given a private valuation β_{t_1} is positive so that it dominates NT:

$$
(\beta_{t_1} - P^{cum.LB}) \times Pr(\theta_{t_1}^{x^{LBP}}|\Xi, L_{t_0}) > 0 \tag{6}
$$

[8] The "max" and "min" ensure the probability is between 0 and 1.

[9] The book opens empty at t_1 and therefore the only possible limit buy a seller at t_2 can hit is a limit buy posted at t_1

and, second, it is greater than the expected payoff from all other alternative limit buys LBP_{alt}:

$$(\beta_{t_1} - P^{cum,LB}) \times Pr(\theta_{t_1}^{x^{LBP}} | \Xi, L_{t_0}) > (\beta_{t_1} - P_{alt}^{cum,LB}) \times Pr(\theta_{t_1}^{x^{LBP_{alt}}} | \Xi, L_{t_0}) \tag{7}$$

at an alternative price P_{alt} where where $P^{cum,LB} = P + MF$ and $P_{alt}^{cum,LB} = P_{alt} + MF$ are the associated cum-fee limit-buy prices. Hence, the order-submission probability of LBP at t_1 is the probability that conditions (6) and (7) are both satisfied:

$$Pr(x_{t_1}^{LBP} | \Xi, L_{t_0}) = Pr(\beta_{t_1} \text{ s.t. both (6) and (7) hold}) \tag{8}$$

A potential seller at t_1 with $\beta_{t_1} < v$ submits a limit sell LSP at time t_1 if symmetric conditions hold:

$$(P^{cum,LS} - \beta_{t_1}) \times Pr(\theta_{t_1}^{x^{LSP}} | \Xi, L_{t_0}) \quad > \quad 0 \tag{9}$$

$$(P^{cum,LS} - \beta_{t_1}) \times Pr(\theta_{t_1}^{x^{LSP}} | \Xi, L_{t_0}) \quad > \quad (P_{alt}^{cum,LS} - \beta_{t_1}) \times Pr(\theta_{t_1}^{x^{LSP_{alt}}} | \Xi, L_{t_0}) \tag{10}$$

where $P^{cum,LS} = P + MF$ and $P_{alt}^{cum,LS} = P_{alt} + MF$ are the cum-fee limit-sell prices. The associated sell limit-order submission probabilities are analogous to (8) using (9) and (10)

Fees Ξ in our model are set either by an exchange or, alternatively, by a Social Planner. In doing so, both take into account optimal investor trading behavior given the fee pricing Ξ. In other words, fee pricing by the exchange or Social Planner is subject to an incentive compatibility constraint on the orders investors choose to submit given the fees that are set.

An exchange chooses its fees Ξ to maximize its expected payoff from completed transactions:[10]

$$\max_{\substack{MF,TF \\ \{MF,TF\}\in R}} W^{Ex}(MF,TF) = \left[\sum_{x_{t_1}\in X^L} Pr(x_{t_z}|\Xi,L_{t_{z-1}})\,Pr(\theta_{t_z}^{x_{t_z}}|\Xi,L_{t_{z-1}})\right](MF+TF) \tag{11}$$

where the product of the order-submission probabilities $Pr(x_{t_z}|\Xi,L_{t_{z-1}})$ and the order-execution probabilities $Pr(\theta_{t_z}^{x_{t_z}}|\Xi,L_{t_{z-1}})$ gives the transaction probabilities induced by the exchange fees Ξ and the optimal investor order–submission strategies from (2) for different limit orders $x_{t_z}\in X^L$ at time t_1. In other words, the formula in (11) reflects the fact that limit orders are submitted first and then executed later. The exchange has non-negative profits since $TF = MF = 0$ is feasible and gives zero profits. A profit-maximizing exchange faces a trade-off. Trading fees and rebates affect both the probability $Pr(x_{t_z}|\Xi,L_{t_{z-1}})\,Pr(\theta_{t_z}^{x_{t_z}}|\Xi,L_{t_{z-1}})$ of transactions — which reflects the net impact of access pricing on both order-submission probabilities $Pr(x_{t_z}|\Xi,L_{t_{z-1}})$ and order-execution probabilities $Pr(\theta_{t_z}^{x_{t_z}}|\Xi,L_{t_{z-1}})$ for different orders x_{t_1} — the net fee $MF+TF$ the exchange receives per transaction.

The expression $\{MF,TF\}\in R$ in (11) allows for possible regulatory restriction on fee pricing. One possibility is that fee pricing by the exchange is unrestricted. However, another possibility is that, consistent with current practice and with Foucault et al. (2013), a regulator may impose restrictions on fee pricing by the exchange. For notational simplicity, we assume the maximum allowable fee (whether take or make) is one tick (i.e., rather than a fraction of a tick as in, e.g., Reg NMS). Thus, the regulatory constraint on fees is more binding for smaller tick sizes. There are no direct regulatory constraints on rebates in our model, but Lemma 5 in Appendix A shows that if fees are capped at one tick, then in equilibrium exchange rebates are never larger than one tick. The welfare impact of a regulatory cap on fees is considered in Section 4.

A Social Planner chooses fees to maximize the total welfare of all market participants:[11]

[10] The expression here is for a discrete price grid. A similar formulation holds with continuous prices.
[11] Again, this expression is for discrete prices.

$$\max_{\substack{MF,TF \\ MF+TF\geq 0}} \left(\sum_{t_z \in \{t_1,t_2\}} W^{INV}_{t_z}(MF,TF) \right) + W^{Ex}(MF,TF) \tag{12}$$

$$= \sum_{x_{t_1} \in X^L} Pr(x_{t_1}|\Xi, L_{t_0}) \times Pr(\theta^{x_{t_1}}_{t_1}|\Xi, L_{t_0}) \times I(x_{t_1}) \times \left[E[\beta_{t_1}|x_{t_1}] - E[\beta_{t_2}|\theta^{x_{t_1}}_{t_1}] \right]$$

where $W^{INV}_{t_z}(MF,TF) = E[\pi_{t_z}(x_{t_z}|\beta_{t_z}, \Xi, L_{t_{z-1}})]$ is the ex ante expected profit of the investor at time t_z given randomness in their private value β_{t_z} and where the indicator function $I(x_t) = 1$ for limit buy orders and -1 for limit sells. The expression on the right in (12) is derived in Section B of the Appendix given that the Social Planner optimally sets the exchange's profit to zero. The intuition for (12) is that the total welfare associated with each possible order x_{t_1} at time t_1 is the probability that order x_{t_1} is submitted times the probability it is executed at time t_2 times the conditional expected gains-from-trade between the two investors at times t_1 and t_2.

Given the optimization problems for investors and the exchange or Social Planner, an equilibrium is:

Definition. A *Subgame Perfect Nash Equilibrium* of the trading game is a collection $\{x_{t_z}(\beta_{t_z}|\Xi, L_{t_{z-1}}),\ \Xi^*\}$ of order-submission strategies and trading fees such that conditions 1, 2 and 3 or conditions 1, 2 and 4 hold:

1. The equilibrium order-submission strategies $x_{t_z}(\beta_{t_z}|\Xi, L_{t_{z-1}})$ solve investors' optimization problems (2) given the equilibrium execution probabilities $Pr(\theta^{x_{t_z}}_{t_z}|\Xi^*, L_{t_{z-1}})$.

2. The order-execution probabilities $Pr(\theta^{x_{t_z}}_{t_z}|\Xi^*, L_{t_{z-1}})$ for an order x_{t_z} submitted at time t_z are consistent with the subsequent equilibrium order-submission strategies $x_{t_{z'}}(\beta_{t_{z'}}|\Xi, L_{t_{z'-1}})$ at times $t_{z'} > t_z$.

3. The trading fees Ξ^* are optimal for the exchange given its optimization problem (11).

4. The trading fees Ξ^* are optimal for the Social Planner given its optimization problem (12).

As in Foucault et al. (2013), a discrete tick size guarantees traders cannot neutralize fee pricing by adjusting posted limit prices to exactly offset the impact of trading fees and rebates on their net cum-fee transaction prices. Our model differs from Foucault et al. (2013) in that investors in our model have random gains-from-trade and exogenous arrival/monitoring timing, whereas in Foucault et al. (2013) buyers and sellers arrival alternate. Our model also differs from both Foucault et al. (2013) and Chao et al. (2018) in that the endogenous choice of limit order prices is central in our model.[12]

Optimal fee pricing, for the exchange or the Social Planner, may or may not be unique with discrete prices, as shown by Chao et al. (2018) and Foucault et al. (2013). This follows because the identical cum-fee prices $P+MF$ ($P-MF$) for buy (sell) limit orders and $P-TF$ ($P+TF$) for sell (buy) market orders at a posted price P given a fee breakdown with MF and TF are also available given make and take fees $MF+\varepsilon$ and $TF-\varepsilon$ for any positive or negative integer multiple of the tick size ε via limit and market orders at posted prices $P-\varepsilon$ (when submitting limit buys) and $P+\varepsilon$ (when submitting limit sells). The restriction of ε to integer multiples of the tick size (i.e., equal to 1 here given our normalization) ensures that the adjusted posted prices $P-\varepsilon$ $P+\varepsilon$ are on the discrete price grid. For ease of future reference, we state this property as a formal lemma.

Lemma 1. *The identical trading outcomes and total fee given make and take fees MF and TF are also achievable with adjusted make and take fees $MF+\varepsilon$ and $TF-\varepsilon$ — called* equivalent *fees and rebates — for any integer ε allowed by regulation.*

This property has two immediate implications. One implication is that if a pair of non-positive make and take fees $MF \geq 0$ and $TF \geq 0$ is optimal, then fee pricing with either make (or take) rebates $MF-\varepsilon$ and $TF+\varepsilon$ ($MF+\varepsilon$ and $TF-\varepsilon$) have the same trading outcomes for sufficiently large negative (positive)

[12] Foucault et al. (2013) has an extension in which the limit price is determined by Nash Bargaining. In contrast, limit price choice in our model is a decision of the limit order submitter. Chao et al. (2018) includes one example of endogenous limit order choice, but otherwise their analysis is focused more on exchange competition than on investor order choice.

integers ε. However, if one fee is between in the interval $(0,1)$ and the other is a rebate in the interval (-1,0), then all equivalent fee pricing require rebates as well as fees. A second implication is that optimal fee pricing is unique given a sufficiently tight regulatory restriction on fees or rebates — i.e., so that the minimal integer adjustments $\varepsilon = 1$ or -1 would cause the adjusted fees to violate the regulatory constraint.

2.1 Equilibrium with exchange fees and continuous prices

As a starting reference point we first present the equilibrium in a market with continuous prices. Results are symmetric for buyers and sellers, so we consider an investor arriving at t_1 with a private value $\beta_{t_1} > v$. In this discussion, the total fee is $T = MF + TF$.

For an interior execution probability from (4), the expected payoff from posting a limit buy at P at t_1 is:

$$\begin{aligned} \pi_{t_1}(LBP) &= (\beta_{t_1} - P - (T - \mathrm{TF}))\frac{(P-v)+0.5\Delta - \mathrm{TF}}{\Delta} \\ &= -P^2 + [(\beta_{t_1} + v - 0.5\Delta - T + 2TF]P + [\beta_{t_1} - T + \mathrm{TF}][-v + 0.5\Delta - TF] \end{aligned} \tag{13}$$

The first-order condition of (13) gives the optimal limit-buy price given β_{t_1}:

$$P(\beta_{t_1}) = \frac{\beta_{t_1} + v - 0.5\Delta - T + 2\mathrm{TF}}{2} \tag{14}$$

Substituting (14) into (4) gives the execution probability of a limit buy posted at price $P_(\beta_{t_1})$:

$$\begin{aligned} Pr(x_{t_2}^{MSP}|P(\beta), T, L_{t_1}) &= \frac{\left(\frac{\beta_{t_1} + v - 0.5\Delta - T + 2\mathrm{TF}}{2} - v\right) + 0.5\Delta - \mathrm{TF}}{\Delta} \\ &= \frac{\beta_{t_1} - v + 0.5\Delta - T}{2\Delta} \end{aligned} \tag{15}$$

The total probability of transactions from limit buys at t_1 and market sells at t_2 is

$$\int_{v}^{(v+\frac{1}{2}\Delta)} Pr(x_{t_1}^{LBP}, \theta_{t_1}^{x^{LBP}}|T, L_{t_0})\, d\beta_{t_1} = \int_{v}^{(v+\frac{1}{2}\Delta)} \frac{\beta_{t_1} - v + 0.5\Delta - T}{2\Delta}\frac{1}{\Delta}\, d\beta_{t_1} \tag{16}$$
$$= \Big[\frac{T\left(\beta^2 + \beta\Delta - 2\beta(T+v)\right)}{4\Delta^2}\Big]_{v}^{(v+\frac{1}{2}\Delta)}$$
$$= \frac{3\Delta - 4T}{16\Delta}$$

where $Pr(x_{t_1}^{LBP}, \theta_{t_1}^{x^{LBP}}|T, L_{t_0})$ is the probability of a limit buy at $P(\beta)$ being submitted (i.e., $1/\Delta$) times the probability it is executed (from (15)). The corresponding total probability of transactions resulting from limit sells at t_1 and market buys at t_2 is symmetric. Thus, the exchange's ex ante expected profit is:

$$W^{Ex}(T) = 2 \times \int_{v}^{(v+\frac{1}{2}\Delta)} \Big[Pr(x_{t_1}^{LBP}, \theta_{t_1}^{x^{LBP}}|T, L_{t_0})\Big]\, d\beta_{t_1} \times T \tag{17}$$
$$= \frac{T(3\Delta - 4T)}{8\Delta}$$

The exchange chooses T to maximize $W^{Ex}(T)$. The first-order condition from (17) (from differentiating with respect to T) gives the optimal total fee:

$$T^* = \frac{3}{8} \times \Delta \tag{18}$$

When the support width is equal to the tick size, $\Delta = 1$, this solution coincides with the one obtained by Chao et al. (2018). Substituting (18) into (16) and multiplying by 2 (to allow for both limit buys and sells) gives a constant equilibrium transaction probability of $\frac{3}{32}$ and substituting (18) into (17) gives an expected exchange profit of $\frac{9}{256}\Delta$ given optimal fees. Since prices are continuous, as in Colliard and Foucault (2012), only the total fee matters. The fee breakdown does not matter.

The solution here depends on the width of the private-value support Δ, which represents the ex ante demand for trading. To develop some intuition, Figure 1 shows the exchange's expected profit, the probability of transactions, and the total fee as a function of the support width, Δ, which correspond to equations (17), (16) (times 2 for both limit buys and sells), and (18), respectively. Both the optimal total fee and the expected profit for the exchange are increasing in the support width (demand for trading). In particular, the exchange increases its rent extraction by increasing its fee. In addition, the probability of transactions is constant. However, consistent with Colliard and Foucault (2012), there is no role for the fee breakdown to improve welfare given continuous prices.

2.2 Results with discrete prices

Once trading is restricted to a discrete price grid, then buyers and sellers may be unable to find mutually acceptable prices at which to trade. As shown in Foucault et al. (2013) and Chao et al. (2018), fee pricing with rebates can ameliorate this discrete-price friction. Our analysis now considers a market in which posted limit-order prices are restricted to a discrete price grid $\{\ldots, P_{-k}, \ldots, P_{-1}, P_1, \ldots, P_k, \ldots\}$ centered around the mean private valuation v with a fixed tick size normalized to 1.[13] In numerical calculations, we center prices and valuations a $v = 10$ with $P_{-2} = 8.5$, $P_{-1} = 9.5\ldots$ and $P_1 = 10.5$, $P_2 = 11.5, \ldots$ and then vary the support width Δ.

Our analysis examines the relation between fee pricing and the investor valuation support $S = [\underline{\beta}, \bar{\beta}]$ and investor trading behavior. Given the normalized tick size, we vary the investor private-value support width Δ to show the effects of changes in ex ante trading demand on both total welfare and on the breakdown between investors welfare and exchange profit. Thus, our results can be interpreted in terms of stocks with varying degrees of high and low ex ante trading demand.

Our analysis builds on the important paper of Chao et al. (2018), so we highlight here how our analysis extends their results. First, we formulate our analysis to give comparative statics for the effect of varying

[13] We allow, in principle, for all possible prices, but from Lemma 3 in Appendix A only a range of prices around v is feasible.

amounts of ex ante trading demand given a normalized tick size. In particular, we show how limit-price choice and incentive compatibility affect fee pricing. Chao et al. (2018) instead normalize the valuation support to 1, and vary the price grid. Second, we study the effects of a regulatory cap that allows for the possibility of strictly positive fees as well as rebate-based fee pricing. In contrast, Chao et al. (2018) constrain the take fee to be non positive.[14] Third, we derive results for a Social Planner and contrast them with a profit-maximizing exchange with and without regulatory restrictions. Fourth, in Section 3 we extend the analysis beyond two periods and provide analytic formulation for a model that allows market participants to choose between market and limit orders, as well as limit order prices.

2.2.1 Equilibrium in the generic trading subgame

We start with a generic solution to the trading subgame given hypothetical fees MF and TF. Given the subgame equilibrium, we then solve via backwards induction in Sections 2.2.2, 2.2.3, and 2.3 for optimal fees for the exchange and for the Social Planner given the trading behavior their fees will induce.

With a discrete price grid, the investor optimization in (2) is a discrete-choice problem given a finite set of possible orders. The optimal order-submission strategy $x_{t_z}(\beta_{t_z}|\Xi, L_{t_{z-1}})$ assigns orders that maximize (2) to each possible investor valuations β_{t_z} in the support $[\underline{\beta}, \overline{\beta}]$ at time t_z conditional on the standing book $L_{t_{z-1}}$ at t_z and the fee pricing Ξ. Consequently, the maximized expected profit in (2) for different possible β_{t_z} valuations in the support S is the upper envelope of expected payoff functions for different orders from (1) that are linear in the valuations β_{t_z}. Each optimal order is associated with an interval of β_{t_z} valuations in between points where the optimized linear expected payoff functions intersect. The specific values that separate intervals of private valuations for one order from intervals for other orders are called *thresholds*.

Definition. *Suppose limit buys at prices P_j and $P_k > P_k$ both have positive execution probabilities, then the*

[14] See Chao et al. (2018), footnote 14 on page 1090.

threshold $\beta_{t_z}^{j,k}$ denotes the private value such that the expected profit is greater on a limit buy at P_k than at P_j when $\beta_{t_z} > \beta_{t_1}^{LB,j,k}$. A symmetric definition applies to thresholds $\beta_{t_1}^{LS,j,k}$ for limit sells.

These thresholds are computed in closed-form in Lemma 6. Thresholds for adjacent prices P_j and P_k where $k = \text{prior}(j)$ or $\text{next}(j)$ index the next price above or below P_j (i.e., where $|P_k - P_j| = 1$) play a key role in the structure of equilibrium trading subgame with discrete prices.[15]

Lemma 2. *Given hypothetical fees MF and TF, the equilibrium trading strategies in the 2-period trading subgame are as follows: At time t_2 optimal market orders are as in (3), and optimal limit order at time t_1 are*

$$x_{t_1} = \begin{cases} LSP_j & \text{if } \beta_{t_1} \in [\beta^{LS,prior(j),j}, \beta^{LS,j,next(j)}] \;\; \text{for } j < j^* \\ LSP_{j^*} & \text{if } \beta_{t_1} \in [\beta^{LS,prior(j^*),j^*}, \min\{v, P_{j^*}^{cum}\}] \\ LBP_{j^{**}} & \text{if } \beta_{t_1} \in [\max\{v, P_{j^{**}}^{cum}\}, \beta^{LS,j^{**},next(j^{**})}] \\ LBP_j & \text{if } \beta_{t_1} \in [\beta^{LB,prior(j),j}, \beta^{LB,j,next(j)}] \;\; \text{for } j > j^{**} \\ NT & \text{otherwise} \end{cases} \tag{19}$$

where the index j^{**} for the lowest limit buy order used in equilibrium is determined as follows: If the limit buy with the lowest limit price P_j with a positive execution probability has a cum-fee price $P^{cum,LBP_j} > v$, then the j^{**} is the index of that posted price. If instead $P^{cum,LBP_j} < v$, then j^{**} is the index of the posted limit buy price with the maximum expected profit $\pi_{t_z}(x_{t_z}|\beta_{t_z}, \Xi, L_{t_{z-1}})$ evaluated at $\beta_{t_1} = v$. The construction is symmetric for j^* for limit sells.

Simply put, limit orders used in equilibrium are associated with successive intervals of private values for which they are optimal. The optimal limit buys start with a lowest "used" limit price $P_{j^{**}}$ that is used for β_{t_1} realizations in the lowest interval, and then switch to successively higher "used" limit prices for β_{t_1}

[15] The index of the next highest price above P_j is $\text{next}(j) = j+1$ if $j \neq -1$ and $\text{next}(j) = 1$ if $j = -1$ (since there is no price P_0 below P_1 on the grid). The treatment of prior prices below P_j is symmetric.

realizations in higher intervals until a maximum possible limit price is reached. The switches happen for private valuations at the thresholds for adjacent "used" limit orders. A symmetric structure describes optimal limit sells. Figure 2 provides an illustration. The trading subgame equilibrium is in closed-form since the thresholds in (19) are determined in Lemma 6.

Fee pricing by an exchange or Social Planner takes into account the trading behavior its fees induce in the associated trading subgame. Inequalities (6) and (9) are Individual Rationality constraints that the exchange (or Social Planner) cannot force investor to trade. Inequalities (7) and (10) are Incentive Compatibility constraints that investors, not the exchange (or Social Planner), decide which orders to submit. Incentive Compatibility has a significant impact on optimal fee pricing. In particular, order-submission decisions by investors depend on a trade-off between execution probability and the impact of price improvement on the payoff conditional on order execution. In contrast, the exchange (and Social Planner) care about execution probabilities but not about zero-sum transfers between investors due to price improvement on one side of the trade. Figure 2 illustrates the Incentive Compatibility problem. For private valuations β_{t_1} between P_j^{cum} and the threshold $\beta^{LB,prior(j),j}$, a limit buy LBP_j is Individually Rational for the investor at t_1 and has a higher execution-probability than $LBP_{\text{prior}(j)}$ but it is not incentive compatible and, thus, would not be used by the investor at t_1.

Thus, both the exchange and Social Planner prefer that investors with private valuations $\beta_{t_1} \in [P_j^{cum}, \beta^{LB,j,\text{next}(j)}]$ submit limit buys with a posted price $P_{\text{next}(j)}$ given its higher execution probability. However, a limit buy $LBP_{\text{next}(j)}$ is not Inventive Compatible for the investor given the higher payoff conditional on execution of the LBP_j. As a result, we show below that both an exchange and the Social Planner use fee pricing to deter investors from using orders with better price improvement but worse execution probabilities.

2.2.2 Equilibrium with fees set by an unrestricted exchange

With no regulatory restrictions, the exchange is a monopolist in trading facilities. Hence, given a discrete tick size, the fee breakdown affects order-submission and -execution probabilities both via its impact on the discrete-price trading friction and via rent extraction from investors wanting to trade.

Figure 3 shows the profit-maximizing make and take fees MF, and TF, the total fee $MF + TF$ received by the exchange, the exchange's expected profit, and the transaction probability as a function of the valuation support width Δ. The various quantities are computed for unit increases in the support width (i.e., $\Delta = 1, 2, \dots$) on the horizontal axis holding the normalized unit tick size fixed.[16] The figure shows two large-scale patterns. Generally rising fees but with oscillations in MF and TF. When the support width Δ is small (ex ante trading demand is low), rebates are necessary for trading to occur. Moreover, since the fees and rebates are both less than one in absolute value, all equivalent fee pricings involve rebates given the first implication of Lemma 1. As the valuation support width Δ increases (and ex ante trading demand increases), the exchange increases both make and take fees and eventually rebates are no longer necessary. In other words, the impact of the discrete-price friction on fees and rebates shrinks relative to monopolistic rent extraction as ex ante trading demand becomes stronger. However, price-discreteness effects do not disappear. This is clear from another feature in Figure 3.

Unlike with continuous prices, the investor decision of which discrete price to use when posting limit orders has a discrete effect on the order-execution probability. In particular, if the change in Δ induces an investor to submit a limit buy at a one-tick worse price, this leads to a discrete reduction in the set of investors who are willing to sell to such a limit order via market orders at date t_2. For example, buyers at time t_1 have the option to lower the posted limit-buy price to gain price improvement conditional on execution while lowering the order-execution probability. This is the Incentive Compatibility effect. As a result, an exchange

[16] For visual presentation purposes, the unit valuations are connected with line.

may adjust its fee pricing to deter price-improvement seeking behavior by limit-order submitters. The result is oscillation in the make and take fees around the rising trend to growing rent extraction in Figure 3. The resulting expected profit for the exchange is growing in the support width Δ with discrete prices. Moreover, the overall expected-profit level is very similar to the market with continuous prices. Table 1 zooms in to provide greater insight into the oscillation phenomenon. The table reports optimal fees and rebates, cum-fee prices, and various probabilities for different support widths Δ. A key variable of interest is the order threshold between a limit buy order at a particular limit price and the next best order (including possibly NT). Between $\Delta = 3$ and just below $\Delta = 4$, the exchange steadily increases its take fee so as to depress the execution probability for LBP_{-1} limit buys so that investors at time t_1 will not submit LBP_{-1} limit buys and instead continue to submit higher execution-probability LBP_1 limit buys.[17] However, once Δ reaches or exceeds 4, the exchange stops deterring LBP_{-1} — which now becomes part of the equilibrium strategy for some β_{t_1} realizations — and increases the make fee and reduces the take fee. As Δ then becomes even larger, the exchange once again starts increasing the take fee to deter LBP_{-2}. This continues until a value of Δ between 8 and 8.8, at which point LBP_{-2} becomes part of the equilibrium, and there is another oscillation in the MF and FT fees. Thus, the pattern of rising fees is due to rent extraction, and the oscillations are due to managing the Incentive Compatibility problem in investor order choice given the exchange's fees .

2.2.3 Equilibrium with fees set by a Social Planner

This section considers optimal fees in the two-period model set by a Social Planner rather than by a profit-maximizing exchange. Lemma 8 in Appendix A.1 establishes that a Social Planner optimally uses fees and rebates such that $MF = -TF$ (i.e., the total fee for the exchange is $TF + MF = 0$). Thus, our analysis

[17] Note that the thresholds in Table 1 from $\Delta = 3$ to $\Delta = 3.5$ indicate that between $v = 10$ and 10.50 the equilibrium strategy is NT, whereas between 10.50 and $\overline{\beta}$ investors use LBP_1, which is the only equilibrium strategy for this support width. The next threshold reported, 12.50, is the hypothetical threshold between LBP_1 and LBP_2: as it falls beyond $\overline{\beta}$, it indicates that LBP_2 is not an equilibrium strategy. Within this support width investors do not use LBP_{-1}.

determines the equilibrium take fee TF. We first provide a characterization result for general private-value distributions. We then illustrate that result for uniformly distributed private values.

Theorem 1. *The optimal take fee TF for a Social Planner to maximize total welfare in (12) is a function $F(\Delta)$ of the valuation support width Δ that is determined either i) as the local optimizer for a particular total welfare objective given a set of "used" orders that are incentive-compatible for the investor at t_1 to submit or ii) as a linear function of Δ that keeps the threshold between the best unused order and the adjacent used order equal to v.*

The key part in the proof is the fact that the optimal fee for the Social Planner, depending on Δ, is determined one of two ways: One possibility is that the optimal fee is the local maximizer of a social value function given an associated set of used orders that are locally incentive compatible with the optimal fee. The other possibility is that the optimal fee is the fee value that deters investor use of lower limit buy prices (higher limit sell prices) by keeping the threshold for the lowest "used" price vs. the next lowest "non–used" price equal to v. An important implication here is that in each of the two cases, the optimal fee $F(\Delta)$ is continuous in Δ, but it changes differently in Δ depending on which of the two ways $F(\Delta)$ is being determined.

This phenomenon is illustrated in Figure 4, which shows the welfare-maximizing MF and TF for different investor valuation support widths Δ (with unit increases), and Table 2, which reports additional detail about equilibrium strategies, cum fee buy and sell prices, submission, execution and transaction probabilities, and total welfare. In the figure, there is again a clear oscillation in fees. However, here the oscillation is around a constant rather than around a rising trend line due to increasing rent-extraction. In addition, with uniform private values, the Social-Planner fee oscillations are actually piece-wise linear. For the segments associated with order-deterrence, the slope is always identical since it comes from adjusting the fees to keep the order thresholds (which are linear in Δ and in F) equal to v. Another important property of optimal

Social Planner fees is that total welfare can be maximized using fees that are less than 1 tick in size. This is in contrast to the large fees used by the unrestricted profit-maximizing exchange in Section 2.2.2.

To elaborate, the pattern in the Social Planner's welfare-maximizing fees follows from four intuitions: The first is that when $\Delta \leq 2$, the Social Planner uses fees and rebates to facilitate trading at a mid-quote equal to v. The second intuition is that, as the valuation support width Δ widens, mid-quote trading is no longer possible. In particular, investors at time t_2 eventually become potentially willing to hit limit orders with worse limit prices. This creates an incentive for investors at t_1 with valuations β_{t_1} close to v to submit worse limit orders, which give them a private gain from price improvement, but which lower total welfare due to their lower execution probability. This is the Incentive Compatibility effect again. To maximize total welfare, the Social Planner adjusts fee pricing to prevent investors from using worse limit prices for a range of Δs. For example, for Δ between 2 and 3, the Social Planner in the Maker-Taker equilibrium increases the take fee to deter execution of limit buys at P_{-1} at t_2 so that potential buyers at t_1 will not submit them and instead continue submitting limit buys at P_1. The third intuition is that expected welfare on submitted limit orders is maximized if the probability of limit order submission and the probability of submission of market orders that execute standing limit orders are equal. However, deterring latent limit orders at worse prices skews these probabilities away from equality. Thus, there is a trade-off between these two effects. For example, Table 2 shows that once Δ exceeds 3, the probability distortion is so large, that the Social Planner switches and begins to adjust fees to reduce the probability distortion. As a result, potential buyers at t_1 with valuation above but close to v start submitting limit buys at P_{-1} (while investors at t_1 with higher valuations β_{t_1} continue to submit limit orders at P_1 for the higher order-execution probability). The fourth intuition is that with a wider support Δ there are more investors at t_1 with extreme valuations who value higher execution probability more than price improvement. The Social Planner also uses fee pricing to increase the endogenous number of such investors. For example, that is why the Social Planner continues decreasing take fee below 0.5 when Δ is between 4 and 5. The various patterns continue to repeat for Δs that are even larger.

2.3 Equilibrium with a regulatory cap on fees

Regulatory restrictions can have a significant impact in equilibrium on optimal fee pricing by an exchange. The large differences between fees in Sections 2.2.2 and 2.2.3 suggest that regulators might be concerned about unconstrained profit-maximization by a monopolistic exchange. However, while optimal welfare-maximizing fees may be difficult for a regulator to impose, Section 2.2.3 shows that the welfare-maximizing fees tied to the price-discreteness friction are always less than one-tick in size. Thus, this section solves for equilibrium in a market in which a regulator imposes a one-tick cap on trading fees, and then the exchange sets its fee pricing to maximize its expected profit given that investors choose what orders to use given the exchange's fees. A regulatory restriction $\Xi \in R$ in (11) tied to the tick size in this way is qualitatively similar to US regulation (and also with Foucault et al. (2013)).[18] Appendix B presents the equilibrium construction and shows that the equilibrium fees and rebates for the exchange are given by the simple close-form expressions for support widths Δ in different ranges.

Our first result is an analytic solution for optimal fee pricing for a range of valuation supports that extends beyond the unit support in Chao et al. (2018) to include supports with larger widths $\Delta > 1$.

Theorem 2. *When the valuation support width is $\Delta \leq 3$ (given a tick size normalized to 1), the equilibrium fee pricing for a profit-maximizing exchange in the two-period market given the one-tick regulatory constrain on fees can be achieved by either Taker-Maker with fees and rebates*

$$MF^* = \frac{\Delta + 3}{6} \qquad TF^* = \frac{\Delta - 3}{6} < 0, \tag{20}$$

[18] Using a cap of 0.3 of the tick size from Reg NMS makes our results tighter. In Europe, there is no formal regulatory cap but informal regulatory understandings and industry norms following US markets lead exchanges usually to set fees less than one tick.

or Maker-Taker with fees and rebates

$$MF^* = \frac{\Delta - 3}{6} < 0 \qquad TF^* = \frac{\Delta + 3}{6}. \tag{21}$$

When the valuation support with is $\Delta \in [3,5]$, the equilibrium fee pricing is achieved with strictly positive fees (i.e., no rebates) that are unique

$$MF^* \& TF^* = \begin{cases} 1 \ \& \ \frac{1}{2}(\Delta - 3) & \text{if } 3 < \Delta \leq 4 \\ 1 \ \& \ \frac{1}{4(\Delta-2)}\left(\Delta^2 - 5\Delta + 8\right) & \text{if } 4 < \Delta \leq 4.7 \\ \frac{1}{2} \ \& \ 1 & \text{if } 4.7 < \Delta \leq 5 \end{cases} \tag{22}$$

Fee pricing depends on the size of the support of traders' valuation relative to the normalized tick size. Rebates are necessary in two-period markets with low valuation dispersion (i.e., the support width is $\Delta \leq 3$), and so the exchange uses equivalent Maker-Taker and Taker-Maker fee pricing (which are one-tick perturbations of each other as per Lemma 1, but *only* one tick given the one-tick regulatory fee cap). This is the same as in Chao et al. (2018). However, once valuation dispersion is higher (i.e., $\Delta \in [3,5]$), rebates are no longer necessary. As the support of investor valuations increases, the ex ante potential gains-from-trade increase, which increases investor trading demand. As a result, the exchange has less need to incentivize trading. In equilibrium, the exchange exploits investors' greater ex ante gains-from-trade by increasing fees and reducing rebates. In addition, the potential fee multiplicity via perturbations as in Lemma 1 is eliminated for positive fees by the one-tick regulatory cap on fees. The different fee pricing in the different ranges of Δ are due to the fact that, as the valuation support widens, the set of ex ante feasible prices expands to include more possible limit prices, and the execution probabilities of limit orders at more extreme limit prices increases. The following proposition summarizes these results:

Proposition 1. *When an exchange optimally uses Maker-Taker or Taker-Maker fee pricing in the two-period market, then rebates are decreasing and fees are increasing as the valuation support width* Δ *increases.*

Another property of equilibrium fees follows from the fees and rebates in (20) and (21) in Theorem 2.

Proposition 2. *The sum of the make and take fees is one third of the support width,* $MF + TF = \Delta/3$ *for all support widths* $\Delta < 3$ *in the two-period model.*

The key part of the proof is that the exchange's expected profit in (11) can be expressed as

$$\pi^{Ex}(MF,TF) = 2 \max\left\{0, \frac{\overline{\beta} - P_{-1}^{cum,LB}}{\Delta}\right\}(MF+TF) \max\left\{0, \frac{P_{-1}^{cum,MS} - \underline{\beta}}{\Delta}\right\} \tag{23}$$

which is the product of the relevant limit-order submission probability at time t_1, the net fee, and the relevant market-order submission probability at time t_2. The specific functional form of (23) follows from there just being two periods and from the uniform valuation distribution assumption and symmetry between the buy and sell sides of the market. The three components $\overline{\beta} - P_{-1}^{cum,LB}$, $MF + TF$ (which equals $P_{-1}^{cum,LB} - P_{-1}^{cum,MS}$), and $P_{-1}^{cum,MS} - \underline{\beta}$ in (23), when they are positive, sum to the valuation support width Δ. Proposition 2 shows that the product in (23) is maximized by the exchange choosing MF and TF to set these three components equal to each other, which implies that $MF + TF = \Delta/3$.

Figure 5 shows equilibrium fees and rebates for different support widths (i.e., $\Delta = 1, 2, \ldots$) — both for the range of support widths in Theorem 2 and also for larger Δ widths — holding the normalized unit tick size fixed.[19] The figure shows two large-scale patterns. First, rebates are still necessary to mitigate the price-discreteness friction when Δ is small (and the ex ante demand to trade is low). However, when Δ grows (and ex ante trading demand become large) eventually rebates are no longer optimal for the exchange to maximize its expected profit, and fees are both strictly positive. As a result, now the regulatory cap binds

[19]For visual presentation purposes, the unit valuations are again connected with line.

eventually so that both make and take fees converge to the regulatory cap.

The fact that the tick size of 1 is a normalization means our results about how fee pricing changes when the valuation support is varied holding the tick size constant translate immediately into corresponding results about fee pricing when varying the tick size holding the support Δ fixed.

Theorem 3. *The equilibrium fee pricing in a two-period market with a tick size equal to a fraction τ of the normalized unit tick size, for $0 < \Delta \leq 5$, is given by:*

$$\{mf^*, tf^*\} = \{\tau MF^*, \tau TF^*\} \tag{24}$$

where MF^* and TF^* are the optimal fee pricing given the unit tick size in Theorem 2.

Markets with different tick sizes are isomorphic in the sense that optimal fee pricing scales linearly in the tick size τ. Intuitively, a tick of "one" can be a tick of one penny, one eighth or one dollar. This isomorphism is illustrated both in Figure A1 and A2, in the Online Appendix for a small tick market (STM) with a tick size set to $\frac{1}{3}$ of the normalized unit tick size. The patterns in the STM are qualitatively identical to the unit tick-size market for the exchange wth the unit fee cap and for the Social Planner except that the magnitudes and the speed of the oscillations are rescaled. The associated fees and rebates are one third of those in the market with the unit tick size, and relation to the support width is rescaled to one third of that with the unit tick size market. In other words, the optimal fee pricing in this STM is $\frac{1}{3}$ of the pricing in a unit-tick-size market with the same relative valuation support ratio Δ/τ. Given this rescaling, all the results unit tick size market are qualitatively the same in the STM. This analysis leads to the following empirical prediction:

Empirical Prediction: *When, holding the trading population constant, the tick size increases (decreases), the exchange has an incentive to offer greater (smaller) fees and rebates.*

Our empirical prediction can be tested by investigating how a change in the tick size alters the incentive for the exchange to offer rebates. Our model predicts that when, all else equal, the tick size increases, the exchange, to attract volume, should increase the rebates offered to the same population of market participants. However, with competition, if the exchange does not adjust the rebates to the new tick size, it runs the risk of seeing orders migrating to other more profitable venues. Comerton-Forde et al. (2019) investigate the effects of an increase in the tick size within the U.S. tick size pilot program started in October 2016 and, interestingly, find that following the increase in the U.S. tick size from 1 penny to 5 pennies a substantial amount of orders migrated from the maker-taker to the taker-maker inverted fees platforms. This finding is consistent with our model's prediction that following an increase in the tick size the exchange should offer greater rebates to ensure that volume is maximized within a trading platform.

3 Three-Period model

Our analysis is readily extended to a richer market environment with N investor arrivals at times $t_z \in \{t_1, \ldots, t_N\}$. The basic economics can be illustrated with $N = 3$. This multiperiod extension lets us describe the effect of increased investor arrival activity on fee pricing. In particular, trading activity can refer either to potentially longer trading horizons or to more frequent investor arrival over a fixed horizon (e.g. over a trading day). From a modeling viewpoint, there are two new elements: First, the limit order book can potentially accumulate depth at a given price or at different prices in the multiperiod market whereas there is at most only one limit order in the book in the two-period model. In particular, the arrival of new limit and market orders augments or reduces the depth of the limit order book respectively, leading to dynamics:

$$L_{t_z} = L_{t_{z-1}} + Q_{t_z} \quad z = 1, \ldots, N \tag{25}$$

where $Q_{t_z} = [Q_{t_z}^{P_k}]$ is a vector of changes in the limit order book due to an arriving investor's action x_{t_z} at t_z. The change $Q_{t_z}^{P_k}$ in depth at price P_k is "+1" when an arriving limit order adds an additional share and "−1" when a market order executes a limit order where P_k is the best bid or offer (BBO), and otherwise is zero (at other prices unaffected by arriving orders). The changes $Q_{t_z}^{P_k}$ are all zero if no order is submitted. Second, investors arriving after t_1 and before t_N have a non-trivial choice between market and limit orders. Once again, an arriving investor at t_1 still only chooses between different limit orders at different possible limit prices and NT, and the investor in the final time t_N still chooses between buy and sell market orders and NT. For tractability, we assume limit orders cannot be modified or cancelled after submission and that investors can only send one order of unitary size at a time.[20]

The objectives for fee pricing with N periods are analogous to those in the two-period model. An exchange chooses its fees, Ξ, to maximize its expected payoff from completed transactions:

$$\max_{\substack{MF,TF \\ \{MF,TF\}\in R}} \sum_{t_z\in\{t_1,\ldots,t_N\}} W_{t_z}^{Ex}(MF,TF) = \left[\sum_{t_z\in\{t_1,\ldots,t_N\}} \sum_{x_{t_z}\in X^L} Pr(x_{t_z},\theta_{t_z}^{x_{t_z}}|\Xi)\right](MF+TF) \tag{26}$$

given transaction probabilities

$$Pr(x_{t_z},\theta_{t_z}^{x_{t_z}}|\Xi) = \sum_{L_{t_{z-1}}} Pr(L_{t_{z-1}}|\Xi)\,Pr(x_{t_z}|\Xi,L_{t_{z-1}})\,Pr(\theta_{t_z}^{x_{t_z}}|\Xi,L_{t_{z-1}}) \tag{27}$$

where now the limit-order submissions can occur at multiple dates and the execution probabilities $Pr(\theta_{t_z}^{x_{t_z}}|\Xi,L_{t_{z-1}})$ take into account the fact that a limit order submitted at time t_z can potentially be executed at multiple possible dates in the future.

A Social Planner maximizes the total welfare, which generalizes to the N-period model to:

[20] As noted in Parlour and Seppi (2008), such limit orders are essentially "take it or leave it" offers of liquidity.

$$\max_{\substack{MF,TF\\MF+TF\geq 0}} \sum_{t_z\in\{t_1,\ldots,t_N\}} \left(W_{t_z}^{INV}(MF,TF)+W_{t_z}^{Ex}(MF,TF)\right) \tag{28}$$

$$= \sum_{t_z\in\{t_1,\ldots,t_N\}} \sum_{x_{t_z}\in X^L} Pr(x_{t_z},\theta_{t_z}^{x_{t_z}}|\Xi)\times I(x_{t_z})\times\left[E[\beta_{t_z}|x_{t_z}]-E[\beta_{t_{z'}}|\theta_{t_z}^{x_{t_z}}]\right]$$

where $t_{z'}$ denotes the execution time of a limit order posted at t_z and where the last line follows because the Social Planner optimally sets $MF=-TF$ (i.e., sets the exchange's expected profit to zero). The optimization in (28) is subject to a non-negative net fee constraint (individual rationality) for the exchange ($MF+TF\geq 0$) and a regulatory tick-size constraint on fees.

The existence of equilibrium for a general N-period model with fees set either by an exchange or a Social Planner follows from first principles:

Theorem 4. *The equilibrium of a trading game with N periods and a price grid with a fixed number of prices exists and can be constructed analytically via backward induction.*

Proofs for general N-period models are in Appendix A.2. The functional forms of both the exchange profit function and the Social Planner total welfare function can be complex as the number of periods grows and as the number of possible limit prices increases — i.e., as more limit orders become a priori feasible as larger investor valuation supports encompass more prices. Therefore, rather than explicitly differentiating the analytic exchange expected profit function or the analytic Social Planner total welfare, we report results using a search algorithm to solve the first-order conditions for Ξ^*.[21]

To illustrate optimal fee pricing and trading in a multi-period market, we consider a three-period market

[21] Section B in the Online Appendix describes the Simulated Annealing Algorithm (SA) and Grid Search Algorithm (GS) we use to find numerical results. Although we obtain optimal fees of the three-period model numerically, in Online Appendices (Section C) we show how to obtain a closed-form solution for the three-period benchmark model. As a robustness check, we confirmed that optimal fees computed using the SA and GS algorithms in the two-period model agree with the analytic ones.

with investor arrival dates $\{t_1, t_2, t_3\}$. Two key intuitions drive our results: First, in the two-period model in Section 2, investors in the first period are monopolists in supplying liquidity since there is no opportunity for later traders to compete against the first-period trader's limit orders. In particular, investors at t_2 only accept or decline liquidity offered by a limit order posted at time t_1 since the game ends after t_2. With more than two periods, the first-period liquidity supply is no longer monopolistic, and some amount of intertemporal competition in liquidity supply is possible. Second and relatedly, a higher level of market activity with more rounds of investor-arrival increases the opportunities for limit order execution.

Figures 6 and 7 show equilibrium fee pricing for the three-period model for different investor valuation supports in the restricted and unrestricted regulatory regimes respectively.[22] Many of the results for the three-period model are similar to the two-period model. There is still a region of valuation supports with both Taker-Maker and Maker-Taker equilibria and, again, as the valuation support width Δ increases, the exchange optimally increases both MF and TF (possibly subject to the regulatory cap), and eventually there is an equilibrium with strictly positive fees (which is unique if there is a regulatory cap). However, the next proposition highlights a difference relative to the 2-period market:

Proposition 3. *The set of valuation supports associated with rebates is smaller in the three-period model, and fees are larger, and rebates are smaller than in the two-period model.*

Comparing the two-period model results with the three-period results for both the unrestricted regime (Figures 3 with 6) and for the restricted regime (Figures 5 with 7) shows that the region with rebate-based fee pricing (Maker-Taker or Taker-Maker) is smaller in the three-period market. The largest valuation support width associated with rebate-based pricing is 2.31 in the three-period market vs. 3 in the two-period framework. In addition, because trading volume is higher in the three-period model, exchange profits are higher.

[22] Once again, the 3-period exchange profit functions look qualitatively similar to those for the two-period exchange modulo the asymmetry discussed below.

The levels of fees (rebates) in the three-period model are also larger (smaller). The intuition for the effect of the number of trading periods on the use of rebates and the level of fee pricing is the following: Holding everything fixed, the probability limit orders are executed increases because there are more opportunities for investors with complementary reasons to trade to arrive and trade with each other. As a result, the exchange can set larger fees and has less of an incentive to offer rebates.

Proposition 4. *Maker-Taker and Taker-Maker pricing is asymmetric in the three-period model with smaller rebates in the Maker-Taker equilibrium than in the Taker-Maker equilibrium.*

This asymmetry in rebate-based fee pricing is new and is in contrast to the symmetry in our two-period model and also in Chao et al. (2018). The equilibrium fees are asymmetric because in the three-period model the investor at time t_1 is no longer a monopolist in liquidity provision. An arriving investor at time t_2 can undercut the t_1 limit order (by submitting a limit order in the same direction as the t_1 limit order with a better price) or may seek price improvement (by submitting a limit order in the opposite direction of the t_1 limit order rather than hitting it with a market order).

Consider, for example, the equilibrium strategies in Row 1 of Table 3 for a support width $\Delta = 0.33$. In the Taker-Maker equilibrium, when the investor in period t_1 limit buys (LBP_{-1}) at the price P_{-1}, an incoming seller in period t_2 has the option of either market selling (MSP_{-1}) at P_{-1} or limit selling (LSP_1) at the higher price P_1. In contrast, in the Maker-Taker equilibrium, the investor at t_1 limit buys (LBP_1) at P_1 (because of the rebate $MF = -0.428$), which consequently means a seller at t_2 has no other trading option than market selling (MSP_1) at the high price P_1 — since limit selling at P_{-1} is not allowed given the pre-existing limit buy at P_1 in order to prevent a locked market — and therefore will be charged a positive fee $TF = 0.557$.[23] This theoretical result about the connection between fee pricing and locked markets is new.[24]

[23]The state of the book when the seller arrives at t_2 has a limit order at P_1, hence the seller does not compete for the provision of liquidity as a limit sell order at P_{-1} is dominated by the market sell order at P_1.

[24] We thank Mao Ye for insights on this point.

The comparison between the two-period and three-period markets shows how optimal fee pricing differs for stocks with different rates of trading activity. In particular, high investor arrival is associated with a reduced need for rebates to encourage trading. However, in present day markets, a large portion of trading involves HFT investors who, arguably, have small private valuation dispersion. Thus, our model suggests that rebates can help facilitate trading intermediated by HFT investors. This observation is consistent with the empirical connection between HFT trading and rebates (see Menkveld (2013), Cardella et al. (2015), and O'Hara (2015)).

4 Welfare

Fee pricing that maximizes exchange profits does not necessarily improve the overall welfare of other market participants. This section investigates how fee pricing by exchanges (under the unrestricted and restricted regulatory regimes) affects the welfare of market participants relative to the regime in which trading fees are set by a Social Planner, and relative to a "benchmark" regime with no fees or rebates (i.e., $MF = TF = 0$). We focus on the 2-period framework but analogous results can be obtained for the 3-period model.

Figure 8 shows our welfare results for different investor valuation support widths Δ. Total welfare is computed for all investors (INV) and for the exchange (Ex) at all dates:

$$TW = \sum_{t_z \in \{t_1,\ldots,t_N\}} \left(W_{t_z}^{INV}(MF,TF) + W_{t_z}^{Ex}(MF,TF) \right) \tag{29}$$

$Restric.ExchangeTW$ and $Unrestric.ExchangeTW$ are total welfare for the restricted and unrestricted exchange regime respectively, given the optimal trading fees set by the profit-maximizing exchange. $BenchmarkTW$ is computed using zero fees and rebates $\{MF^{\dagger} = 0, TF^{\dagger} = 0\}$, and $SocialPlannerTW$ ($SPTV$) is total welfare given optimal fees $\{MF^{*}, TF^{*}\}$ chosen by the Social Planner. Figure 8 also shows the welfare breakdown for investors (INV) and the exchange (EXG) for different investor valuation supports. There are three different regions in the figure: The *PIW* region in which optimal fees by the profit-maximizing exchange are Pareto-improving with all market participants better off relative to the no-fee benchmark. The *RW* region in which optimal fee pricing by an exchange increases total welfare, but reallocations (i.e., Pareto transfers) from the exchange to investors are needed for investors to be better off. The *DL* region in which total welfare is lower due to deadweight losses, but the exchange is better off. Our findings are consistent across the two market settings of unrestricted and restricted exchange pricing.

- The PIW region happens for small valuation support widths Δ. This is expected, since, when the support is small relative to the tick size, there is no-trade without a take or make rebate. In general, the reason why rebate-based pricings Pareto improve welfare, even when there are gains-from-trade, is the following: Individual investors care about both the probability of order execution (which increases total welfare) and also about their execution price (which affects their personal payoff but is neutral for total welfare). When the valuation support is small relative to the tick size, there are many investors for whom the probability of execution of orders posted at the available prices makes them unwilling to trade at these prices. An exchange can increase its expected profit and simultaneously improve total welfare by setting fees and rebates to increase the order-execution probabilities.

- The RW region occurs for somewhat larger valuation supports. As the valuation support widens, a growing share of arriving investors have sufficiently strong trading demands (extreme private valuations) that rebates are not needed for trading. However, there is also an externality in exchange behavior. Exchange expected profits and, thus, their fee pricing depend on the total fee (which reduces investor welfare) as well as on the order-execution probability. When valuation dispersion becomes larger relative to the tick size, exchanges set larger net fees to increase their expected profit although this reduces order-execution probabilities. For a range of support widths Δ, the net effect of rebate-based exchange fees is to increase total welfare relative to the no-fee-and-rebate equilibrium, but with the exchange capturing a growing share of the gains-from-trade at the expense of investors.

- The Deadweigth Loss (DL) region happens when the investor valuation support is larger than 1.88. Once the dispersion in investor valuations is large relative to the tick size, the exchange's profit-maximizing net fee becomes so large that it reduces total welfare. The shaded area reported in Figure 8 shows the DL region due to rebate-based pricing — as opposed to positive pricing — set by a profit-maximizing exchange.

Our welfare results have policy implications. Figure 8 shows the welfare improvement by the Social Planner. In the DL regions in which an exchange uses rebate-based pricing, a Social Planner also sets rebate-based pricing but total welfare associated with the Social-Planner fee pricing is much higher compared to the total welfare associated to the exchange pricing. Thus, the deadweight loss is due to the fact that the exchange sets total fees too high in order to maximize its own profits. When setting rebates, the exchange faces a trade-off. The smaller the investor gains-from-trade are, the more rebates are necessary to induce them to participate and the smaller the exchange net revenue from each trade. Hence, in equilibrium to some extent the exchange subsidizes investors who have smaller gains-from-trade. However, the Social Planner pricing subsidizes traders with smaller gains-from-trade to a greater extent as its objective function is to maximize investors' welfare as opposed to the exchange profit. This is the reason why in Figure 8 *SocialPlannerTW* is always greater or equal than *BenchmarkTW*: In correspondence of all regions, the Social Planner rebate-based pricing leads to an improvement in total welfare. Clearly, as the relative tick size becomes smaller, the frictions due to price discreteness become less relevant and the SP pricing tends to converge to the Benchmark pricing.

Our model shows that rebate-based pricing is not detrimental per se to investors given price frictions. It is how exchanges set fees in combination with rebates that can generate deadweight losses. Our results, therefore, suggest a positive role for regulators both when the relative tick size is large, and when the relative tick size is small due to the increase in the investors' gains-from-trade. When the gains-from-trade are small (the relative tick size is large) a regulatory cap that sets the fees no larger than the tick size may be insufficient to affect welfare. For example, to Pareto improve welfare when $\Delta \leq 2$, regulators should impose a cap on fee smaller than the tick size. In this case, capping the positive fee to 0.5 tick moves the exchange fee pricing towards the Social-Planner fee pricing (which sets the fee to 0.5). When gains-from-trade are instead large relative to the tick size, regulators can improve welfare by limiting the ability of exchanges to extract rents (by setting fees that are too large) by imposing a cap on trading fees equal to the tick size (since the Social Planner never uses fees larger than 1 tick).[25] In the large gains-from-trade scenario, due to the cap on fees, the positive change in investors welfare ($\Delta InvestorsWelfare$) is proportionally greater than the negative change in exchange profit ($\Delta ExchangeProfit$). When the gains-from-trade increase, the exchange by increasing the total fees extract rent from investors with large gains-from-trade at the expense of investors with smaller gains-from-trade

Proposition 5. *Optimal fee pricing by an exchange: When Δ is sufficiently small such that welfare is increasing in rebate-based pricing, a regulatory cap equal to half a tick size would improve total welfare. When instead the valuation support Δ is sufficiently large such that the exchange optimally sets strictly positive fees, a regulatory cap on fees equal to the tick size increases total welfare.*

For optimal fee pricing by an exchange, this proposition follows immediately from the existence of welfare-increasing and deadweight loss regions with rebate-based pricing. The intuition is that when rebates are needed to encourage trading, a regulatory action imposing fees equal to half the tick size would increase total welfare as it would drive the exchange fee pricing to match the SP pricing. In contrast, when the exchange uses strictly positive fees, which increases its profits but which leads to deadweight welfare losses, a cap on its fees equal to the tick size would alleviate this problem and increase total welfare.

[25] In both cases, the cap is not too small (i.e., a way that precludes sufficient rebates).

Figure 8 finally shows that as the support increases, the zero fees regime and the SP regime tend to coincide. Nevertheless, the fee breakdown remains key to increase total welfare as in real markets there is a mixture of population and when there are a lot of agents with small gains-from-trade the role of the fee breakdown is crucial to increase total welfare. This is relevant as although competition drives total fees to zero an important role remains for fee pricing.

5 Conclusion

This paper models optimal fee pricing for an exchange or Social Planner and gives new insights about fee pricing, its drivers, welfare effects and regulatory actions. Our analysis shows investor valuation dispersion relative to the tick size is a key driver of optimal fee pricing. When the market is mainly populated by investors with valuations close together so that ex ante gains-from-trade are small relative to the tick size, the equilibrium fee pricing by a profit-maximizing exchange is rebate-based. The exchange alleviates the trading frictions generated by price discreteness by reallocating gains-from-trade. When instead the market is populated by long-term investors with ex ante valuations dispersion (large gains-from-trade) that is large relative to the tick size, trading frictions become less relevant, and the exchange chooses jointly positive make and take fees to increase its rent extraction.

Our model shows that also the welfare effects of optimal fee pricing by a profit-maximizing exchange vary with the amount of the investor gains-from-trade. When the market is populated by investors with small gains-from-trade, and frictions from price discreteness are severe, rebate-based pricing by the exchange reduces pricing frictions and Pareto improves total welfare for both investors and the exchange. When instead investor gains-from-trade are large, and the tick size friction is less severe, optimal exchange fee pricing — without and sometimes even with rebates — can lead to total deadweight losses as increased profits for the exchange are less than welfare losses for investors.

From a policy perspective, regulation may crucially affects fee pricing by imposing a cap on fees that takes into account the investor gains-from-trade. Importantly for regulators, our analysis shows that rebate-based pricing is not welfare reducing per se, but rather that the welfare effects of rebate-based pricing depend on the incentives of who sets trading fees and rebates and on the magnitude of trading frictions relative to investor ex ante trading demand. In particular, we show that a Social Planner always uses rebate-based fee pricing to increase total welfare, but differently from the profit maximizing exchange it sets the total fees equal to zero. This explains why when the support is small, the social planner fee pricing generates a substantially greater increase in total welfare. This also explains why a regulatory cap on fees equal to half a tick would induce the exchange to set a rebate-based pricing similar to the welfare maximising social planner fee pricing, thus Pareto improving total welfare. When instead the support is large and the relative tick size is small it would be sufficient for regulators to cap trading fees to the tick size to substantially Pareto improve total welfare. A cap on trading fees would limit the ability of the exchange to extract rents from the investor large gains-from-trade.

Our model has a number of other "firsts" from a modeling perspective. Our model is the first to provide a complete analysis of the effect of endogenous limit price choice and market/limit order choice on fee pricing. Our model also is the first to consider more than two periods. This extension shows that fee pricing changes with greater market activity. We also are the first to formally model the Social Planner's fee pricing and show the relevance of the fee breakdown that maximizes total welfare. Finally, our model allows us to understand how exchanges strategically manage total fee and fee breakdown to maximize profits.

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Figure 1: Model with Continuous Prices This figure reports the Expected Exchange Profit (gray), the Equilibrium Total Fee (green) and the Transaction Probability (orange) of the model with continuous prices as a function of the Support With (Δ)

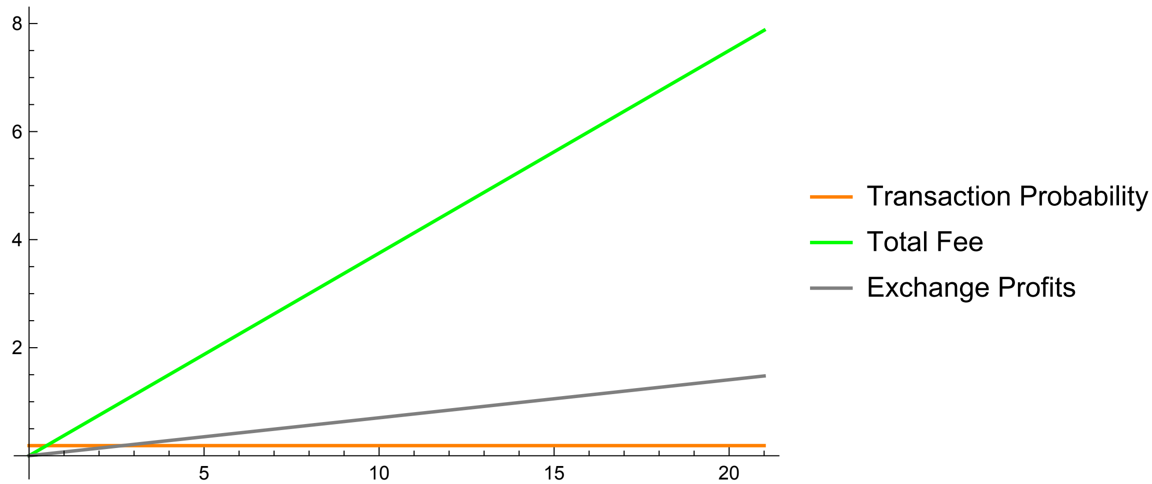


Figure 2: 2-Period Model with Profit-Maximizing Exchange: Upper Envelope. This figure reports for investor at time t_1 with $v = 10$ and support width $\Delta = 5$ the plot of the expected payoffs of the investor at time t_1 for each order type LSP_1, LSP_{-1}, LBP_1, LBP_{-1}, NT.

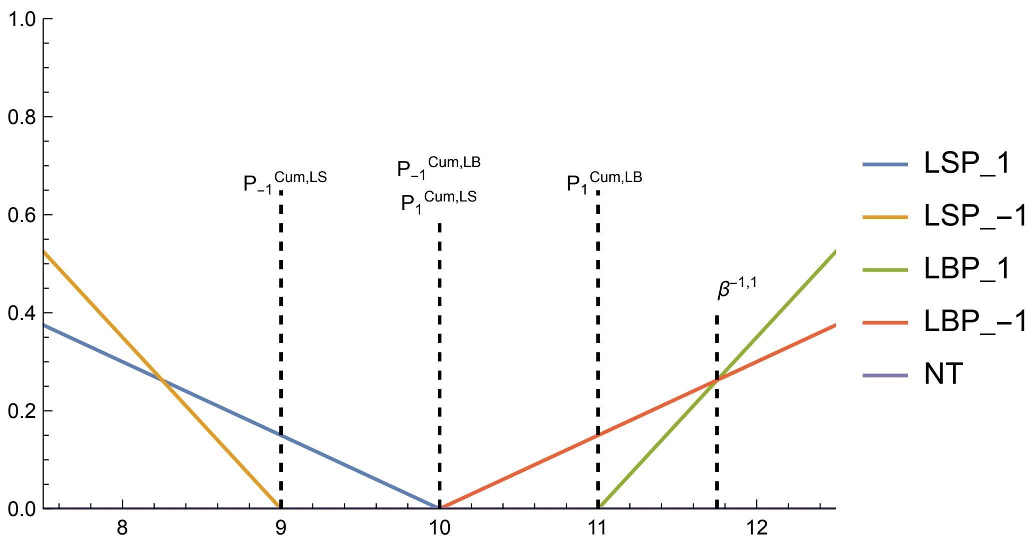


Figure 3: 2-Period Model with Profit-Maximizing Exchange. Unrestricted fee pricing regime. This figure reports the equilibrium make fees (MF) and take fees (TF) corresponding to different investor valuation supports with widths ranging from 0.33 to 21 on the horizontal axes. The support is expressed in tick unit of measure (τ). The figure reports in blue (red) the equilibrium fees MF (TF). The Taker-Maker and Maker-Taker pricing structures are optimal and symmetric for a support widths ranging from 0.33 to 3. We report the Maker-Taker pricing: for example, in correspondence of the smallest support we consider, 0.33, the figure reports the equilibrium symmetric Maker-Taker MF and TF, -0.444 and 0.556. For supports larger than 3, fee pricing also include positive MF and TF. The figure also reports the sum of MF and TF, Total Fee (green), as well as the Transaction Probability (orange) and the Exchange Profit (gray).

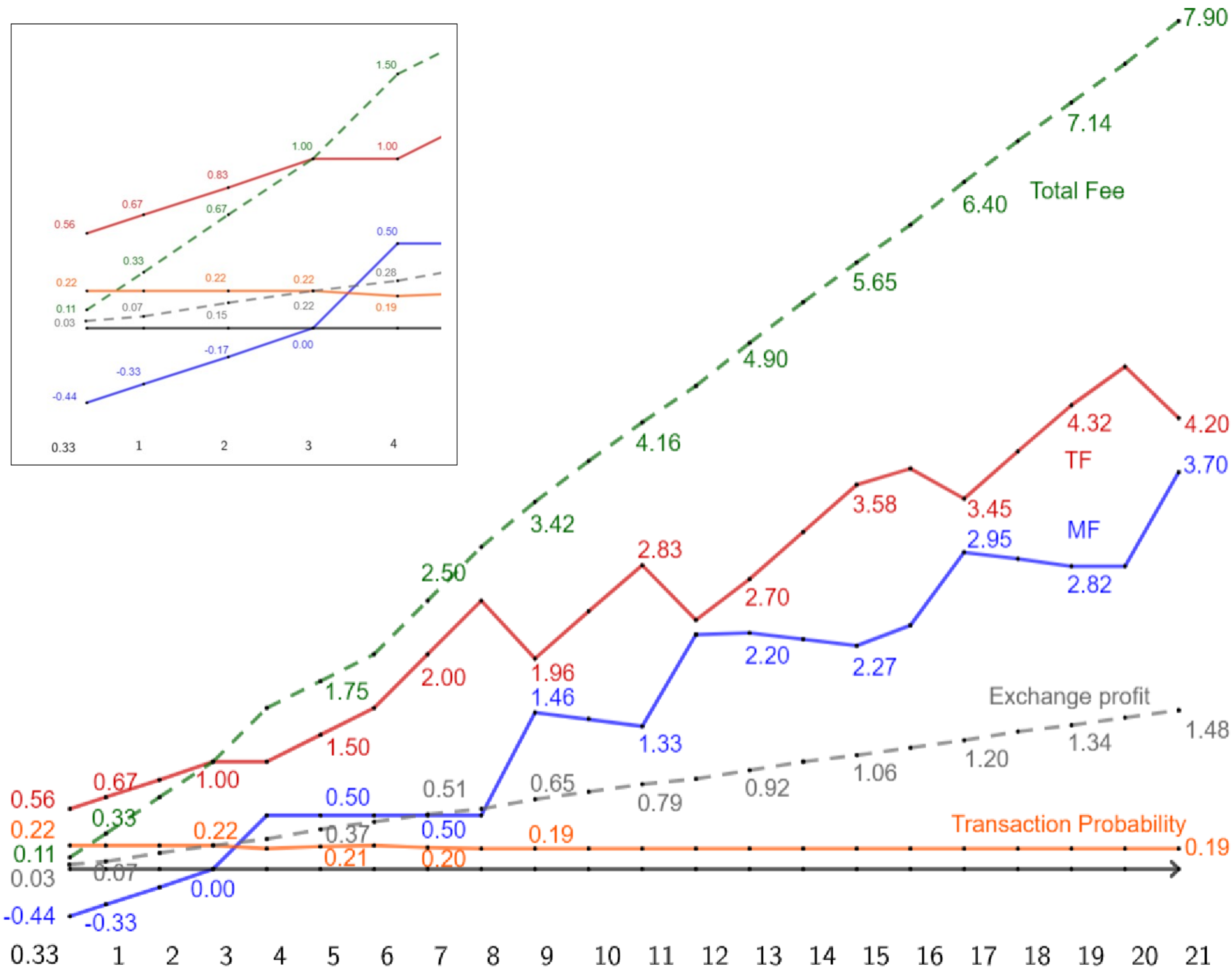


Table 1: 2-Period Model with Profit-Maximizing Exchange: Equilibrium Fees and Trading Strategies. Unrestricted fee pricing regime. This table reports for different investor valuation support width, $\Delta = \overline{\beta} - \underline{\beta}$ expressed in terms of the tick size, τ (column 1), the extreme values of the support, $\underline{\beta}$ and $\bar{\beta}$ (column 2), the equilibrium make and take fees, MF^* and TF^* (column 3 and 4), the sum of the equilibrium MF^* and TF^* (column 5), the cum-fee buy and sell prices $P_k^{cum,LB}$ and $P_k^{cum,MS}$ (columns 6 and 7), the thresholds between NT and LBP_{-2}, LBP_{-2} and LBP_{-1}, LBP_{-1} and LBP_1, LBP_1 and LBP_2 (column 8), the buyer's equilibrium trading strategies at t_1, x_{t_1} other than No Trade (column 9) and the associated probability of submission at t_1, $Pr(x_{t_1}|\Xi^*, L_{t_0})$ (column 10). The table also shows the equilibrium probability of execution of the buyer's order posted at t_1, $Pr(\theta_{t_1}^{x_{t_1}}|\Xi^*, L_{t_0})$ (column 11), the equilibrium transaction probability $Pr(x_{t_z}, \theta_{t_z}^{x_{t_z}}|S, \tau, \Xi)$ (column 12), and the exchange expected profit from both buyers and sellers, $\pi^{Ex}(MF^*, TF^*)$ (column 13). Up to a support equal to 3 the equilibrium pricing is symmetric, with both Taker-Maker and Maker-Taker pricing, and we report only the Maker-Taker pricing. Results are rounded to the third decimal.

Support width $\Delta = \overline{\beta} - \underline{\beta}$	$\underline{\beta}, \bar{\beta}$	MF^*	TF^*	MF^*+TF^*	$P_k^{cum,LB}$	$P_k^{cum,MS}$	Threshold	Eq.Strategy x_{t_1} at t_1	Pr. Submission $Pr(x_{t_1}\|\Xi^*, L_{t_0})$	Pr. Execution $Pr(\theta_{t_1}^{x_{t_1}}\|\Xi^*, L_{t_0})$	Pr.Trans	Exchange E[Profit] $\pi^{Ex}(MF^*, TF^*)$
0.33	9.833, 10.167	-0.444	0.556	0.111	10.056	9.944	n.a, n.a, 10.056,11.556	LBP_1	0.333	0.333	0.222	0.025
1	9.500, 10.500	-0.333	0.667	0.333	10.167	9.833	n.a, n.a, 10.167,11.667	LBP_1	0.333	0.333	0.222	0.074
2	9.000, 11.000	-0.167	0.833	0.666	10.333	9.667	n.a, n.a, 10.333,12.000	LBP_1	0.333	0.333	0.222	0.148
3	8.500, 11.500	0	1	1	10.5	9.5	n.a, n.a, 10.500,12.500	LBP_1	0.333	0.333	0.222	0.222
3.1	8.450, 11.550	0	1.05	1.05	10.5	9.45	n.a, n.a, 10.500,12.500	LBP_1	0.339	0.323	0.219	0.229
3.5	8.250, 11.750	0	1.25	1.25	10.5	9.25	n.a, n.a, 10.500,12.500	LBP_1	0.357	0.285	0.204	0.255
3.6	8.200, 11.800	0	1.30	1.30	10.5	9.20	n.a, n.a, 10.500,12.500	LBP_1	0.361	0.278	0.201	0.261
3.8	8.100, 11.900	0	1.40	1.40	10.5	9.10	n.a, n.a, 10.500,12.500	LBP_1	0.368	0.263	0.194	0.271
3.9	8.050, 11.950	0	1.45	1.45	10.5	9.05	n.a, n.a, 10.500,12.500	LBP_1	0.372	0.256	0.191	0.276
4	8.000, 12.000	0.5	1	1.5	10.000, 11.000	8.500, 9.500	n.a, 10.000, 11.500, 13.500	LBP_{-1} , LBP_1	0.375, 0.125	0.125, 0.375	0.188	0.281
4.1	7.950, 12.050	0.5	1.025	1.525	10.000, 11.000	8.475, 9.475	n.a, 10.000,11.525,13.525	LBP_{-1} , LBP_1	0.372, 0.128,	0.128, 0.372	0.191	0.291
4.4	7.800, 12.200	0.5	1.1	1.6	10.000, 11.000	8.400, 9.400	n.a, 10.000,11.600,13.600	LBP_{-1} , LBP_1	0.364, 0.136	0.136, 0.364	0.198	0.317
4.5	7.750, 12.250	0.5	1.125	1.625	10.000, 11.000	8.375, 9.375	n.a, 10.000,11.625,13.625	LBP_{-1} , LBP_1	0.361, 0.139	0.139, 0.361	0.201	0.326
4.7	7.650, 12.350	0.5	1.175	1.675	10.000 , 11.000	8.325 , 9.325	n.a, 10.000, 11.675, 13.675	LBP_{-1} , LBP_1	0.356, 0.144	0.144, 0.356	0.205	0.343
4.8	7.600, 12.400	0.5	1.2	1.7	10.000, 11.000	8.300, 9.300	n.a, 10.000,11.700,13.700	LBP_{-1} , LBP_1	0.354, 0.146,	0.146,0.354	0.207	0.351
5	7.500, 12.500	0.5	1.25	1.75	10.000 , 11.000	8.250 , 9.250	n.a, 10.000,11.750,13.750	LBP_{-1} , LBP_1	0.350, 0.150	0.150 , 0.350	0.21	0.368
6	7.000, 13.000	0.5	1.5	2	10.000 , 11.000	8.000 , 9.000	n.a, 10.000,12.000,14.000	LBP_{-1} , LBP_1	0.333, 0.167	0.167, 0.333	0.222	0.444
7	6.500, 13.500	0.5	2	2.5	10.000 , 11.000	7.500 , 8.500	n.a, 10.000,12.000,14.000	LBP_{-1} , LBP_1	0.286, 0.214	0.143, 0.286	0.204	0.51
8	6.000, 14.000	0.5	2.5	3	10.000 , 11.000	7.000 , 8.000	n.a, 10.000,12.000,14.000	LBP_{-1} , LBP_1	0.250 , 0.250	0.125, 0.250	0.188	0.563
8.8	5.600, 14.400	1.468	1.868	3.336	9.968 , 10.968, 11.968	6.632 , 7.632 , 8.632	9.968,12.000,14.000,16.000	LBP_{-2} , LBP_{-1}, LBP_1	0.227, 0.227, 0.045	0.117 , 0.231, 0.345	0.190	0.632
9	5.500, 14.500	1.458	1.958	3.416	9.958 , 10.958, 11.958	6.542 , 7.542, 8.542	9.958,12.000,14.000,16.000	LBP_{-2} , LBP_{-1}, LBP_1	0.222, 0.222, 0.056 ,	0.116, 0.227, 0.338	0.19	0.649
12	4.000, 16.000	2.184	2.315	4.499	9.684, 10.684 , 11.684, 12.684	5.185 , 6.185, 7.185, 8.185	10.684,13.869,15.869,17.869	LBP_{-3} , LBP_{-2}, LBP_{-1}, , LBP_1	0.156, 0.167, 0.167, 0.011	0.099 , 0.182, 0.265, 0.349	0.190	0.844

Figure 4: 2-Period Model with Social Planner: Equilibrium Fees, Exchange Profit and Transaction Probability. This figure reports the equilibrium make fees (MF) and take fees (TF) corresponding to different investor valuation supports with widths ranging from 0.33 to 21 on the horizontal axes. The support is expressed in tick unit of measure (τ). The figure reports in blue (red) the equilibrium fees MF (TF). The Taker-Maker and Maker-Taker pricing structures are optimal and asymmetric. The figure also report Total Welfare (green) which is the sum of investors welfare and exchange profit, and Transaction Probability (orange).

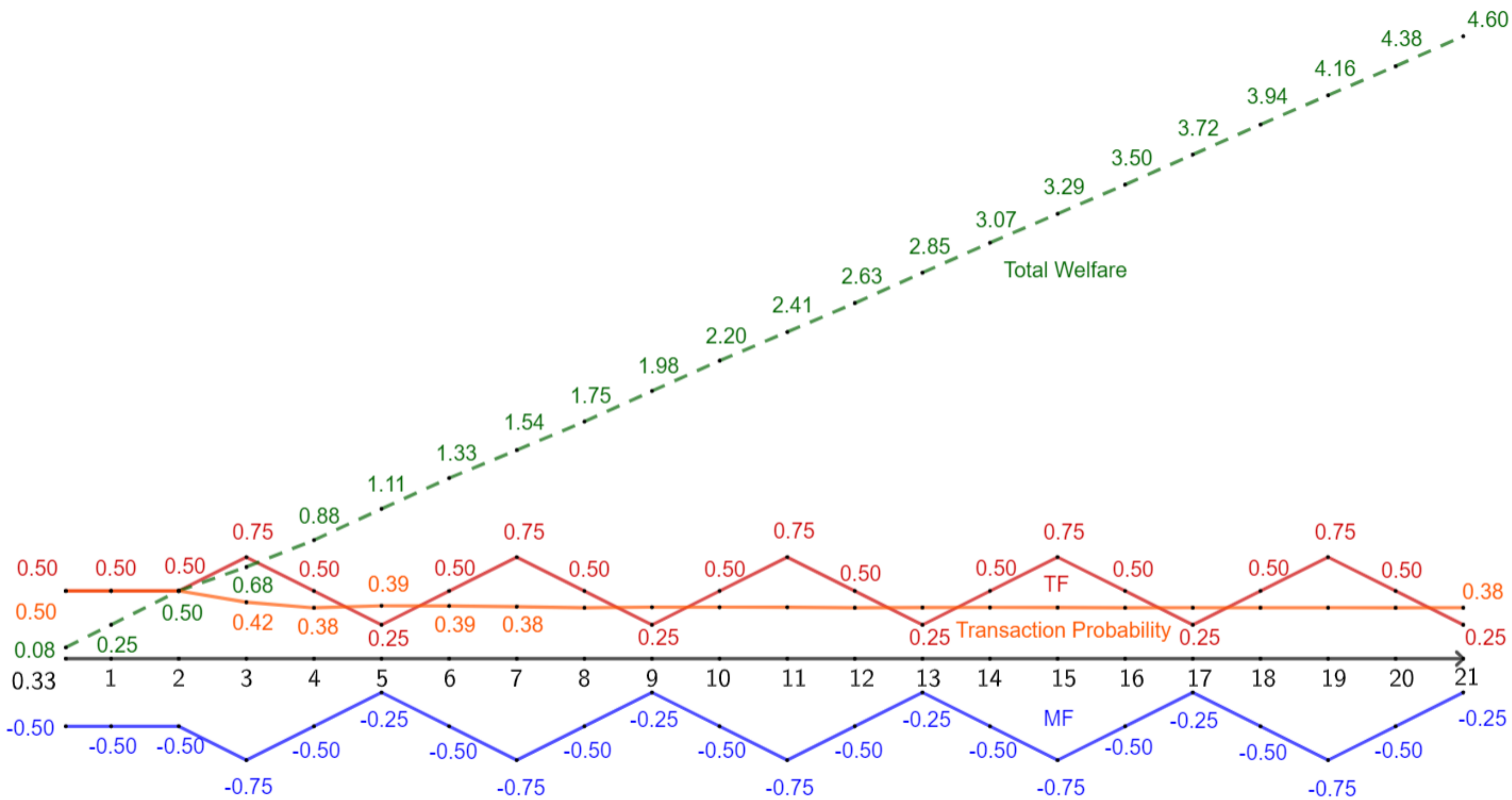


Table 2: 2-Period Model with Social Planner: Equilibrium Fees and Trading Strategies. This table reports for different investor valuation support width, $\Delta = \overline{\beta} - \underline{\beta}$ expressed in terms of τ (column 1), the extreme values of the support, $\underline{\beta}$ and $\bar{\beta}$ (column 2), the equilibrium make and take fees, MF and TF (column 3 and 4), the cum-fee buy prices $P_k^{cum,LB}$ (column 5), the thresholds (column 6), the buyer's equilibrium trading strategies at t_1, x_{t_1} other than No Trade (column 7) and the associated probability of submission and execution (column 8 and 9). Finally the table reports in column 10 the total welfare which is the sum of investors welfare and exchange profit. Results are rounded to the third decimal.

Support width $\Delta = \overline{\beta} - \underline{\beta}$	$\underline{\beta}, \bar{\beta}$	MF	TF	$P_k^{cum,LB}$	Thresholds	Eq.Strategy	Pr. Submission	Pr. Execution	Pr. Trans	Total Welfare
0.33	9.833,10.167	-0.500	0.500	8.000, 9.000, 10.000	n.a, n.a, 10.000	LBP_1	0.000, 0.000, 0.500	0.000, 0.000, 0.500	0.500	0.083
1	9.500, 10.500	-0.5	0.5	8.000, 9.000, 10.000	n.a, n.a, 10.000	LBP_1	0.000, 0.000, 0.500	0.000, 0.000, 0.500	0.500	0.250
2	9.000, 11.000	-0.5	0.5	8.000, 9.000, 10.000	n.a, n.a, 10.000	LBP_1	0.000, 0.000, 0.500	0.000, 0.000, 0.500	0.500	0.500
2.1	8.950, 11.050	-0.525	0.525	7.975, 8.975, 9.975	n.a., 8.975, 10.000	LBP_1	0.000, 0.000, 0.500	0.000, 0.012, 0.488	0.488	0.519
3	8.500, 11.500	-0.75	0.75	7.750, 8.750, 9.750	n.a., 8.750, 10.000	LBP_1	0.000, 0.000, 0.500	0.000, 0.033, 0.417	0.417	0.677
3.1	8.450, 11.550	-0.725	0.725	7.775, 8.775, 9.775	n.a., 8.775, 10.100	LBP_{-1}, LBP_1	0.000, 0.032, 0.468	0.000, 0.105, 0.427	0.407	0.694
4	8.000, 12.000	-0.5	0.5	8.000, 9.000, 10.000	n.a., 9.000, 11.000	LBP_{-1}, LBP_1	0.000, 0.250, 0.250	0.000, 0.250, 0.500	0.375	0.875
4.1	7.950, 12.050	-0.475	0.475	8.025, 9.025, 10.025	8.025, 9.100, 11.100	LBP_{-1}, LBP_1	0.000, 0.268, 0.232	0.018, 0.262, 0.506	0.375	0.885
4.5	7.750, 12.250	-0.325	0.325	8.125, 9.125, 10.125	8.175, 9.600, 11.600	LBP_{-1}, LBP_1	0.000, 0.333, 0.166	0.083, 0.306, 0.528	0.380	0.988
5	7.500, 12.500	-0.25	0.25	8.250, 9.250, 10.250	8.250, 10.000, 12.000	LBP_{-1}, LBP_1	0.000, 0.400, 0.100	0.150, 0.350, 0.550	0.390	1.106
5.1	7.450, 12.550	-0.275	0.275	8.225, 9.225, 10.225	8.225, 10.000, 12.000	LBP_{-1}, LBP_1	0.000, 0.108, 0.392	0.152, 0.348, 0.544	0.390	1.130
7	6.500, 13.500	-0.75	0.75	7.750, 8.750, 9.750	7.750, 10.000, 12.000	LBP_{-1}, LBP_1	0.000, 0.286, 0.214	0.179, 0.321, 0.464	0.383	1.540
7.1	6.450, 13.550	-0.725	0.725	7.775, 8.775, 9.775	7.775, 10.100, 12.100	LBP_{-2}, LBP_{-1}, LBP_1	0.014, 0.282, 0.204	0.187, 0.327, 0.468	0.381	1.130

Figure 5: 2-Period Model with Profit-Maximizing Exchange: Equilibrium Fees, Exchange Profit and Transaction Probability. Restricted fee pricing regime. This figure reports the equilibrium make fees (MF) and take fees (TF) corresponding to different investor valuation supports with widths ranging from 0.33 to 21 on the horizontal axes. In the restricted fee pricing model trading fees are capped to the tick size value which is set equal to 1τ. The support is expressed in tick unit of measure (τ). The figure reports in blue (red) the equilibrium fees MF (TF). The Taker-Maker and Maker-Taker pricing structures are optimal and symmetric for a support widths ranging from 0.33 to 3. We report the Maker-Taker pricing: for example, in correspondence of the smallest support we consider, 0.33, the figure reports the equilibrium symmetric Maker-Taker MF and TF, -0.444 and 0.556. For supports larger than 3, fee pricing also include positive MF and TF. The figure also reports the sum of MF and TF, Total Fee (green), as well as the Transaction Probability (orange) and the Exchange Profit (gray).

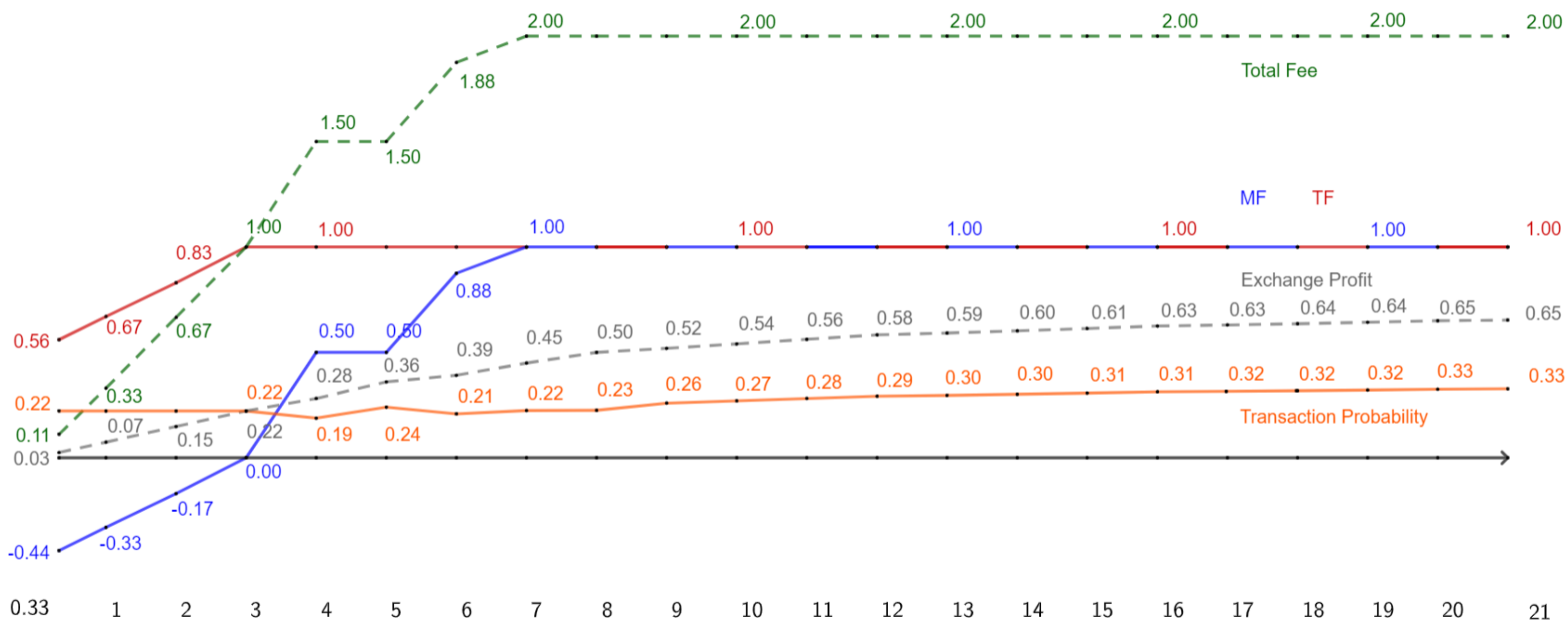


Figure 6: 3-Period Model with Profit-Maximizing Exchange. Unrestricted fee pricing regime. This figure reports the equilibrium make fees (MF) and take fees (TF) corresponding to different investor valuation supports with widths ranging from 0.33 to 21 on the horizontal axes. The support is expressed in tick unit of measure (τ). The figure reports in blue (red) the equilibrium fees MF (TF). For supports ≤ 2.4 , the Taker-Maker and Maker-Taker pricing structures are optimal and asymmetric. We report the Maker-Taker pricing: for example, in correspondence of the smallest support we consider, 0.33, the figure reports the equilibrium asymmetric Maker-Taker MF and TF, -0.428 and 0.556. For supports ≥ 2.4, fee pricing also include positive MF and TF. The figure also reports the sum of MF and TF, Total Fee (green), as well as the Transaction Probability (orange) and the Exchange Profit (gray).

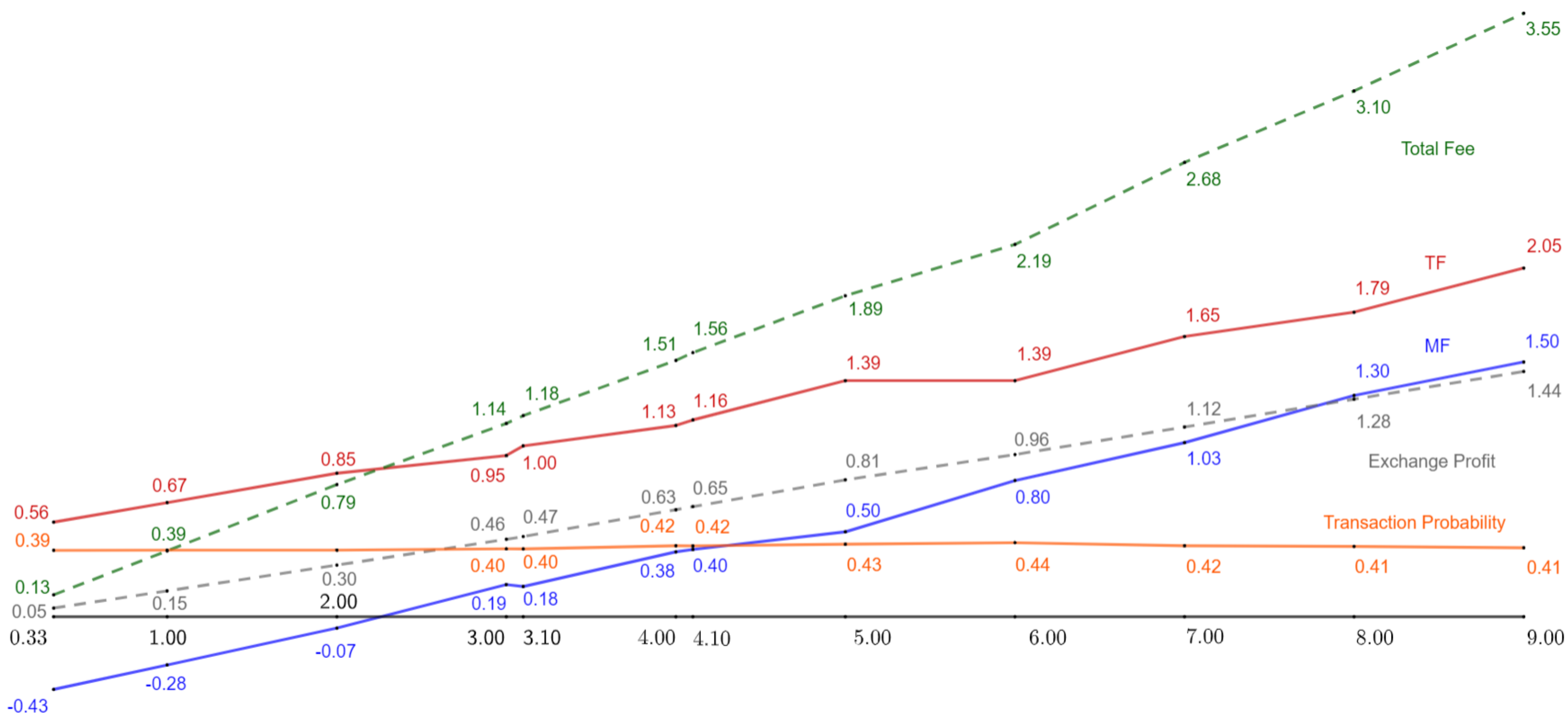


Table 3: 3-Period Model with Profit-Maximizing Exchange: Equilibrium Fees and Trading Strategies. Unrestricted fee pricing regime. This table reports for different investor valuation support width, $\Delta = \bar{\beta} - \underline{\beta}$ expressed in terms of the tick size, τ (column 1), the extreme values of the support, $\underline{\beta}$ and $\bar{\beta}$ (column 2), the equilibrium make and take fees, MF^* and TF^* (column 3 and 4), the sum of the equilibrium MF^* and TF^* (column 5), the thresholds (column 6), the buyer's equilibrium trading strategies at t_1 and at t_2 other than No Trade (columns 7 and 8) and the associated probability of submission at t_1 and at t_2, (columns 9 and 10). The table also shows the equilibrium transaction probability (column 11), and the exchange expected profit (column 12). For supports $\leq 2.4\tau$ the equilibrium pricing is asymmetric, with both Taker-Maker and Maker-Taker pricing, and we report only the Maker-Taker pricing. The third and fourth gray rows report results (marked with a *) for off-equilibrium fees that symmetrically flip the corresponding equilibrium fees. When the equilibrium pricing is rebate-based for a given support, we report the Taker-Maker fees on the first row and then the Maker-Taker fees on the second row. For supports ≤ 2.4 the equilibrium pricing is asymmetric, with both Taker-Maker and Maker-Taker pricing. When, for a given support and set of fees, there are multiple optimal orders given different valuations β_{t_1} for the investor at t_1, these orders are shown on different rows along with the optimal potential responses at t_2. Results are rounded to the third decimal.

Support width $\Delta = \bar{\beta} - \underline{\beta}$	$\underline{\beta}, \bar{\beta}$	MF^*	TF^*	MF^*+TF^*	Threshold	Eq.Strategies x_{t_z} $Pr(x_{t_z} \vert \Xi^*, L_{t_{z-1}})$		Pr. Submission		Pr. Trans	Exchange E[Profit] $\pi^{Ex}(MF^*, TF^*)$
						t_1	t_2	t_1	t_2		
0.33	9.833, 10.167	0.572	-0.443	0.129	10.072	LBP_{-1}	MSP_{-1}	0.284	0.328	0.391	0.051
		-0.428	0.557	0.129	10.072	LBP_1	MSP_1	0.284	0.328	0.391	0.051
1	9.500, 10.500	0.716	-0.328	0.388	10.216	LBP_{-1}	MSP_{-1}	0.284	0.328	0.392	0.152
		-0.284	0.672	0.388	10.216	LBP_1	MSP_1	0.284	0.328	0.392	0.152
2	9.000, 11.000	0.933	-0.156	0.777	10.433	LBP_{-1}	MSP_{-1}	0.284	0.328	0.392	0.304
		-0.067	0.845	0.777	10.433	LBP_1	MSP_1	0.284	0.328	0.392	0.304
2.31	8.850, 11.150	1.000	-0.102	0.898	10.500	LBP_{-1}	MSP_{-1}	0.284	0.328	0.392	0.351
		-0.001	0.898	0.898	10.500	LBP_1	MSP_1	0.284	0.328	0.392	0.351
3	8.500, 11.500	0.189	0.949	1.138	10.000	LBP_{-1}	MSP_{-1}, LSP_2	0.241	0.341, 0.270	0.400	0.456
					10.723	LBP_1	MSP_1	0.259	0.350		
4	8.000, 12.000	0.382	1.126	1.508	10.000	LBP_{-1}	LBP_1, MSP_{-1}, LSP_2	0.244	0.030, 0.324, 0.205	0.418	0.630
					11.131	LBP_1	LBP_2, MSP_1	0.256	0.030, 0.344		
					10.000	LBP_{-1}	LBP_1, MSP_{-1}, LSP_2	0.246	0.038, 0.320, 0.206		
5	7.500, 12.500	0.500	1.390	1.890	10.000	LBP_{-1}	LSP_2, MSP_{-1}, LBP_1	0.236	0.203, 0.297, 0.010	0.427	0.806
					11.405	LBP_1	LBP_2, MSP_1	0.263	0.100, 0.322		
6	7.000, 13.000	0.802	1.390	2.192	10.000	LBP_{-1}	LBP_1, LSP_2, LSP_1, MSP_{-1},	0.289	0.283, 0.185, 0.306, 0.125	0.436	0.955
					10.437	LBP_{-2}	LBP_{-1}, LBP_1, LSP_2, LSP_1, , LSP_{-1}	0.058	0.346, 0.098, 0.125, 0.333, 0.098		
					12.189	LBP_1	LBP_2, MSP_1, LSP_2,	0.153	0.116, 0.346, 0.270		
7	6.500, 13.500	1.025	1.650	2.675	10.655	LBP_{-1}	LBP_1, LSP_2, LSP_1, MSP_{-1},	0.264	0.282, 0.193, 0.238, 0.138	0.418	1.117
					10.000	LBP_{-2}	LBP_{-1}, LBP_1, LSP_2, LSP_1, LSP_{-1}	0.094	0.336, 0.089, 0.193, 0.286, 0.089		
					12.505	LBP_1	LBP_2, MSP_1, LSP_2,	0.142	0.139, 0.323, 0.244		
8	6.000, 14.000	1.303	1.793	3.096	11.033	LBP_{-1}	LBP_1, LSP_2, LSP_1, MSP_{-1},	0.259	0.274, 0.213, 0.148, 0.163	0.414	1.281
					10.000	LBP_{-2}	LBP_{-1}, LBP_1, LSP_2, LSP_1, LSP_{-1}	0.129	0.338, 0.061, 0.213, 0.250, 0.061		
					13.108	LBP_1	LBP_2, MSP_1, LSP_2,	0.111	0.150, 0.320, 0.204		
9	5.500, 14.500	1.500	2.053	3.553	11.405	LBP_{-1}	LBP_1, LSP_2, LSP_1, MSP_{-1},	0.233	0.278, 0.216, 0.114, 0.169	0.406	1.444
					10.000	LBP_{-2}	LBP_{-1}, LBP_1, LSP_2, LSP_1, LSP_{-1}	0.156	0.327, 0.061, 0.216, 0.222, 0.040		
					13.510	LBP_1	LBP_2, MSP_1, LSP_2,	0.110	0.167, 0.307, 0.193		

Figure 7: 3-Period Model with Profit-Maximizing Exchange. Restricted fee pricing regime. This figure reports the equilibrium make fees (MF) and take fees (TF) corresponding to different investor valuation supports with widths ranging from 0.33 to 21 on the horizontal axes. The support is expressed in tick unit of measure (τ). The figure reports in blue (red) the equilibrium fees MF (TF). The Taker-Maker and Maker-Taker pricing structures are optimal and asymmetric for small support widths. We report the Maker-Taker pricing: for example, in correspondence of the smallest support we consider, 0.33, the figure reports the equilibrium symmetric Maker-Taker MF and TF, -0.428 and 0.556. For large supports, fee pricing also include positive MF and TF. The figure also reports the sum of MF and TF, Total Fee (green), as well as the Transaction Probability (orange) and the Exchange Profit (gray). In the restricted fee pricing model trading fees are capped to the tick size value which is set equal to 1τ. The support is expressed in tick unit of measure (τ).

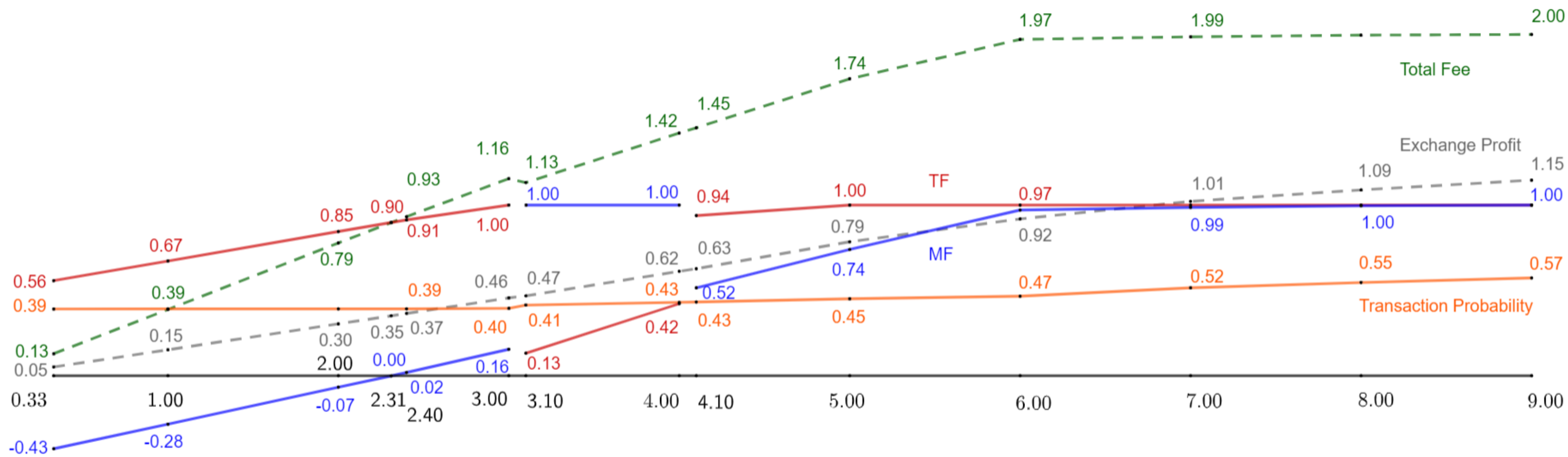
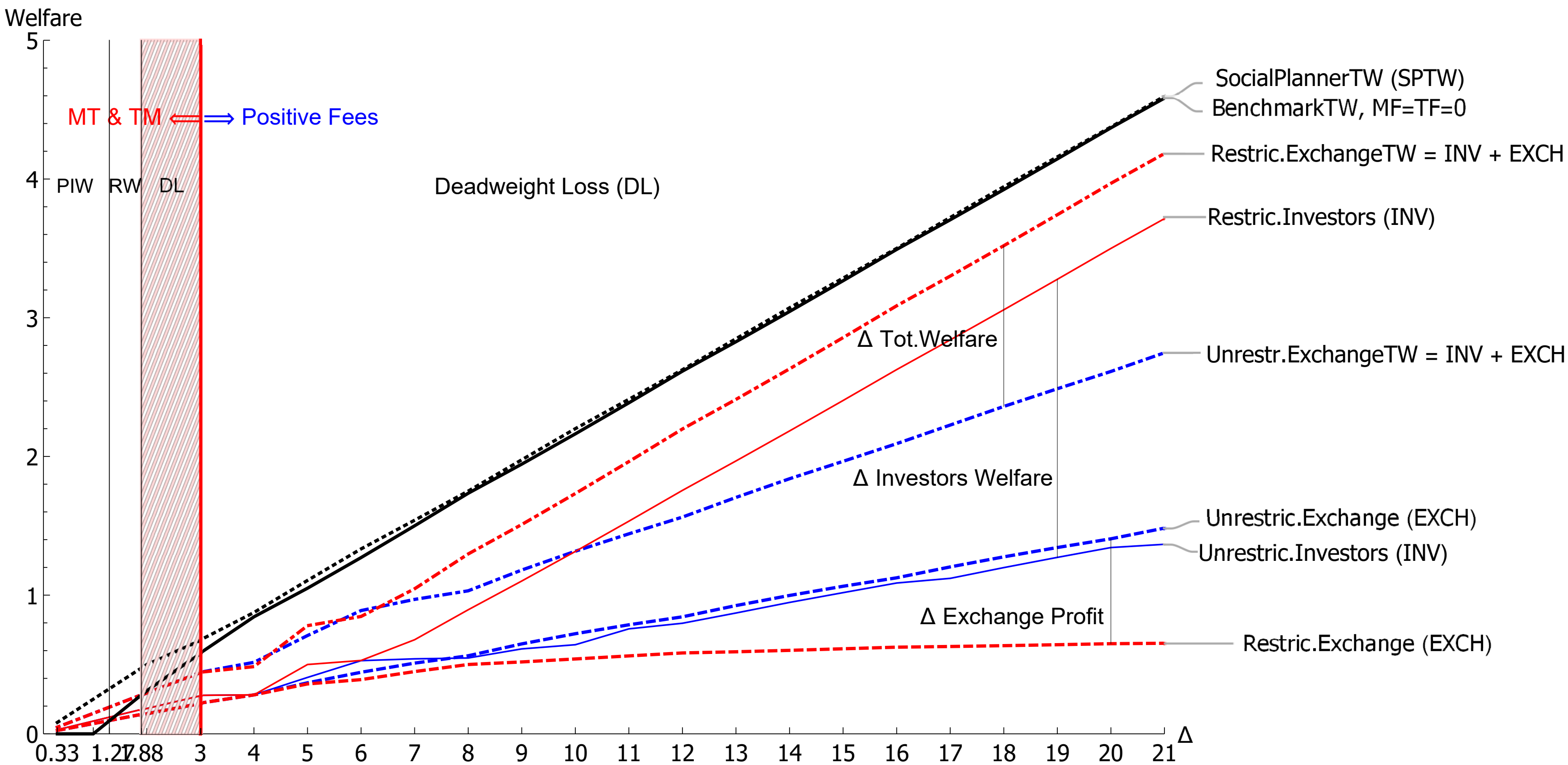


Figure 8: Welfare Comparison This figure shows how the following variables evolve with the support width Δ. For both the restricted (red) and the unrestricted (blue) exchange fee pricing regime the figure reports: profit of the exchange (EXCH - dashed line), investors welfare , (INV - thin line), total welfare (ExchangeTW=INV + EXCH - dashed-dotted line). In addition the figure reports the social planner total welfare (SocialPlannerTW,SPTW - black dotted line) and the benchmark regime with no trading fees (BenchmarkTW, MF=TF=0 - - black solid thick line). Finally, the figure shows the results for three regions: Pareto Improvement Welfare (PIW), Redistribution Welfare (RW) and Deadweight Loss (DL); and also report the difference between the Restricted and the Unrestricted regime in total welfare ($\Delta Tot.Welfare$), in investors welfare ($\Delta InvestorsWelfare$), and in exchange profit ($\Delta ExchangeProfit$). The support is expressed in tick unit of measure (τ).



Appendices

A Proofs

Lemma 3. *Given fee pricing such that the exchange has a non-negative profit $MF + TF \geq 0$ per transaction, a priori arriving investors only consider submitting limit buys (sells) at posted prices P_k that are sufficiently low (high) such that the associated cum-fee prices for market sells (buys) satisfies $P_k^{cum,MS} < \overline{\beta}$ ($P_k^{cum,MB} > \underline{\beta}$).*

Proof for Lemma 3: : This follows immediately from the fact that submitting a limit buy is only profitable for an investor if the cum-fee limit-buy price $P_k^{cum,LB} = P_k + MF \leq \overline{\beta}$ and the fact that $P_k^{cum,MS} = P_k - TF = P_k + MF - (MF + TF)$ where $MF + TF \geq 0$ for an exchange with a non-negative profit per transaction and, thus, the cum-fee market sell price satisfies $P_k^{cum,MS} \leq \overline{\beta}$. The argument for limit sells is symmetric. Q.E.D.

Lemma 4. *If the standing limit order book is symmetric at a time t_z, then investors with $\beta_{t_z} > v$ are potential buyers at time t_z (i.e., they either submit limit buy orders or NT but they never submit limit sell orders). Similarly, investors with $\beta_{t_z} < v$ are potential sellers at time t_z. In particular, this is true at time t_1 when the opening book L_{t_0} is empty.*

Proof of Lemma 4: This result follows from the fact that the investor expected profit functions from limit buy and sell orders are symmetric and increasing in the distance of β_{t_z} from the posted limit prices. Q.E.D.

Comment about Lemma 4: In particular, Lemma 4 applies at time t_1 since the initial book is empty.

Lemma 5. *If trading fees are capped at one tick by regulation, then an exchange never sets rebates larger than one tick in equilibrium.*

Proof for Lemma 5 If the trading rebate is larger than the trading fee, then the exchange's net profit per transaction is negative. However, an exchange can always earn a zero net profit per transaction by setting its rebates equal to its fee. Thus setting rebates larger than fees does not maximize exchange profits (if profit-maximizing exchanges set access pricing) and violates incentive compatibility (if a welfare-maximizing Social Planner sets fee pricing). Q.E.D.

Lemma 6. *Consider limit buys at prices P_j and $P_k > P_j$, which both have positive execution probabilities and positive execution payoffs given hypothetical fees MF and TF. With uniformly distributed private values β_{t_2} at time t_2, the threshold for limit buys at P_j versus P_k on a discrete price grid is*

$$\beta_{t_1}^{LB,j,k} = P_j + P_k - \underline{\beta} - TF + MF \tag{30}$$

and the corresponding threshold for limit sells at P_j and $P_k < P_j$ is

$$\beta^{LS,k,j} = P_j + P_k - \overline{\beta} + TF - MF \tag{31}$$

Proof of Lemma 6: Given hypothetical take and make fees TF and MF, consider two limit buy orders posted at P_j and $P_k > P_j$, where $P_k^{MS,cum} \in (\underline{\beta}, \overline{\beta})$ and $P_j^{MS,cum} \in (\underline{\beta}, \overline{\beta})$, so that there is a positive well-defined probability of investors arriving at t_2 who would be willing to submit a market sell MSP_j and MSP_k to hit these limit buys. Define $n_{j,k} = P_k - P_j > 0$, which is the integer number of ticks between P_j and P_k. Using the above notation, we have $P_k^{MS,cum} = P_k - TF = P_j + n_{j,k} - TF$ and $P_k^{LB,cum} = P_k + MF = P_j + n_{j,k} + MF$. In addition, we can write $P_k^{LB,cum} = P_j^{LB,cum} + n_{j,k}$ and $P_k^{MS,cum} = P_j^{MS,cum} + n_{j,k}$.

We now solve for the critical value $\beta^{LB,j,k}$ of the private valuation β_{t_1} such that the expected profit of a limit buy at P_k is greater than for a limit buy at P_j:

$$\begin{aligned}
\pi_{t_1}^{INV}(LBP_k) = \frac{P_k^{MS,cum} - \underline{\beta}}{\overline{\beta} - \underline{\beta}}(\beta_{t_1} - P_k^{LB,cum}) &\geq \frac{P_j^{MS,cum} - \underline{\beta}}{\overline{\beta} - \underline{\beta}}(\beta_{t_1} - P_j^{LB,cum}) = \pi_{t_1}^{INV}(LBP_j) \quad (32)\\
\rightarrow [(P_k^{MS,cum} - \underline{\beta}) - (P_j^{MS,cum} - \underline{\beta})]\,\beta_{t_1} &\geq (P_k^{MS,cum} - \underline{\beta})\,P_k^{LB,cum} - (P_j^{MS,cum} - \underline{\beta})\,P_j^{LB,cum}\\
\rightarrow n_{j,k}\,\beta_{t_1} &\geq n_{j,k}P_j^{LB,cum} + (P_j^{MS,cum} - \underline{\beta} + n_{j,k})\,n_{j,k}\\
\rightarrow \beta_{t_1} &\geq P_j^{LB,cum} + (P_k^{MS,cum} - \underline{\beta})\\
\rightarrow \beta_{t_1} &\geq P_j + [(P_k - \underline{\beta}) - TF] + MF
\end{aligned}$$

Thus, the threshold $\beta^{j,k}$ for limit buys is as given in (30).

Similar calculations give the threshold $\beta_{k,j}$ for limit sells at P_j and $P_k < P_j$ where $n_{k,j} = P_j - P_k < 0$ is the signed integer number of price ticks between P_k and P_j:

$$\begin{aligned}
\pi_{t_1}^{INV}(LSP_k) = \frac{\overline{\beta} - P_k^{MB,cum}}{\overline{\beta} - \underline{\beta}}(P_k^{LS,cum} - \beta_{t_1}) &\geq \frac{\overline{\beta} - P_j^{MB,cum}}{\overline{\beta} - \underline{\beta}}(P_j^{LS,cum} - \beta_{t_1}) = \pi_{t_1}^{INV}(LSP_j) \quad (33)\\
\rightarrow [(\overline{\beta} - P_j^{MB,cum}) - (\overline{\beta} - P_k^{MB,cum})]\,\beta_{t_1} &\geq (\overline{\beta} - P_j^{MB,cum})\,P_j^{LS,cum} - (\overline{\beta} - P_k^{MB,cum})\,P_k^{LS,cum}\\
\rightarrow n_{k,j}\,\beta_{t_1} &\geq n_{k,j}P_j^{LS,cum} - (\overline{\beta} - P_j^{MB,cum} - n_{k,j})\,n_{k,j}\\
\rightarrow \beta_{t_1} &\leq P_j^{LS,cum} - (\overline{\beta} - P_k^{MB,cum})\\
\rightarrow \beta_{t_1} &\leq P_j + [(P_k - \overline{\beta}) + TF] - MF
\end{aligned}$$

so that, the threshold $\beta^{LS,j,k}$ for limit sells is given in (31). Q.E.D.

Proof of Lemma 2: The subgame existence result follows from the following construction: First, fix a finite support width Δ. It can be arbitrarily large or small relative to the normalized tick size of 1. Second, for any hypothetical take and make fees TF and MF, market-order cum prices can be calculated analytically and thus, limit-order execution probabilities and limit-order expected profits can be determined analytically for each possible limit order price in the a priori relevant portion of the price grid. Third, given the limit-order expected profit functions in β_{t_1}, the limit-order thresholds can be computed analytically from the upper envelope of the investor expected profit functions for the different hypothetical limit orders at the different hypothetical posted prices and, thus, the associated limit-order submission strategy can be determined analytically. Fourth, the upper envelope for limit buy orders starts with the limit order with the lowest cum-fee price above v with positive expected profits or at the limit order with the maximum expected profit for $\beta_{t_1} = v$ given that, from Lemma 4, limit buys are only used when $\beta_{t_1} > v$. The envelope then continues to successively higher limit buys since limit buy thresholds from Lemma 6 are sequential (i.e., does not skip orders). In particular, we know in the construction of the upper envelope that P_{j+1} is the next optimal limit buy price after P_j rather than some higher price P_k because, holding price P_j fixed, we have from (30) above that $\beta_{t_1}^{j,j+1} < \beta_{t_1}^{j,k}$ for $k > j+1$. Thus, the interval for which LBP_{j+1} is optimal extends from $\beta_{t_1}^{j,j+1}$ to $\beta_{t_1}^{j+1,j+2}$ and so on. Q.E.D.

Comment on order-submission probabilities in 2-period equilibrium: If β is uniformly distributed, the order-submission probabilities for optimal limit buys at t_1 are just the width of the interval for which they are optimal divided by the length Δ of the full valuation support. To see this, consider first an arbitrary "interior" interval $[\beta_{t_1}^{prior(k),k}, \beta_{t_1}^{k,next(k)}]$ for $1 < k < m$. The thresholds, from above, are

$$\begin{aligned}
\beta_{t_1}^{prior(k),k} &= P_{prior(k)} + P_k - \underline{\beta} - TF + MF \\
\beta_{t_1}^{k,next(k)} &= P_k + P_{next(k)} - \underline{\beta} - TF + MF
\end{aligned} \tag{34}$$

Thus, the width of any "interior" interval for β_{t_1} is

$$\beta_{t_1}^{k,next(k)} - \beta_{t_1}^{prior(k),k} = next(k) - prior(k) = 2 \tag{35}$$

and the associated interval probability of $\beta_{t_1} \in [\beta_{t_1}^{prior(k),k}, \beta_{t_1}^{k,next(k)}]$ is $\frac{2}{\Delta}$. The construction for the upper envelope of limit sells is symmetric.

Lemma 7. *i) In a Taker-Maker regime with $-1 \leq TF < 0 \leq MF \leq 1$, limit orders are never posted at prices P_k outside of the interval $[\underline{\beta}-1, \bar{\beta}]$ for limit buys or outside of the interval $[\underline{\beta}, \bar{\beta}+1]$ for limit sells. ii) In a Maker-Taker regime with $-1 \leq MF < 0 \leq TF \leq 1$, limit orders are never posted at prices P_k outside of the interval $[\underline{\beta}, \bar{\beta}+1]$ for limit buys or outside of the interval $[\underline{\beta}-1, > \bar{\beta}]$ for limit sells. iii) In a positive-fee regime with $0 \leq TF \leq 1$ and $0 \leq MF \leq 1$, limit buy and sell orders are never posted at prices P_k outside of the interval $[\underline{\beta}, \bar{\beta}]$ for both limit buys or limit sells.*

Proof of Lemma 7: In a Taker-Maker regime, the highest possible cum price $P_k^{cum,MS} = P_k - TF$ for a market sell given a limit buy at a posted price $P_k < \underline{\beta} - 1$ is $P_k + 1 < \underline{\beta}$ given the bounded take rebate $-TF \leq 1$. Thus, no investor arriving at t_2 will be willing to submit a market sell given a limit buy at posted prices $P_k < \underline{\beta} - 1$. Similarly, the lowest possible cum price $P_k^{cum,LB} = P_k + MF$ for a limit buy at a posted price $P_k > \bar{\beta}$ is $P_k > \bar{\beta}$ given the non-negative fee $MF > 0$ in a Taker-Maker regime. As a result, no investor arriving at t_1 will be willing to post a limit buy at prices $P_k > \bar{\beta}$. A similar logic applies for the result for potential posted limit prices in the Maker-Taker regime and the positive-fee regime. Q.E.D.

A.1 Optimal fees for the Social Planner

Lemma 8. *The fees that maximize total welfare are such that if the optimal the take fee (rebate) $TF = A > 0$ ($TF = A < 0$), then the associated optimal make fee is a rebate (fee) $MF = -FT = -A < 0$ so that the net fee for the exchange is zero. This means that the total net fee is $TF + MF = 0$.*

Proof: Given an optimal take fee (or rebate) F, the Social Planner optimally set the make rebate (or fee) to maximize order submissions. Q.E.D.

Proof of Theorem 1: The proof starts given the trading subgame equilibrium in Lemma 2 for arbitrary MF and TF and then follows the following steps: First, from Lemma 8, if $TF = F$, the make fee is $MF = -F$ submission strategy can be determined analytically. Second, given the optimal limit order strategy at time t_1 and the associated optimal market-order submission strategy at time t_2, total social welfare in (12) can be computed analytically. Third, the optimal take fee F (and make rebate $-F$) is found by repeating these computations for each possible hypothetical fee F to find the fee F^* that maximizes (12). Fourth, the optimal fee function $F(\Delta)$ can then be constructed by repeating the above steps for each support width Δ.

From the preceding argument, the fee $F(\Delta)$ that maximizes the total welfare function is either the solution to a local optimization given the associated set of equilibrium "used" orders, or it is a fee that makes it incentive compatible for investors to use orders that are associated with a higher local social value function given that set of "used" orders than the set of "used"" orders associated with a next-best (i.e., adjacent) lower local social value function. Thus, thus the changes in $F(\Delta)$ given a change in Δ are due either to continuous changes in the maximum of the highest local social value function or to continuous changes in the threshold associated with some order than needs to be deterred. Q.E.D.

Proof of Theorem 3: The proof follows from rescaling all of the variables in the model relative to an absolute tick size $\tau > 0$. In particular, we define scaled quantities $\hat{MF} = MF/\tau$, $\hat{TF} = TF/\tau$, $\hat{\beta}_{t_z} = \beta_{t_z}/\tau$, $\hat{\bar{\beta}} = \bar{\beta}/\tau$, $\hat{\underline{\beta}} = \underline{\beta}/\tau$, and prices $\hat{P}_j = P_j/\tau$ for all h. Given this rescaling, we next observe that all of the order-submission and order-execution probabilities are homogeneous of degree zero in the absolute tick size τ. In particular, the absolute tick size factors out of both the numerator and denominator and cancels. Similarly, the conditional payoffs on executed orders for the exchange and investors is homogeneous of order one. Thus, changing the tick size does not change relative comparisons for prices centered around a given fundamental valuation v. The rescaled optimization problems for the investors and exhange are, therefore, equivalent to the optimization problem with a tick of 1. This gives us the solutions to the exchange's optimal scaled fees. Multiplying the scaled fees by τ gives the corresponding absolute fees $MF = \hat{M}F\tau$ and $MF = \hat{M}F\tau$. Q.E.D.

A.2 Proof for the 3-period market

Proof for Theorem 4: The proof strategy is standard for finite sequential games and consists of three steps: The recursion step for deriving analytic investor strategies is the following: Given access pricing fees Ξ, the order-execution probabilities $Pr(\theta_{t_z}^{x_{t_z}}|\Xi, L_{t_{z-1}})$ for computing the investor expected profit for each possible order $x_{t_z} \in X_{t_z}$ at any time t_z in the investor maximization problem (2) are either 1 for market orders at the BBO or are determined recursively for limit orders from the order-submission probabilities $Pr(x_{t_z}|\Xi, L_{t_{z-1}})$ at later dates. The upper envelope of the expected investor payoffs for the different possible actions at a generic time t_z determines the optimal investor actions at t_z and, given the distribution over the investor valuation β_{t_z} the associated order-submission probabilities for the optimal actions in terms of intervals on the investor valuation support S for any incoming book $L_{t_{z-1}}$. Given the assumptions of a discrete number of possible investor actions and discrete tine, the set of possible incoming books is finite.

The initiation step starts the recursion at the terminal period t_N, at which time the order-execution probabilities take a simple form: They are zero for new limit orders (since the game ends after time t_N) and one for market orders (which can only be submitted if the book is non-empty). Thus, investor expected profit for different orders, the upper envelope, the optimal orders, and the order-submission probabilities at time t_N can be derived directly.

The exchange profit optimization step is then as follows: The order-submission and order-execution probabilities from the first two steps can then be used to construct the exchange's expected profit in (26) analytically given arbitrary fees Ξ. Given the analytic exchange expected profit function, the profit-maximizing fees Ξ^* can then be found analytically since the set of possible fees and rebates is compact given the regulatory cap on access fees. Q.E.D.

A.3 Derivations of selected formulas

In a two-period model, the Social Planner's problem is

$$\max_{\substack{MF,TF \\ -\tau<MF,TF<+\tau \\ MF+TF\geq 0}} \sum_{t_z\in\{t_1,t_2\}} \left(W_{t_z}^{INV}(MF,TF)+W_{t_z}^{Ex}(MF,TF)\right) \tag{36}$$

$$=\sum_{x_{t_1}\in X^L}\left(W_{t_1}^{INV}(x_{t_1}|\Xi,L_{t_0})+W_{t_2}^{INV}(\tilde{x}_{t_2}(x_{t_1})|\Xi,L_{t_0})+Pr(x_{t_1},\theta_{t_1}^{x_{t_1}}|\Xi)(MF+TF)\right)$$

given that limit orders x_{t_1} are only submitted at t_1 and lead to investor welfare:

$$\begin{aligned}
W_{t_1}^{INV}(x_{t_1}|\Xi,L_{t_0}) &= \int_{\beta_{t_1}\in B_{t_1}(x_{t_1},\Xi,L_{t_0})}\left[I(x_{t_1})\times(\beta_{t_1}-P(x_{t_1}))-MF\right]f(\beta_{t_1})d\beta_{t_1}\times Pr(\theta_{t_1}^{x_{t_1}}|\Xi,L_{t_0}) \\
&= I(x_{t_1})\times\int_{\beta_{t_1}\in B_{t_1}(x_{t_1},\Xi,L_{t_0})}\beta_{t_1}f(\beta_{t_1})d\beta_{t_1}\times Pr(\theta_{t_1}^{x_{t_1}}|\Xi,L_{t_0}) \\
&- [I(x_{t_1})\times P(x_{t_1})+MF)]\times\left(\int_{\beta_{t_1}\in B_{t_1}(x_{t_1},\Xi,L_{t_0})}f(\beta_{t_1})d\beta_{t_1}\right)\times Pr(\theta_{t_1}^{x_{t_1}}|\Xi,L_{t_0}) \\
&= Pr(x_{t_1}|\Xi,L_{t_0})\times Pr(\theta_{t_1}^{x_{t_1}}|\Xi,L_{t_0})\times I(x_{t_1})\times E[\beta_{t_1}|x_{t_1}] \\
&- Pr(x_{t_1}|\Xi,L_{t_0})\times Pr(\theta_{t_1}^{x_{t_1}}|\Xi,L_{t_0})\times[I(x_{t_1})\times P(x_{t_1})+MF)]
\end{aligned} \tag{37}$$

where I_{t_z} is an indicator variable equal to 1 (-1) for buy (sell) orders at t_z, $B_{t_1}(x_{t_1},\Xi,L_{t_0})$ is the interval of the β_{t_1} realizations for which a given limit order x_{t_1} is optimal at t_1, and where market orders $\tilde{x}_{t_2}(x_{t_1})$ at t_2

executing earlier limit orders x_{t_1} from t_1 lead to investor welfare:

$$\begin{aligned}
W_{t_2}^{INV}(\tilde{x}_{t_2}(x_{t_1})\,|\Xi, L_{t_0}) &= Pr(x_{t_1}|\Xi, L_{t_0}) \times \int_{\beta_{t_2} \in B_{t_2}(\theta_{t_1}^{x_{t_1}}, \Xi, L_{t_1})} [I(x_{t_1}) \times (P(x_{t_1}) - \beta_{t_2}) - TF] f(\beta_{t_2}) d\beta_{t_2} \\
&= Pr(x_{t_1}|\Xi, L_{t_0}) \times [I(x_{t_1}) \times P(x_{t_1}) - TF)] \times \int_{\beta_{t_2} \in B_{t_2}(\theta_{t_1}^{x_{t_1}}, \Xi, L_{t_1})} f(\beta_{t_2}) d\beta_{t_2} \\
&- Pr(x_{t_1}|\Xi, L_{t_0}) \times I(x_{t_1}) \times \int_{\beta_{t_2} \in B_{t_2}(\theta_{t_1}^{x_{t_1}}, \Xi, L_{t_1})} \beta_{t_2} f(\beta_{t_2}) d\beta_{t_2} \\
&= Pr(x_{t_1}|\Xi, L_{t_0}) \times Pr(\theta_{t_1}^{x_{t_1}}|\Xi, L_{t_0}) \times [I(x_{t_1}) \times P(x_{t_1}) - TF)] \\
&- Pr(x_{t_1}|\Xi, L_{t_0}) \times Pr(\theta_{t_1}^{x_{t_1}}|\Xi, L_{t_0}) \times I(x_{t_1}) \times E[\beta_{t_2}|\theta_{t_1}^{x_{t_1}}]
\end{aligned} \tag{38}$$

given the interval $B_{t_2}(\theta_{t_1}^{x_{t_1}}, \Xi, L_{t_1})$ of β_{t_2} realizations for which a market order is optimal at t_2 that leads to execution the order x_{t_1} submitted at t_1. The third term in (36) is the exchange's profit $W_{t_z}^{Ex}(MF, TF)$ from (11). Adding (37) and (38) and using $MF = -TF$ for the Social Planner gives the addends in (12).

In a general multiperiod model, a Social Planner maximizes the total welfare, which generalizes to the N-period model to:

$$\max_{\substack{MF,TF \\ -\tau < MF,TF < +\tau \\ MF+TF \geq 0}} \sum_{t_z \in \{t_1,\dots,t_N\}} \left(W_{t_z}^{INV}(MF,TF) + W_{t_z}^{Ex}(MF,TF) \right) \tag{39}$$

$$\begin{aligned}
= \quad & \sum_{t_z \in \{t_1,\dots,t_N\}} \sum_{\forall L_{t_{z-1}}} Pr(L_{t_{z-1}}|\Xi) \times \Bigg[\sum_{x_{t_z} \in X^L} \Bigg(\int_{\beta_{t_z} \in B_{t_z}(x_{t_z}, \Xi, L_{t_{z-1}})} I_{t_z} \times [\beta_{t_z} - P(x_{t_z}) - MF)] f(\beta_{t_z}) d\beta_{t_z} \times Pr(\theta_{t_z}^{x_{t_z}}|\Xi, L_{t_{z-1}}) \\
& + Pr(x_{t_z}|\Xi, L_{t_{z-1}}) \times \sum_{t_n \in \{t_{z+1},\dots,t_N\}} \sum_{\forall L_{t_{n-1}}} Pr(L_{t_{n-1}} \cap \mathcal{N}_{t_{n-1}}^{x_{t_z}}|x_{t_z}, L_{t_{z-1}}, \Xi) \int_{\beta_{t_n} \in B_{t_n}(\tilde{x}_{t_n}(x_{t_z}), \Xi, L_{t_{n-1}})} I_{t_z} \times [P(x_{t_z}) - \beta_{t_n} - TF)] f(\beta_{t_n}) d\beta_{t_n} \\
& + Pr(x_{t_z}, \theta_{t_z}^{x_{t_z}}|\Xi)\,(MF + TF) \Bigg) \Bigg]
\end{aligned}$$

where $W_{t_z}^{INV}(MF, TF)$ is the expected welfare of arriving investors at each date t_z and $W_{t_z}^{Ex}(MF, TF)$ is the exchange's expected profit from (26) from limit orders submitted at dates t_z and subsequently executed at later dates. The first term on the right is the welfare of investors who submitted different possible limit orders $x_{t_z} \in X^L$ at all dates t_z where $B_{t_z}(x_{t_z}, \Xi, L_{t_{z-1}})$ is the interval of private value realizations β_{t_z} at time t_z

for which a limit order x_{t_z} is optimal given the book $L_{t_{z-1}}$ and fees Ξ. The second term on the right is the welfare of investors who subsequently submit market orders $\tilde{x}_{t_n}(x_{t_z})$ at later dates $t_n > t_z$ that execute earlier limit orders x_{t_z}, where $B_{t_n}(\tilde{x}_{t_n}(x_{t_z}), \Xi, L_{t_{n-1}})$ is the corresponding interval of private value realizations β_{t_n} at times $t_n > t_z$ for which the market order $\tilde{x}_{t_n}(x_{t_z})$ is optimal at date t_n. Investor private valuations at different dates (i.e., β_{t_z} or β_{t_n} at t_z or t_n) are i.i.d. random variables with uniform distributions $U[\underline{\beta}, \overline{\beta}]$. The indicator function I_{t_z} denotes limit buys ($I_{t_z} = +1$) and sells ($I_{t_z} = -1$) at t_z. The conditioning information at date t_n includes both the incoming book $L_{t_{n-1}}$ and also the fact that the limit order from t_z is still unexecuted as of date t_{n-1}. In particular, $\mathscr{N}_{t_{n-1}}^{x_{t_z}}$ denotes the set of states in which the limit order x_{t_z} from t_z is not executed before time t_n.

A similar logic to the 2-period model simplifies the 3-period Social Planner objective to (28).

B Equilibrium of Two-Period Model and Proof of Theorem 2

This Appendix solves for the profit-maximizing exchange's optimal access pricing in the two-period model given the regulatory constraint and given optimal trading by investors. Since the investors' optimal orders are solutions to discrete choice problems, their trading strategies and the exchange's optimal fees change qualitatively in different regions of the parameter space in terms of the size of the investor valuation support Δ relative to the tick size τ, which is normalized to 1 in the LTM.

B.1 Case 1: $0 < \Delta \leq 3\tau$

Our analysis of this first case shows that, when $\Delta \leq 3\tau$, optimal access pricing by the exchange in equilibrium takes the functional forms in (20) for Taker-Maker pricing and (21) for Maker-Taker pricing.

Figure B1: Taker-Maker Pricing: $\Xi_{TM} = \{0 \leq MF \leq 1, -1 \leq TF \leq 0\}$ This Figure provides a graphical representation of how to obtain the equilibrium probabilities of order submission and execution for the Taker-Maker pricing structure and the support $\Delta \in [\underline{\beta}, \bar{\beta}]$. P_2 and P_{-2} are the outside quotes of the LTM, whereas P_1 and P_{-1} are the inside quotes of the LTM. $P_{-1}^{cum,LB}$ and $P_{-1}^{cum,MS}$ are the cum-fee buy and sell prices, respectively. LBP_{-1,t_1} is a limit buy order posted at P_{-1} at t_1, and MSP_{-1,t_2} is a market sell order posted at P_{-1} at t_2.

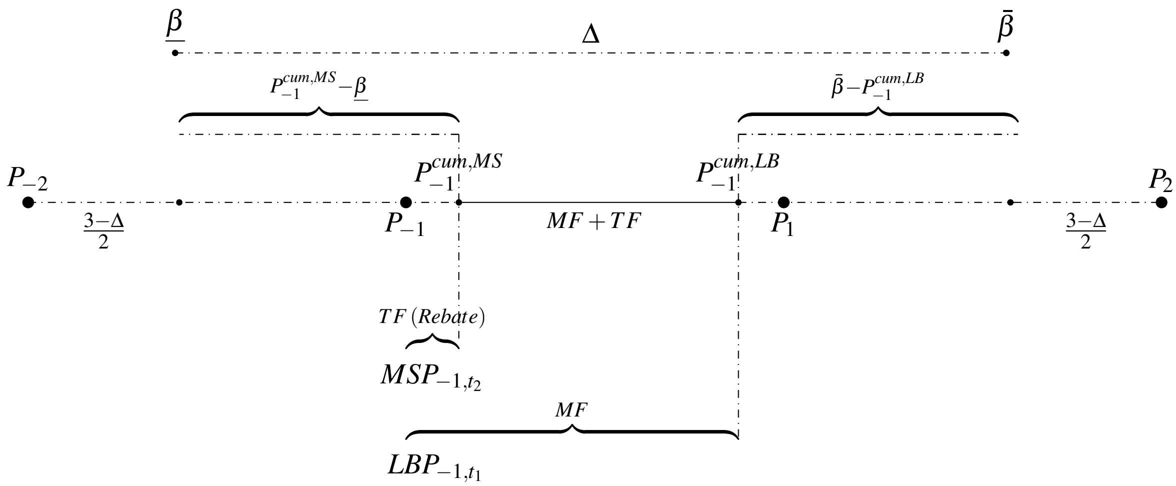


Figure B2: Maker-Taker Pricing: $\Xi_{MT} = \{-1 \leq MF \leq 0, 0 \leq TF \leq 1\}$ This Figure provides a graphical representation of how to obtain the equilibrium probabilities of order submission and execution for the Maker-Taker pricing structure and the support $\Delta \in [\underline{\beta}, \bar{\beta}]$. P_2 and P_{-2} are the outside quotes of the LTM, whereas P_1 and P_{-1} are the inside quotes of the LTM. $P_1^{cum,LB}$ and $P_1^{cum,MS}$ are the cum-fee buy and sell prices, respectively. LBP_{1,t_1} is a limit buy order posted at P_1 at t_1, and MSP_{1,t_2} is a market sell order posted P_1 at t_2.

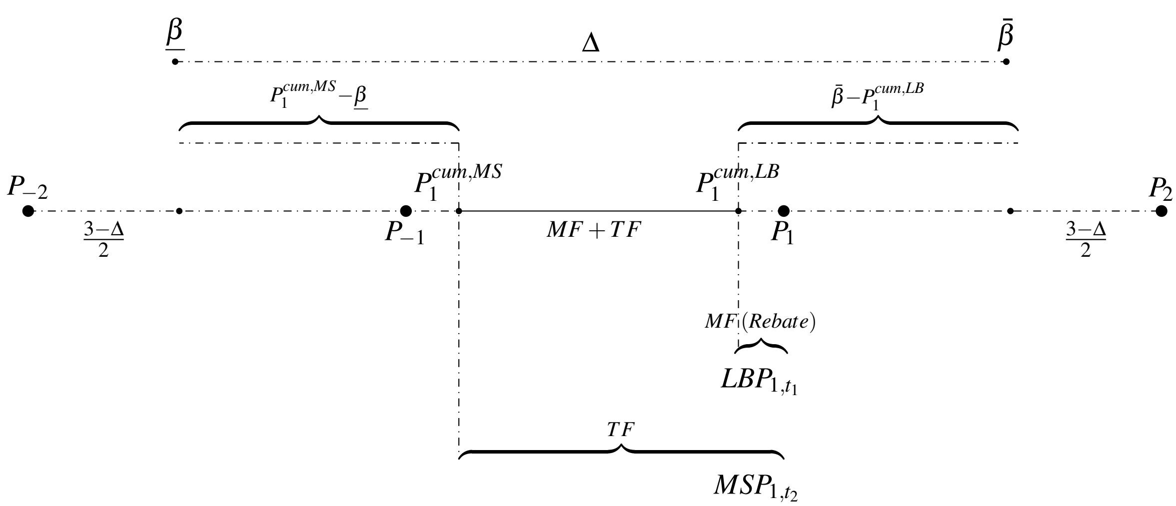


Table B1: Submission and Execution Probability. This table reports the price levels on the LTM price grid (column 1) and the associated probabilities $Pr(\beta_{t_1} > P_k^{cum,LB}) = \max\{0, \frac{\bar{\beta}-P_k^{cum,LB}}{\Delta}\}$ and $Pr(P_{-k}^{cum,LS} > \beta_{t_1}) = \max\{0, \frac{P_{-k}^{cum,LS}-\underline{\beta}}{\Delta}\}$, which, in equilibrium, correspond to the submission probabilities for limit orders posted at P_k and at P_{-k} at t_1 (columns 2 and 3). In addition, the table reports the associated limit order execution probabilities, $Pr(\theta_{t_1}^{x_k^{LB}}|\Xi, L_{t_0}) = Pr(x_{k,t_2}^{MS}|\Xi, L_{t_0}) = \max\{0, \frac{P_k^{cum,MS}-\underline{\beta}}{\Delta}\}$ and $Pr(\theta_{t_1}^{x_{-k}^{LS}}|\Xi, L_{t_0}) = Pr(x_{-k,t_2}^{MB}|, \Xi, L_{t_0}) = \max\{0, \frac{\bar{\beta}-P_{-k}^{cum,MB}}{\Delta}\}$ (columns 4 and 5).

P_k	$Pr(\beta_{t_1} > P_k^{cum,LB})$	$Pr(P_{-k}^{cum,LS} > \beta_{t_1})$	$Pr(\theta_{t_1}^{x_k^{LB}}\|\Xi, L_{t_0})$	$Pr(\theta_{t_1}^{x_{-k}^{LS}}\|\Xi, L_{t_0})$
P_3	$\max\left\{0, \frac{1}{\Delta}[\frac{\Delta}{2} - \frac{5}{2} - MF]\right\}$	$\max\left\{0, \frac{1}{\Delta}[\frac{\Delta}{2} - \frac{5}{2} - MF]\right\}$	$\max\left\{0, \frac{1}{\Delta}[\frac{\Delta}{2} + \frac{5}{2} + TF]\right\}$	$\max\left\{0, \frac{1}{\Delta}[\frac{\Delta}{2} + \frac{5}{2} + TF]\right\}$
P_2	$\max\left\{0, \frac{1}{\Delta}[\frac{\Delta}{2} - \frac{3}{2} - MF]\right\}$	$\max\left\{0, \frac{1}{\Delta}[\frac{\Delta}{2} - \frac{3}{2} - MF]\right\}$	$\max\left\{0, \frac{1}{\Delta}[\frac{\Delta}{2} + \frac{3}{2} - TF]\right\}$	$\max\left\{0, \frac{1}{\Delta}[\frac{\Delta}{2} + \frac{3}{2} - TF]\right\}$
P_1	$\max\left\{0, \frac{1}{\Delta}[\frac{\Delta}{2} - \frac{1}{2} - MF]\right\}$	$\max\left\{0, \frac{1}{\Delta}[\frac{\Delta}{2} - \frac{1}{2} - MF]\right\}$	$\max\left\{0, \frac{1}{\Delta}[\frac{\Delta}{2} + \frac{1}{2} - TF]\right\}$	$\max\left\{0, \frac{1}{\Delta}[\frac{\Delta}{2} + \frac{1}{2} - TF]\right\}$
P_{-1}	$\max\left\{0, \frac{1}{\Delta}[\frac{\Delta}{2} + \frac{1}{2} - MF]\right\}$	$\max\left\{0, \frac{1}{\Delta}[\frac{\Delta}{2} + \frac{1}{2} - MF]\right\}$	$\max\left\{0, \frac{1}{\Delta}[\frac{\Delta}{2} - \frac{1}{2} - TF]\right\}$	$\max\left\{0, \frac{1}{\Delta}[\frac{\Delta}{2} - \frac{1}{2} - TF]\right\}$
P_{-2}	$\max\left\{0, \frac{1}{\Delta}[\frac{\Delta}{2} + \frac{3}{2} - MF]\right\}$	$\max\left\{0, \frac{1}{\Delta}[\frac{\Delta}{2} + \frac{3}{2} - MF]\right\}$	$\max\left\{0, \frac{1}{\Delta}[\frac{\Delta}{2} - \frac{3}{2} - TF]\right\}$	$\max\left\{0, \frac{1}{\Delta}[\frac{\Delta}{2} - \frac{3}{2} - TF]\right\}$
P_{-3}	$\max\left\{0, \frac{1}{\Delta}[\frac{\Delta}{2} + \frac{5}{2} - MF]\right\}$	$\max\left\{0, \frac{1}{\Delta}[\frac{\Delta}{2} + \frac{5}{2} - MF]\right\}$	$\max\left\{0, \frac{1}{\Delta}[\frac{\Delta}{2} - \frac{5}{2} - TF]\right\}$	$\max\left\{0, \frac{1}{\Delta}[\frac{\Delta}{2} - \frac{5}{2} - TF]\right\}$

Taker-Maker: We first consider Taker-Maker pricing $\Xi_{TM} = \{0 \leq MF \leq 1, -1 \leq TF \leq 0\}$ with a take rebate and a positive make fee. Given $\Delta \leq 3$, the lower investor-valuation bound in this case is $\underline{\beta} = P_{-2} + \frac{3-\Delta}{2}$, and the upper bound is $\bar{\beta} = P_2 - \frac{3-\Delta}{2}$, as illustrated in Figures B1 and B2. Consider first a potential buyer arriving at t_1 with $\beta_{t_1} > v$. The logic for a potential seller arriving at t_1 is symmetric.

Order-submission probabilities for each possible market order at t_2 can be computed using (4) and (5) given the valuation-support restriction $\Delta \leq 3$ and Taker-Maker pricing. Columns 4 and 5 in Table B1 report the market order submission probabilities for the price levels in Column 1:

$$Pr(x_{k,t_2}^{MS}|\Xi, L_{t_1}) = \max\left\{0, \frac{P_k - TF - \underline{\beta}}{\Delta}\right\} = \max\left\{0, \frac{1}{\Delta}\left[\frac{\Delta}{2} + \frac{P_k - P_{-k}}{2} - TF\right]\right\} \tag{40}$$

$$Pr(x_{-k,t_2}^{MB}|\Xi, L_{t_1}) = \max\left\{0, \frac{\bar{\beta} - P_{-k} - TF}{\Delta}\right\} = \max\left\{0, \frac{1}{\Delta}\left[\frac{\Delta}{2} - \frac{P_k - P_{-k}}{2} - TF\right]\right\} \tag{41}$$

For example, Row 5 in Column 4 and Row 5 in Column 5 in Table B1 gives the order-submission probability

Table B2: Submission and Execution Probability V2. This table reports the price levels on the LTM price grid (column 1) and the associated probabilities $Pr(\beta_{t_1} > P_k^{cum,LB}) = \max\{0, \frac{\bar{\beta}-P_k^{cum,LB}}{\Delta}\}$ and $Pr(P_{-k}^{cum,LS} > \beta_{t_1}) = \max\{0, \frac{P_{-k}^{cum,LS}-\underline{\beta}}{\Delta}\}$, which, in equilibrium, correspond to the submission probabilities for limit orders posted at P_k and at P_{-k} at t_1 (columns 2 and 3). In addition, the table reports the associated limit order execution probabilities, $Pr(\theta_{t_1}^{x_k^{LB}}|\Xi, L_{t_0}) = Pr(x_{k,t_2}^{MS}|\Xi, L_{t_0}) = \max\{0, \frac{P_k^{cum,MS}-\underline{\beta}}{\Delta}\}$ and $Pr(\theta_{t_1}^{x_{-k}^{LS}}|\Xi, L_{t_0}) = Pr(x_{-k,t_2}^{MB}|, \Xi, L_{t_0}) = \max\{0, \frac{\bar{\beta}-P_{-k}^{cum,MB}}{\Delta}\}$ (columns 4 and 5).

P_k	$Pr(\beta_{t_1} > P_k^{cum,LB})$	$Pr(P_{-k}^{cum,LS} > \beta_{t_1})$	$Pr(\theta_{t_1}^{x_k^{LB}}|\Xi, L_{t_0})$	$Pr(\theta_{t_1}^{x_{-k}^{LS}}|\Xi, L_{t_0})$
P_3	$\max\left\{0, \frac{1}{\Delta}[\frac{\Delta}{2} - \frac{5}{2} - MF]\right\}$	$\max\left\{0, \frac{1}{\Delta}[\frac{\Delta}{2} + \frac{5}{2} - MF]\right\}$	$\max\left\{0, \frac{1}{\Delta}[\frac{\Delta}{2} + \frac{5}{2} + TF]\right\}$	$\max\left\{0, \frac{1}{\Delta}[\frac{\Delta}{2} - \frac{5}{2} + TF]\right\}$
P_2	$\max\left\{0, \frac{1}{\Delta}[\frac{\Delta}{2} - \frac{3}{2} - MF]\right\}$	$\max\left\{0, \frac{1}{\Delta}[\frac{\Delta}{2} + \frac{3}{2} - MF]\right\}$	$\max\left\{0, \frac{1}{\Delta}[\frac{\Delta}{2} + \frac{3}{2} - TF]\right\}$	$\max\left\{0, \frac{1}{\Delta}[\frac{\Delta}{2} - \frac{3}{2} - TF]\right\}$
P_1	$\max\left\{0, \frac{1}{\Delta}[\frac{\Delta}{2} - \frac{1}{2} - MF]\right\}$	$\max\left\{0, \frac{1}{\Delta}[\frac{\Delta}{2} + \frac{1}{2} - MF]\right\}$	$\max\left\{0, \frac{1}{\Delta}[\frac{\Delta}{2} + \frac{1}{2} - TF]\right\}$	$\max\left\{0, \frac{1}{\Delta}[\frac{\Delta}{2} - \frac{1}{2} - TF]\right\}$
P_{-1}	$\max\left\{0, \frac{1}{\Delta}[\frac{\Delta}{2} + \frac{1}{2} - MF]\right\}$	$\max\left\{0, \frac{1}{\Delta}[\frac{\Delta}{2} - \frac{1}{2} - MF]\right\}$	$\max\left\{0, \frac{1}{\Delta}[\frac{\Delta}{2} - \frac{1}{2} - TF]\right\}$	$\max\left\{0, \frac{1}{\Delta}[\frac{\Delta}{2} + \frac{1}{2} - TF]\right\}$
P_{-2}	$\max\left\{0, \frac{1}{\Delta}[\frac{\Delta}{2} + \frac{3}{2} - MF]\right\}$	$\max\left\{0, \frac{1}{\Delta}[\frac{\Delta}{2} - \frac{3}{2} - MF]\right\}$	$\max\left\{0, \frac{1}{\Delta}[\frac{\Delta}{2} - \frac{3}{2} - TF]\right\}$	$\max\left\{0, \frac{1}{\Delta}[\frac{\Delta}{2} + \frac{3}{2} - TF]\right\}$
P_{-3}	$\max\left\{0, \frac{1}{\Delta}[\frac{\Delta}{2} + \frac{5}{2} - MF]\right\}$	$\max\left\{0, \frac{1}{\Delta}[\frac{\Delta}{2} - \frac{5}{2} - MF]\right\}$	$\max\left\{0, \frac{1}{\Delta}[\frac{\Delta}{2} - \frac{5}{2} - TF]\right\}$	$\max\left\{0, \frac{1}{\Delta}[\frac{\Delta}{2} + \frac{5}{2} - TF]\right\}$

at t_2 of a market sell at P_{-1}, which is equal to the order-submission probability of a market buy at P_1

$$\begin{aligned} Pr(x_{-1,t_2}^{MS}|\Xi, L_{t_1}) &= \max\left\{0, \frac{P_{-1}^{cum,MS} - \underline{\beta}}{\Delta}\right\} \\ &= Pr(x_{1,t_2}^{MB}|\Xi, L_{t_1}) = \max\left\{0, \frac{\bar{\beta} - P_1^{cum,MB}}{\Delta}\right\} = \max\left\{0, \frac{1}{\Delta}\left[\frac{\Delta}{2} - \frac{1}{2} - TF\right]\right\}. \end{aligned} \tag{42}$$

To understand the intuition in the last term in (42), note from Figure B1 that only traders with β_{t_2} in the interval $[\underline{\beta}, P_{-1}^{cum,MS}]$ with width $\frac{\Delta}{2} - \frac{1}{2} - TF$ are willing to use a market order to sell at a posted price P_{-1}. This interval is equal to half of the support minus half the tick size, hence $\frac{1}{2}$, given $\tau = 1$, which is the distance from the fundamental asset value v to P_{-1}, minus TF (negative in the Taker-Maker regime), which increases the interval of the support including βs belonging to sellers. This interval is strictly positive for $\Delta \geq 1$, which means that $Pr(x_{-1,t_2}^{MS}|\Xi, L_{t_1}) > 0$ for $\Delta \geq 1$.

The market-order submission probabilities at t_2 are, in turn, respectively the corresponding order-execution probabilities of limit orders posted at t_1. Thus, we can consider the expected profits for all possible limit orders that a potential buyer and symmetrically a potential seller can post at t_1. We verify the conditions under which (6) and (7) hold — and symmetrically (9) and (10) — and finally compute the limit order submission probabilities at t_1 consistent with both (8) and (11).

To check that conditions (6) and (9) hold, we compute $Pr(\beta_{t_1} > P_k^{cum,LB})$ and $Pr(P_{-k}^{cum,LS} > \beta_{t_1})$ for each order in Columns 2 and 3 of Table B1. For example, for a limit buy at P_{-1} and limit sell at P_1 we have:

$$\begin{aligned} Pr(\beta_{t_1} > P_{-1}^{cum,LB}) &= \max\left\{0, \frac{\bar{\beta} > P_{-1}^{cum,LB}}{\Delta}\right\} \\ &= Pr(P_1^{cum,LS} > \beta_{t_1}) = \max\left\{0, \frac{P_1^{cum,LS} > \underline{\beta}}{\Delta}\right\} = \max\left\{0, \frac{1}{\Delta}\left[\frac{\Delta}{2} + \frac{1}{2} - MF\right]\right\}. \end{aligned} \tag{43}$$

To understand the intuition for the final term in (43), notice, for example, from Figure B1 that only traders with a β_{t_1} in the interval $[P_{-1}^{cum,LB}, \overline{\beta}_{t_1}]$ with width $\frac{\Delta}{2} + \frac{1}{2} - MF$ will be willing to buy at the quoted price P_{-1}. This interval is equal to half of the investor valuation support (consistent with Lemma 4 only traders with a personal evaluation larger than the fundamental value v will be buying) plus half the tick size (the distance between the mid-point of the support/fundamental asset value v and P_{-1}) minus MF, which decreases the interval of the support including βs belonging to buyers.

We also need to check whether both conditions (7) and (10) hold for each possible order at t_1:

- First, consider a limit buy at P_2 and symmetrically a limit sell at P_{-2}. Given the assumed investor valuation support with width $\Delta \leq 3$ and given the positive MF with Taker-Maker pricing, the expected payoff associated with limit orders at P_2 (P_{-2}) would be negative since the associated cum-fee buy (sell) price would be above (below) the maximum (minimum) possible trader valuation. Hence, such limit orders would never be submitted.

- Second, the expected profit $(\beta_{t_1} - P_{-1}^{cum,LB}) \times Pr(\theta_{t_1}^{x_{-1}^{LB}} | \Xi, L_{t_0})$ on a limit buy at P_{-1} for a potential buyer with $\beta_{t_1} > v$ and $(P_1^{cum,LS} - \beta_{t_1}) \times Pr(\theta_{t_1}^{x_1^{LS}} | \Xi, L_{t_0})$ on a limit sell at P_1 for a potential seller with $\beta_{t_1} < v$

is:

$$\left(|\beta_{t_1}-v|+\frac{1}{2}-MF\right)\max\left\{0,\frac{1}{\Delta}\left[\frac{\Delta}{2}-\frac{1}{2}-TF\right]\right\}, \tag{44}$$

which is positive given Take rebates ($0\geq TF\geq -1$) and Make fees ($1\geq MF\geq 0$).

- Third, the expected profit $(\beta_{t_1}-P_{-2}^{cum,LB})\times Pr(\theta_{t_1}^{x_{-2}^{LB}}|\Xi,L_{t_0})$ on a limit buy at P_{-2} for a potential buyer with $\beta_{t_1}>v$ or $(P_2^{cum,LS}-\beta_{t_1})\times Pr(\theta_{t_1}^{x_2^{LS}}|\Xi,L_{t_0})$ on a limit sell at P_2 for a potential seller with $\beta_{t_1}<v$ is:

$$\left(|\beta_{t_1}-v|+\frac{3}{2}-MF\right)\max\left\{0,\frac{1}{\Delta}\left[\frac{\Delta}{2}-\frac{3}{2}-TF\right]\right\}, \tag{45}$$

To characterize when the expected profit on limit buys at P_{-1} (and limit sells at P_1) are greater than on limit buys at P_{-2} (and limit sells at P_2), we write the expected profits in (44) for limit buys at P_{-1} and limit sells at P_1 as $a*b$ where $a=|\beta_{t_1}-v|+\frac{1}{2}-MF$ and $b=\frac{1}{\Delta}\left[\frac{\Delta}{2}-\frac{1}{2}-TF\right]$. Given this, we derive the β threshold between a limit buy at P_{-1} and a limit buy at P_{-2} as the β values for which (46) holds

$$(\beta_{t_1}-P_{-2}^{cum,LB})\times Pr(\theta_{t_1}^{x_{-2}^{LB}}|\Xi,L_{t_0})-(\beta_{t_1}-P_{-1}^{cum,LB})\times Pr(\theta_{t_1}^{x_{-1}^{LB}}|\Xi,L_{t_0})=0 \tag{46}$$

The β_{t_1} values which satisfy (46) are

$$\beta_{t_1}^{x_{P_{-2}}^{LB},x_{P_{-1}}^{LB}}=\begin{cases} v+\frac{\Delta}{2}-2+MF-TF & TF<0\wedge TF<\frac{\Delta-3}{2}\\ v-\frac{1}{2}+MF & \text{Otherwise}\end{cases} \tag{47}$$

We can now compute the order-submission probabilities from (8) for a limit buy at P_{-1} and at P_{-2}

$$Pr(x^{LB}_{-1,t_1}|\Xi, L_{t_0}) = \begin{cases} \frac{1}{\Delta}[\frac{\Delta+1}{2} - MF] & TF \geq 0 \vee TF \geq \frac{\Delta-3}{2} \\ \frac{1}{\Delta}\left[2 - MF + TF\right] & \text{Otherwise} \end{cases} \tag{48}$$

$$Pr(x^{LB}_{-2,t_1}|\Xi, L_{t_0}) = \begin{cases} \frac{1}{\Delta}\left[\frac{\Delta-4}{2} + MF - TF\right] & (MF > \frac{1}{2} \wedge TF < 0 \wedge TF < \frac{\Delta-3}{2}) \vee \\ & \left(MF \leq \frac{1}{2} \wedge MF > TF + \frac{1}{2} \wedge MF > \frac{4-\Delta}{2} + TF\right) \\ 0 & \text{Otherwise} \end{cases} \tag{49}$$

Lastly, the expected profit $(\beta_{t_1} - P^{cum,LB}_{1,LB}) \times Pr(\theta^{x^{LB}_1}_{t_1}|\Xi, L_{t_0})$ on a limit buy at P_1 for a potential buyer with $\beta_{t_1} > 0$ and $(P^{cum,LS}_{-1} - \beta_{t_1}) \times Pr(\theta^{x^{LS}_{-1}}_{t_1}|\Xi, L_{t_0})$ on a limit sell at P_{-1} for a potential seller with $\beta_{t_1} < 0$ is:

$$\left(|\beta_{t_1} - v| - \frac{1}{2} - MF\right) \max\left\{0, \frac{1}{\Delta}\left[\frac{\Delta}{2} + \frac{1}{2} - TF\right]\right\}. \tag{50}$$

Comparing (44) and (50) shows that, in Taker-Maker regimes, limit buys at P_{-1} always have higher expected profit than limit buys at P_1 and that limit sells at P_1 have higher expected profits than limit sells at P_{-1}:[26]

$$\begin{aligned} &(\beta_{t_1} - P^{cum,LB}_1) \times Pr(\theta^{x^{LB}_1}_{t_1}|\Xi, L_{t_0}) - (\beta_{t_1} - P^{cum,LB}_{-1}) \times Pr(\theta^{x^{LB}_{-1}}_{t_1}|\Xi, L_{t_0}) \\ &= (P^{cum,LS}_{-1} - \beta_{t_1}) \times Pr(\theta^{x^{LS}_{-1}}_{t_1}|\Xi, L_{t_0}) - (P^{cum,LS}_1 - \beta_{t_1}) \times Pr(\theta^{x^{LS}_1}_{t_1}|\Xi, L_{t_0}) \\ &= (|\beta_{t_1} - v| - \frac{\Delta}{2}) - MF + TF \leq 0 \end{aligned} \tag{51}$$

where the inequality in the third line follows because $|\beta_{t_1} - v| \leq \frac{\Delta}{2}$ by definition for all β_{t_1} and because $MF \geq 0$ and $TF \leq 0$ in the Taker-Maker regime. We can now set the optimizing function for both the exchange and the Social Planner.

[26] Using the representation for the expected profit for a limit buy at P_{-1} and limit sell at P_1 in (44) as $a * b$ where $a = |\beta_{t_1} - v| + \frac{1}{2} - MF$ and $b = \frac{1}{\Delta}\left[\frac{\Delta}{2} - \frac{1}{2} - TF\right]$, the expected profit for a limit buy LBP_1 at P_1 and limit sell LSP_{-1} at P_{-1} in (50) can be represented as $(a-1)(b+1)$. Taking the difference $a * b - (a-1)(b+1)$ and substituting in for a and b gives the third line in (51).

Comment: The discussion above identifies which orders are possibly used in the two-period trading subgame in the LTM with $\tau = 1$. This analysis is used next to derive optimal fees in the LTM.

Exchange Problem: Taker-Maker $\Delta \in (0,3]$ The exchange chooses MF and TF to maximize its profits given the optimal strategy for potential buyers and sellers posting limit orders LBP_{-1,t_1} and LSP_{1,t_1} and LBP_{-2,t_1} and LSP_{2,t_1} at t_1, which we have derived as a function of the trading fees MF and TF and the investor valuation-support width Δ.[27] From now onward we concentrate on the buy side, the sell side being symmetric. The exchange's expected profit is equal to the submission probability $Pr(x^{LB}_{-1,t_1}|\Xi, L_{t_0})$ of LBP_{-1,t_1} and the submission probability $Pr(x^{LB}_{-2,t_1}|\Xi, L_{t_0})$ of LBP_{-2,t_1}, times the associated execution probability $Pr(\theta^{x^{LB}_{-1}}_{t_1}|\Xi, L_{t_0})$ and $Pr(\theta^{x^{LB}_{-2}}_{t_1}|\Xi, L_{t_0})$ times the per share net fee, MF+TF. Table B1 reports the order-execution probabilities.

$$\max_{\substack{MF,TF \\ 0 \le MF \le \tau \\ TF \le MF}} \pi^{Ex,LTM}(MF,TF) \tag{52}$$

$$= \left[Pr(x^{LB}_{-1,t_1}|\Xi, L_{t_0}) \times Pr(\theta^{x^{LB}_{-1}}_{t_1}|\Xi, L_{t_0}) + Pr(x^{LB}_{-2,t_1}|\Xi, L_{t_0}) \times Pr(\theta^{x^{LB}_{-2}}_{t_1}|\Xi, L_{t_0})\right] \times (MF+TF)$$

$$= \begin{cases} \dfrac{(-\Delta+2MF-1)(MF+TF)(-\Delta+2TF+1)}{4\Delta^2} & TF \ge 0 \vee 2TF+3 \ge \Delta \\ -\dfrac{(MF+TF)(-(\Delta-3)\Delta+4MF+2(\Delta-2)TF-8)}{4\Delta^2} & \text{Otherwise} \end{cases}$$

The first order conditions are:

$$\begin{cases} \dfrac{(-\Delta+2TF+1)(-\Delta+4MF+2TF-1)}{4\Delta^2} = 0 & TF \ge 0 \vee 2TF+3 \ge \Delta \\ \dfrac{-8MF+\Delta(\Delta-2TF-3)+8}{4\Delta^2} = 0 & \text{Otherwise} \end{cases} \tag{53}$$

$$\begin{cases} \dfrac{(-\Delta+2MF-1)(-\Delta+2MF+4TF+1)}{4\Delta^2} = 0 & TF \ge 0 \vee 2TF+3 \ge \Delta \\ \dfrac{\Delta(\Delta-2MF-3)-4(\Delta-2)TF+8}{4\Delta^2} = 0 & \text{Otherwise} \end{cases} \tag{54}$$

From the first-order conditions, the equilibrium optimal Take-Make fees for the exchange are in (20).

[27] The case of a seller posting LSP_{1,t_1} or LSP_{2,t_1} is symmetric. As in real markets, traders arrive sequentially and, hence, either a buyer or seller may arrive at t_1.

The second and mixed partial derivatives $\delta_{TF,TF}$, $\delta_{MF,MF}$ and $\delta_{MF,TF}$ are

$$
\begin{aligned}
&\delta_{TF,TF}, \delta_{MF,MF}, \delta_{MF,TF} \\
&= \begin{cases} \left\{ \frac{1}{\Delta^2}\left[-\Delta - 1 + 2MF\right],\ \frac{1}{\Delta^2}\left[-\Delta + 1 + 2TF\right],\ \frac{1}{\Delta^2}\left[-\Delta + 2MF + 2TF\right] \right\} & TF \geq 0 \vee TF \geq \frac{\Delta-3}{2} \\ \left\{ \frac{1}{\Delta^2}\left[-\Delta + 2\right],\ -\frac{2}{\Delta^2},\ -\frac{\Delta}{2} \right\} & \text{Otherwise} \end{cases}
\end{aligned} \tag{55}
$$

which, together with the equilibrium fees from (20), gives the determinant

$$
Det(MF^*, TF^*) = \delta_{MF,MF}(MF^*, TF^*) \times \delta_{TF,TF}(MF^*, TF^*) - (\delta_{MF,TF}(MF^*, TF^*))^2 = \frac{1}{3\Delta^2} > 0 \tag{56}
$$

Since the second-order conditions for profit-maximizing fees are satisfied, and the MF and TF in (20) maximize the exchange profit. This completes our analytic construction of the Taker-Maker equilibrium for the $\Delta \leq 3\tau$ case.

Social Planner Problem: Taker-Maker $\Delta \in (0,3]$ The Social Planner sets MF and TF to maximize total welfare of market participants, which is the sum of the welfare of investors submitting limit orders at t_1 and market orders at t_2, and expected exchange profits:

$$
\begin{aligned}
&\max_{\substack{MF,TF \\ 0 \leq MF \leq \tau \\ MF+TF \geq 0}} \sum_{t_z \in \{t_1, t_2\}} \left(W_{t_z}^{INV}(MF,TF) + W_{t_z}^{Ex}(MF,TF) \right) \\
= \quad & W_{t_1}^{INV}(x_{-1,t_1}^{LB}|\Xi, L_{t_0}) + W_{t_1}^{INV}(x_{-2,t_1}^{LB}|\Xi, L_{t_0}) + W_{t_2}^{INV}(x_{-1,t_2}^{MS}|\Xi, L_{t_0}) \\
& + W_{t_2}^{INV}(x_{-2,t_2}^{MS}|\Xi, L_{t_0}) + \left[Pr(x_{-1,t_1}^{LB}, \theta_{t_1}^{x_{-1}^{LB}}|\Xi) + Pr(x_{-2,t_1}^{LB}, \theta_{t_1}^{x_{-2,t_1}^{LB}}|\Xi) \right] (MF + TF)
\end{aligned} \tag{57}
$$

where the welfare of investors submitting limit buys and market sells, and of exchange profits are defined in (37) and (38), and (26) and (27). We present the welfare of the buy side (the sell side being symmetric):

$$W_{t_1}^{INV}(x_{-1,t_1}^{LB}|\Xi,L_{t_0}) = \int_{\beta_{t_1}\in B_{t_1}(x_{-1,t_1}^{LB},\Xi,L_{t_0})} [\beta_{t_1} - P(x_{-1,t_1}^{LB}) - MF)]\frac{1}{\Delta}d\beta_{t_1} \times Pr(\theta_{t_1}^{x_{-1}^{LB}}|\Xi,L_{t_0}) \tag{58}$$

$$= \begin{cases} \frac{1}{18} & MF = 1 \wedge TF = 0 \wedge \Delta = 3 \\ \frac{1}{18}(MF-2)^2 & 0 \le MF < 1 \wedge TF = 0 \wedge \Delta = 3 \\ \frac{1}{4\Delta^2}\left[(TF+1)(TF-\Delta+2)(2TF-\Delta+1)\right] & MF = 1 \wedge -1 < TF < 0 \wedge TF \le \frac{\Delta-3}{2} \wedge \Delta \le 3 \\ -\frac{1}{4\Delta^2}\left[(MF-TF-2)(2TF-\Delta+1)(MF+TF-\Delta+1)\right] & 0 \le MF < 1 \wedge -1 < TF < 0 \wedge TF \le \frac{\Delta-3}{2} \wedge \Delta \le 3 \\ -\frac{1}{16\Delta^2}\left[(\Delta-1)^2(2TF-\Delta+1)\right] & MF = 1 \wedge \Delta > 1 \wedge ((TF=0 \wedge \Delta < 3) \vee \left(TF \le \frac{\Delta-3}{2} \wedge TF < 0)\right) \\ -\frac{1}{16\Delta^2}\left[(\Delta-2MF+1)^2(\Delta-2TF-1)\right] & MF < 1 \wedge MF \ge 0 \wedge \Delta > 1 \wedge \left((TF=0 \wedge \Delta < 3) \vee (TF \le \frac{\Delta-3}{2} \wedge TF < 0)\right) \end{cases}$$

where the region of integration is $B_{t_1}(x_{-1,t_1}^{LB},\Xi,L_{t_0}) = [\hat{\beta}_{t_1}^{x_{P-2}^{LB},x_{P-1}^{LB}},\overline{\beta}]$, and $\hat{\beta}_{t_1}^{x_{P-2}^{LB},x_{P-1}^{LB}}$ is defined in (46).

$$W_{t_1}^{INV}(x_{-2,t_1}^{LB}|\Xi,L_{t_0}) = \int_{\beta_{t_1}\in B_{t_1}(x_{-2,t_1}^{LB},\Xi,L_{t_0})} [\beta_{t_1} - P(x_{-2,t_1}^{LB}) - MF)]\frac{1}{\Delta}d\beta_{t_1} \times Pr(\theta_{t_1}^{x_{-2,t_1}^{LB}}|\Xi,L_{t_0}) \tag{59}$$

$$= \begin{cases} -\frac{(2TF-\Delta+3)(4-2MF+2TF-\Delta)(2(MF+TF-1)-\Delta)}{16\Delta^2} & (2TF+3<\Delta \wedge ((0<MF<\frac{1}{2} \\ & \wedge((MF = TF+\frac{1}{2} \wedge \Delta < 3)\vee \\ & (MF < TF+\frac{1}{2} \wedge TF < 0 \wedge \Delta \le 3)\vee \\ & (MF > TF+\frac{1}{2}\wedge \\ & TF \ge -1 \wedge \Delta + 2MF < 2TF+4)))\vee \\ & (-1 \le TF < 0 \wedge \frac{1}{2} < MF \le 1 \wedge \Delta \le 3)))\vee \\ & (0 \le MF \le \frac{1}{2} \wedge TF \ge -1 \wedge \\ & \Delta \le 3 \wedge \Delta + 2MF > 2TF+4) \\ 0 & \text{Otherwise} \end{cases}$$

where the region of integration is $B_{t_1}(x_{-2,t_1}^{LB},\Xi,L_{t_0}) = [v,\hat{\beta}_{t_1}^{x_{P-2}^{LB},x_{P-1}^{LB}}]$, and $\hat{\beta}_{t_1}^{x_{P-2}^{LB},x_{P-1}^{LB}}$ is defined in (46).

$$W_{t_2}^{INV}(x^{MS}_{-1,t_2}|\Xi,L_{t_0}) = Pr(x^{LB}_{-1,t_1}|\Xi,L_{t_0}) \times \int_{\beta_{t_2}\in B_{t_2}(x^{MS}_{-1,t_2},\Xi,L_{t_1})} [P(x^{LB}_{-1,t_1}) - \beta_{t_2} - TF)]\frac{1}{\Delta} d\beta_{t_2} \tag{60}$$

$$= \frac{1}{8\Delta}\left[(\Delta - 2TF - 1)^2\right] \times \left(\begin{array}{cc}
\frac{1}{2} & (2MF < 1 \wedge (MF < TF + \frac{1}{2} \vee (MF = TF + \frac{1}{2} \wedge \Delta \leq 3) \vee \\
 & (MF > TF + \frac{1}{2} \wedge 2MF + \Delta \leq 2TF + 4))) \vee \\
 & (2MF = 1 \wedge (TF = 0 \vee (2TF + 3 \geq \Delta \wedge -1 < TF \leq 0))) \\
\frac{1}{3} & MF = 1 \wedge TF = 0 \wedge \Delta = 3 \\
\frac{1}{3}\left[2 - MF\right] & \frac{1}{2} < MF < 1 \wedge TF = 0 \wedge \Delta = 3 \\
\frac{1}{\Delta}\left[TF + 1\right] & MF = 1 \wedge -1 < TF < 0 \wedge TF \leq \frac{1}{2}\Delta - \frac{3}{2} \\
\frac{1}{\Delta}\left[TF - MF + 2\right] & (-1 < TF < 0 \wedge \frac{1}{2} < MF < 1 \wedge TF \leq \frac{1}{2}\Delta - \frac{3}{2}) \vee \\
 & (2MF < 1 \wedge MF > TF + \frac{1}{2} \wedge TF \leq \frac{1}{2}\Delta - \frac{3}{2} > 2TF + 4) \\
\frac{1}{2\Delta}\left[2TF + 3\right] & 2MF = 1 \wedge -1 < TF < 0 \wedge TF < \frac{1}{2}\Delta - \frac{3}{2} \\
\frac{1}{2\Delta}\left[\Delta - 1\right] & MF = 1 \wedge ((TF = 0 \wedge \Delta < 3) \vee (TF > \frac{1}{2}\Delta - \frac{3}{2} \wedge -1 < TF < 0)) \\
\frac{1}{2\Delta}\left[\Delta + 1 - 2MF\right] & \frac{1}{2} < MF < 1 \wedge ((TF = 0 \wedge \Delta < 3) \vee (TF > \frac{1}{2}\Delta - \frac{3}{2} \wedge -1 < TF < 0))
\end{array}\right)$$

where the region of integration is $B_{t_2}(x^{LB}_{-1,t_2},\Xi,L_{t_0}) = [\underline{\beta}, P_{-1} - TF]$.

$$W_{t_2}^{INV}(x^{MS}_{-2,t_2}|\Xi,L_{t_0}) = Pr(x^{LB}_{-2,t_1}|\Xi,\beta_{t_2},L_{t_0}) \times \int_{\beta_{t_2}\in B_{t_2}(x^{MS}_{-2,t_2},\Xi,L_{t_1})} [P(x^{LB}_{-2,t_1}) - \beta_{t_2} - TF)]\frac{1}{\Delta} d\beta_{t_2} \tag{61}$$

$$= \begin{cases} \frac{1}{16\Delta^2}\left[(\Delta - 2TF - 3)^2(\Delta + 2MF - 2TF - 4)\right] & TF \geq -1 \wedge TF < \frac{\Delta - 3}{2} \wedge \Delta \leq 3 \wedge \left(MF > \frac{1}{2} \vee \left(MF > TF + \frac{1}{2} \wedge \Delta + 2MF > 2TF + 4\right)\right) \\ 0 & \text{Otherwise} \end{cases}$$

where the region of integration is $B_{t_2}(x^{MS}_{-2,t_2},\Xi,L_{t_1}) = [\underline{\beta}, P_{-2} - TF]$. Substituting (58), (59), (60) and (61) into the welfare function of the Social Planner, (57), we obtain a functional form whose components are subject to different boundary conditions. The Social Planner problem then simplifies to:

$$
\max_{\substack{MF,TF \\ 0\le MF\le\tau \\ MF+TF\ge 0}} \sum_{t_z\in\{t_1,t_2\}} \left(W_{t_z}^{INV}(MF,TF) + W_{t_z}^{Ex}(MF,TF) \right) \tag{62}
$$

$$
= \begin{cases}
\frac{(-2\Delta+2TF+1)(-\Delta+2TF+1)}{16\Delta} & 2MF=1\wedge 0<\Delta\le 1\wedge \\
& \Big((\Delta+2TF+1>0\wedge TF+1\ge 0\wedge \\
& (2TF+1<0\vee \\
& (TF<0\wedge 2TF+1<\Delta)))\vee \\
& (-\tfrac{1}{2}<TF<0\wedge 2TF+1<\Delta)\Big) \\
\frac{\Delta(\Delta-2TF-1)^2-4(MF-TF-2)(-\Delta+2TF+1)(-\Delta+MF+TF+1)}{16\Delta^2} & -1<TF<0\wedge 1<\Delta\le 3\wedge \\
& \Big((2MF=1\wedge 2TF+3=\Delta)\vee \\
& (0\le MF<\tfrac{1}{2}\wedge 2TF+3\le\Delta\wedge \\
& (MF\le TF+\tfrac{1}{2}\vee \Delta+2MF\le 2TF+4))\Big)
\end{cases}
$$

The fees MF^* and TF^* in Table 2 maximize (62) for values of $0<\Delta\le 3$ satisfying the given conditions of (62). For example, for $\Delta=2$ the second expression in (62) is maximized by $MF^*=0.5$ and $TF^*=-0.5$. In the Online Appendix shows plots of the Social Planner's value function for the Taker-Maker case for the different values of the support ($\Delta\in\{\tau,2\tau,2.5\tau,3\tau\}$).

Comment: When $\Delta\in(0,3\tau)$, the logic of the construction of optimal Maker-Taker fees for a profit-maximizing exchange and the Social Planner is similar to the logic for Taker-Maker fees. To conserve space, the details for the Maker-Taker derivation are in Online Appendix D.1.

Table B3: Difference in expected payoff from different orders. This table reports the difference in the expected payoffs from different orders indicated in column 1. Column 2 reports such differences as a function of Δ, whereas columns 3 to 6 reports the same differences for different values of Δ.

	Δ	$\Delta = 1$	$\Delta = 2$	$\Delta = 3$	$\Delta = 4$
$Pr(\beta_{t_1} - P_{2}^{cum,LB}(x_{t_1})) \times Pr(\theta_{t_1}^{x_{2}^{LB}} \mid \Xi, L_{t_0}) - Pr(\beta_{t_1} - P_{1}^{cum,LB}(x_{t_1})) \times Pr(\theta_{t_1}^{x_{1}^{LB}} \mid \Xi, L_{t_0})$ $Pr(P_{1}^{cum,LS}(x_{t_1}) - \beta_{t_1}) \times Pr(\theta_{t_1}^{x_{1}^{LS}} \mid \Xi, L_{t_0}) - Pr(P_{2}^{cum,LS}(x_{t_1}) - \beta_{t_1}) \times Pr(\theta_{t_1}^{x_{2}^{LS}} \mid \Xi, L_{t_0})$	$\frac{4MF-3\Delta+2TF+5}{2\Delta^2}$	$2MF+TF+1$	$\frac{1}{8}(4MF+2TF-1)$	$\frac{1}{9}(2MF+TF-2)$	$\frac{1}{32}(4MF+2TF-7)$
$Pr(\beta_{t_1} - P_{2}^{cum,LB}(x_{t_1})) \times Pr(\theta_{t_1}^{x_{2}^{LB}} \mid \Xi, L_{t_0}) - Pr(\beta_{t_1} - P_{-1}^{cum,LB}(x_{t_1})) \times Pr(\theta_{t_1}^{x_{-1}^{LB}} \mid \Xi, L_{t_0})$ $Pr(P_{-1}^{cum,LS}(x_{t_1}) - \beta_{t_1}) \times Pr(\theta_{t_1}^{x_{-1}^{LS}} \mid \Xi, L_{t_0}) - Pr(P_{2}^{cum,LS}(x_{t_1}) - \beta_{t_1}) \times Pr(\theta_{t_1}^{x_{2}^{LS}} \mid \Xi, L_{t_0})$	$\frac{2MF-3\Delta+4TF+5}{2\Delta^2}$	$MF+2TF+1$	$\frac{1}{8}(2MF+4TF-1)$	$\frac{1}{9}(MF+2TF-2)$	$\frac{1}{32}(2MF+4TF-7)$
$Pr(\beta_{t_1} - P_{2}^{cum,LB}(x_{t_1})) \times Pr(\theta_{t_1}^{x_{2}^{LB}} \mid \Xi, L_{t_0}) - Pr(\beta_{t_1} - P_{-2}^{cum,LB}(x_{t_1})) \times Pr(\theta_{t_1}^{x_{-2}^{LB}} \mid \Xi, L_{t_0})$ $Pr(P_{-2}^{cum,LS}(x_{t_1}) - \beta_{t_1}) \times Pr(\theta_{t_1}^{x_{-2}^{LS}} \mid \Xi, L_{t_0}) - Pr(P_{2}^{cum,LS}(x_{t_1}) - \beta_{t_1}) \times Pr(\theta_{t_1}^{x_{2}^{LS}} \mid \Xi, L_{t_0})$	$\frac{-3\Delta+6TF+9}{2\Delta^2}$	$3(TF+1)$	$\frac{3}{8}(2TF+1)$	$\frac{TF}{3}$	$\frac{3}{32}(2TF-1)$
$Pr(\beta_{t_1} - P_{1}^{cum,LB}(x_{t_1})) \times Pr(\theta_{t_1}^{x_{1}^{LB}} \mid \Xi, L_{t_0}) - Pr(\beta_{t_1} - P_{-1}^{cum,LB}(x_{t_1})) \times Pr(\theta_{t_1}^{x_{-1}^{LB}} \mid \Xi, L_{t_0})$ $Pr(P_{-1}^{cum,LS}(x_{t_1}) - \beta_{t_1}) \times Pr(\theta_{t_1}^{x_{-1}^{LS}} \mid \Xi, L_{t_0}) - Pr(P_{1}^{cum,LS}(x_{t_1}) - \beta_{t_1}) \times Pr(\theta_{t_1}^{x_{1}^{LS}} \mid \Xi, L_{t_0})$	$\frac{TF-MF}{\Delta^2}$	$TF-MF$	$\frac{TF-MF}{4}$	$\frac{TF-MF}{9}$	$\frac{TF-MF}{16}$
$Pr(\beta_{t_1} - P_{1}^{cum,LB}(x_{t_1})) \times Pr(\theta_{t_1}^{x_{1}^{LB}} \mid \Xi, L_{t_0}) - Pr(\beta_{t_1} - P_{-2}^{cum,LB}(x_{t_1})) \times Pr(\theta_{t_1}^{x_{-2}^{LB}} \mid \Xi, L_{t_0})$ $Pr(P_{-2}^{cum,LS}(x_{t_1}) - \beta_{t_1}) \times Pr(\theta_{t_1}^{x_{-2}^{LS}} \mid \Xi, L_{t_0}) - Pr(P_{1}^{cum,LS}(x_{t_1}) - \beta_{t_1}) \times Pr(\theta_{t_1}^{x_{1}^{LS}} \mid \Xi, L_{t_0})$	$\frac{-2MF+2TF+2}{\Delta^2}$	$-2MF+2TF+2$	$\frac{1}{2}(-MF+TF+1)$	$-\frac{2}{9}(MF-TF-1)$	$\frac{1}{8}(-MF+TF+1)$
$Pr(\beta_{t_1} - P_{-1}^{cum,LB}(x_{t_1})) \times Pr(\theta_{t_1}^{x_{-1}^{LB}} \mid \Xi, L_{t_0}) - Pr(\beta_{t_1} - P_{-2}^{cum,LB}(x_{t_1})) \times Pr(\theta_{t_1}^{x_{-2}^{LB}} \mid \Xi, L_{t_0})$ $Pr(P_{-2}^{cum,LS}(x_{t_1}) - \beta_{t_1}) \times Pr(\theta_{t_1}^{x_{-2}^{LS}} \mid \Xi, L_{t_0}) - Pr(P_{-1}^{cum,LS}(x_{t_1}) - \beta_{t_1}) \times Pr(\theta_{t_1}^{x_{-1}^{LS}} \mid \Xi, L_{t_0})$	$\frac{-MF+TF+2}{\Delta^2}$	$-MF+TF+2$	$\frac{1}{4}(-MF+TF+2)$	$\frac{1}{9}(-MF+TF+2)$	$\frac{1}{16}(-MF+TF+2)$

B.2 Case 2: $3\tau < \Delta \leq 5\tau$

We now consider different ranges of β valuations that are characterized by unique equilibrium strategies. As before we first consider the regime with a maximizing exchange and then a regime with a Social Planner setting optimal fees.

Exchange Maximizing Problem: Positive Fees $\Delta \in (3,5]$ With the exchange setting optimal fees there are three β ranges characterized by different equilibrium strategies: $\Delta \in (3,4]$, $\Delta \in (4,4.7]$ and $\Delta \in (4.7,5]$. All these β ranges are characterized by strictly positive fees, $\Xi_{PF} = \{0 \leq MF \leq 1, 0 \leq TF \leq 1\}$.

Subcase $\Delta \in (3,4]$: Given $3\tau < \Delta \leq 4\tau$, traders choose among the same orders as in Case 1. Note that Table B3 shows that a limit order to buy at P_2 (sell at P_{-2}), and a limit order to buy at P_{-2} (sell at P_2) are dominated strategies for this subcase. Hence, to determine the optimal MF and TF, we maximize the exchange profits conditional on the buyer choosing LBP_{-1,t_1}, the case of the seller choosing LSP_{1,t_1} arriving at t_1 being symmetric:

$$\max_{\substack{MF,TF \\ MF \leq \tau \\ TF \leq \tau \\ 3<\Delta\leq 4}} \pi^{Ex,LTM}(MF,TF) = \Big(Pr(x^{LB}_{-1,t_1}|\Xi,L_{t_0}) \times Pr(\theta_{t_1}^{x^{LB}_{-1}}|\Xi,L_{t_0})\Big) \times (MF+TF)$$

$$= -\frac{(\Delta-1)(\text{MF}+\text{TF})\left(-\frac{\Delta}{2}+\text{TF}+\frac{1}{2}\right)}{2\Delta^2} \tag{63}$$

The Kuhn-Tucker Lagrangian is:

$$L(MF,TF,\lambda_k,v_h) = \pi^{Ex,LTM}(MF,TF) - \lambda_1(-MF+1) - \lambda_2(-TF+\frac{\Delta-3}{2}) \tag{64}$$

The Kuhn-Tucker conditions are:

$$\frac{\partial L(MF,TF,\lambda_k,v_h)}{\partial MF} = \lambda_1 + \frac{(\Delta-1)(\Delta-2TF-1)}{4\Delta^2} \geq 0 \;\&\; MF \times \frac{\partial L(MF,TF,\lambda_k,v_h)}{\partial MF} = 0 \tag{65}$$

$$\frac{\partial L(MF,TF,\lambda_k,v_h)}{\partial TF} = \frac{\left(\frac{1}{2}-\frac{\Delta}{2}\right)MF + (1-\Delta)TF + \Delta\left(\Delta\left(\lambda_2+\frac{1}{4}\right)-\frac{1}{2}\right)+\frac{1}{4}}{\Delta^2} \geq 0 \;\&TF \times \frac{\partial L(MF,TF,\lambda_k,v_h)}{\partial TF} = 0 \tag{66}$$

$$\frac{\partial L(MF,TF,\lambda_k,v_h)}{\partial \lambda_1} = (MF-1) \geq 0 \;\&\; \lambda_1 \times \frac{\partial L(MF,TF,\lambda_k,v_h)}{\partial \lambda_1} = 0 \tag{67}$$

$$\frac{\partial L(MF,TF,\lambda_k,v_h)}{\partial \lambda_2} = (TF - \frac{\Delta-3}{2}) \geq 0 \;\&\; \lambda_2 \times \frac{\partial L(MF,TF,\lambda_k,v_h)}{\partial \lambda_2} = 0 \tag{68}$$

The equilibrium MF^* and TF^* that satisfy these conditions are given in the first line of (22): By substituting a given value of Δ into MF^* and TF^* in the first line (22), we obtain the equilibrium fees.

Table B4: Equilibrium Submission Probability This table reports the equilibrium submission probabilities for the buy side, $Pr(x^{LB}_{k,t_1}|\Xi, L_{t_0})$, conditional on the support Δ. Equilibrium submission probabilities for the sell side, $Pr(x^{LS}_{-k,t_1}|\Xi, L_{t_0})$ are symmetric.

	$0 < \Delta \leq 4\tau$		$4 < \Delta \leq 4.7\tau$	$4.7 < \Delta \leq 5\tau$
	Taker-Maker	Maker-Taker	Positive Fees	Positive Fees
$Pr(x^{LB}_{1,t_1}\|\Xi, L_{t_0})$		$\max\left\{0, \frac{1}{\Delta}[\frac{\Delta}{2}-\frac{1}{2}-MF]\right\}$		$\max\left\{0, \frac{1}{\Delta}[TF-MF]\right\}$ for $\beta > \frac{\Delta}{2}+9.5$
$Pr(x^{LB}_{-1,t_1}\|\Xi, L_{t_0})$	$\max\left\{0, \frac{1}{\Delta}[\frac{\Delta}{2}+\frac{1}{2}-MF]\right\}$		$\max\left\{0, \frac{1}{\Delta}[TF+1]\right\}$ for $\beta > MF+\frac{\Delta}{2}-TF+8$	$\max\left\{0, \frac{1}{\Delta}[\frac{\Delta}{2}+\frac{1}{2}-TF]\right\}$ for $MF+9.5 < \beta < MF+\frac{\Delta}{2}+9$
$Pr(x^{LB}_{-2,t_1}\|\Xi, L_{t_0})$			$\max\left\{0, \frac{1}{\Delta}[\frac{\Delta}{2}+MF-TF-2]\right\}$ for $10 < \beta < MF+\frac{\Delta}{2}-TF+8$	

Subcase $\Delta \in (4, 4.7]$: We have shown that for investor valuation supports with widths up $\Delta = 4$, there are dominant orders for potential buyers and sellers, and so the optimal order-submission strategy can be obtained by comparing the expected payoff associated with each possible order, as shown in Tables B1 and B3; in the latter we present the differences in expected payoffs conditional on different supports. However, for investor valuation supports with widths $\Delta > 4$, there are two possible equilibrium limit orders, which we report in Table B4 showing that both a limit buy order at P_{-1} and a limit buy at P_{-2} may be optimal depending on the investors' evaluation, β_{t_1}. We also report conditions on the value of β such that the equilibrium strategies hold. To determine the optimal MF and TF, the exchange maximizes its expected profits conditional on the buyer choosing either LBP_{-2,t_1}, or LBP_{-1,t_1} the case of the seller arriving at t_1 being symmetric:

$$\begin{aligned}
&\max_{\substack{MF,TF \\ MF\leq\tau \\ TF\leq\tau \\ MF+TF\geq 0 \\ 4<\Delta\leq 4.7}} \pi^{Ex,LTM}(MF,TF) \\
&= \left[Pr(x^{LB}_{-1,t_1}|\Xi,L_{t_0}) \times Pr(\theta_{t_1}^{x^{LB}_{-1}}|\Xi,L_{t_0}) + Pr(x^{LB}_{-2,t_1}|\Xi,L_{t_0}) \times Pr(\theta_{t_1}^{x^{LB}_{-2}}|\Xi,L_{t_0})\right] \times (MF+TF) \\
&= \frac{(\mathrm{MF}+\mathrm{TF})\left(\left(\frac{3}{4}-\frac{\Delta}{4}\right)\Delta+\mathrm{MF}+\left(\frac{\Delta}{2}-1\right)\mathrm{TF}-2\right)}{\Delta^2}
\end{aligned} \tag{69}$$

The Kuhn-Tucker Lagrangian is:

$$L(MF,TF,\lambda_k,v_h) = \pi^{Ex,LTM}(MF,TF) - \lambda_1(-MF+1) - \lambda_2(-TF+\frac{4-\Delta}{2})$$

The Kuhn-Tucker conditions are:

$$\frac{\partial L(MF,TF,\lambda_k,v_h)}{\partial MF} = \frac{8MFTF - 4MF(1+\Delta) + \Delta(-1-2TF+\Delta-4\Delta\lambda_1)}{4\Delta^2} \geq 0 \ \& \ MF \times \frac{\partial L(MF,TF,\lambda_k,v_h)}{\partial MF} = 0 \tag{70}$$

$$\frac{\partial L(MF,TF,\lambda_k,v_h)}{\partial TF} = \frac{4MF^2 + 4(1-3TF)TF - 2MF\Delta + \Delta(-1+\Delta-4\Delta\lambda_2)}{4\Delta^2} \geq 0 \ \& \ TF \times \frac{\partial L(MF,TF,\lambda_k,v_h)}{\partial TF} = 0 \tag{71}$$

$$\frac{\partial L(MF,TF,\lambda_k,v_h)}{\partial \lambda_1} = (-MF+1) \geq 0 \ \& \ \lambda_1 \times \frac{\partial L(MF,TF,\lambda_k,v_h)}{\partial \lambda_1} = 0 \tag{72}$$

$$\frac{\partial L(MF,TF,\lambda_k,v_h)}{\partial \lambda_2} = (-TF + \frac{4-\Delta}{2}) \geq 0 \ \& \ \lambda_2 \times \frac{\partial L(MF,TF,\lambda_k,v_h)}{\partial \lambda_2} = 0 \tag{73}$$

The equilibrium MF^* and TF^* that satisfy these conditions are in the second line of (22): By substituting a given value of Δ into MF^* and TF^* in the second line of (22), we obtain the equilibrium fees.

Subcase $\Delta \in (4.7, 5]$: In this case, the investor valuation support width can be as large as 5τ, which is the difference between P_3 and P_{-3}. So we also consider the investor's profit conditional on orders posted at P_3 and P_{-3}. Table B1 shows that the investor's profit is zero if he buys at P_3 or sells at P_{-3}. Table B4 shows that for this interval of the support the equilibrium strategies are either $x^{LB}_{1,t_1} = LBP_{1,t_1}$, or $x^{LB}_{-1,t_1} = LBP_{-1,t_1}$. Therefore, to determine the optimal MF and TF, we maximize the exchange profits conditional on the buyer optimally using these two strategies, the case of the seller arriving at t_1 being symmetric:

$$\max_{\substack{MF,TF \\ MF \leq \tau \\ TF \leq \tau \\ 4.7 < \Delta \leq 5}} \pi^{Ex,LTM}(MF,TF) \tag{74}$$

$$= \left[Pr(x^{LB}_{1,t_1}|\Xi, L_{t_0}) \times Pr(\theta^{x^{LB}_1}_{t_1}|\Xi, L_{t_0}) + Pr(x^{LB}_{-1,t_1}|\Xi, L_{t_0}) \times Pr(\theta^{x^{LB}_{-1}}_{t_1}|\Xi, L_{t_0}) \right] \times (MF+TF)$$

$$= \frac{(MF+TF)\left((\Delta-1)\Delta - 2(\Delta+1)MF + 4MFTF - 4TF^2 + 2TF\right)}{4\Delta^2}$$

The Kuhn-Tucker Lagrangian is:

$$L(MF,TF,\lambda_k,v_h) = \pi^{Ex,LTM}(MF,TF) - \lambda_1(-MF+1)$$

The Kuhn-Tucker conditions are:

$$\frac{\partial L(MF,TF,\lambda_k,v_h)}{\partial MF} = \frac{8MFTF - 4MF(1+\Delta) + \Delta(-1-2TF+\Delta+4\Delta\lambda_1)}{4\Delta^2} \geq 0 \;\&\; MF \times \frac{\partial L(MF,TF,\lambda_k,v_h)}{\partial MF} = 0 \tag{75}$$

$$\frac{\partial L(MF,TF,\lambda_k,v_h)}{\partial TF} = \frac{4MF^2 + 4(1-3TF)TF - 2MF\Delta + (1-\Delta)\Delta}{4\Delta^2} \geq 0 \;\&\; TF \times \frac{\partial L(MF,TF,\lambda_k,v_h)}{\partial TF} = 0 \tag{76}$$

$$\frac{\partial L(MF,TF,\lambda_k,v_h)}{\partial \lambda_1} = (-MF+1) \geq 0 \;\&\; \lambda_1 \times \frac{\partial L(MF,TF,\lambda_k,v_h)}{\partial \lambda_1} = 0 \tag{77}$$

The equilibrium MF^* and TF^* that satisfy these conditions are given in the third line of (22). Q.E.D.

Social Planner Problem: TM and MT $\Delta \in (3,5]$ Table 2 shows that with the Social Planner setting optimal fees there is a unique β range characterized by both TM and MT pricing.

Subcase $\Delta \in (3,5]$: Under the Taker-Maker regime, to determine the optimal MF and TF, the Social Planner maximizes total welfare from both limit buy orders and market sell orders, and exchange profit as defined in (36), (37) and (38), as well as (26) and (27). We present the welfare of the buy side of the market (the sell side being symmetric):

$$\max_{\substack{MF,TF \\ TF\leq\tau \\ MF\leq\tau \\ MF+TF\geq 0 \\ 3<\Delta\leq 5}} \sum_{t_z\in\{t_1,t_2\}} \left(W_{t_z}^{INV}(MF,TF) + W_{t_z}^{Ex}(MF,TF) \right) \tag{78}$$

$$= W_{t_1}^{INV}(x_{-1,t_1}^{LB}|\Xi,L_{t_0}) + W_{t_2}^{INV}(x_{-2,t_2}^{MS}|\Xi,L_{t_0}) + W_{t_2}^{INV}(x_{-1,t_2}^{MS}|\Xi,L_{t_0})$$
$$+ W_{t_2}^{INV}(x_{-2,t_2}^{MS}|\Xi,L_{t_0}) + \left[Pr(x_{-1,t_1}^{LB}, \theta_{t_1}^{x_{-1}^{LB}}|\Xi) + Pr(x_{-2,t_1}^{LB}, \theta_{t_1}^{x_{-2,t_1}^{LB}}|\Xi) \right](MF+TF)$$

where the welfare from a limit buy at P_{-1} and from a limit buy at P_{-2} with $3 < \Delta \leq 5$ are respectively:

$$W_{t_1}^{INV}(x_{-1,t_1}^{LB}|\Xi,L_{t_0}) = \int_{\beta_{t_1}\in B_{t_1}(x_{-1,t_1}^{LB},\Xi,L_{t_0})} [\beta_{t_1} - P(x_{-1,t_1}^{LB}) - MF)]\frac{1}{\Delta}d\beta_{t_1} \times Pr(\theta_{t_1}^{x_{-1}^{LB}}|\Xi,L_{t_0}) \tag{79}$$

$$
= \begin{cases} \frac{(TF+1)(-\Delta+TF+2)(-\Delta+2TF+1)}{4\Delta^2} & MF=1 \wedge -1<TF\leq 0 \wedge 3<\Delta\leq 5 \\ -\frac{(MF-TF-2)(-\Delta+2TF+1)(-\Delta+MF+TF+1)}{4\Delta^2} & -1\leq TF\leq 0 \wedge 3<\Delta\leq 5 \wedge 0\leq MF<1 \end{cases}
$$

$$
W_{t_2}^{INV}(x_{-2,t_2}^{MS}|\Xi,L_{t_0}) = Pr(x_{-2,t_1}^{LB}|\Xi,L_{t_0}) \times \int_{\beta_{t_2}\in B_{t_2}(x_{-2,t_2}^{MS},\Xi,L_{t_1})} [P(x_{-2,t_1}^{LB})-\beta_{t_2}-TF)]\tfrac{1}{\Delta}d\beta_{t_2} \quad (80)
$$

$$
= \begin{cases} -\frac{(-\Delta+2TF+3)(-\Delta-2MF+2TF+4)(2(MF+TF-1)-\Delta)}{16\Delta^2} & (\Delta\leq 5 \wedge \Delta>3 \wedge TF+1\geq 0 \wedge \\ & ((2MF>1 \wedge MF\leq 1 \wedge TF\leq 0) \vee (MF\geq 0 \wedge \\ & MF\geq TF+\tfrac{1}{2} \wedge 2MF\leq 1))) \vee \\ & (MF\geq 0 \wedge TF\leq 0 \wedge \\ & ((\Delta+2MF>2TF+4 \wedge MF<TF+\tfrac{1}{2} \wedge \\ & \Delta\leq 5) \vee (\Delta>3 \wedge \Delta+2MF<2TF+4))) \\ 0 & \text{Otherwise} \end{cases}
$$

Whereas the welfare from a market sell at P_{-1} and from a market sell at P_{-2} when $3<\Delta\leq 5$ are respectively:

$$
W_{t_2}^{INV}(x_{-1,t_2}^{MS}|\Xi,L_{t_0}) = Pr(x_{-1,t_1}^{LB}|\Xi,L_{t_0}) \times \int_{\beta_{t_2}\in B_{t_2}(x_{-1,t_2}^{MS},\Xi,L_{t_1})} [P(x_{-1,t_1}^{LB})-\beta_{t_2}-TF)]\frac{1}{\Delta}d\beta_{t_2} \quad (81)
$$

$$
= \frac{(\Delta-2TF-1)^2}{8\Delta} \begin{pmatrix} \frac{1}{2} & 2MF+\Delta\leq 2TF+4 \wedge MF<TF+\frac{1}{2} \wedge 2MF\leq 1 \\ \frac{TF+1}{\Delta} & MF=1 \wedge TF>-1 \\ \frac{-MF+TF+2}{\Delta} & MF<1 \wedge \left(2MF>1 \vee MF\geq TF+\frac{1}{2} \vee 2MF+\Delta>2TF+4\right) \end{pmatrix}
$$

$$
W_{t_2}^{INV}(x_{-2,t_2}^{MS}|\Xi,L_{t_0}) = Pr(x_{-2,t_1}^{LB}|\Xi,L_{t_0}) \times \int_{\beta_{t_2}\in B_{t_2}(x_{-2,t_2}^{MS},\Xi,L_{t_1})} [P(x_{-2,t_1}^{LB})-\beta_{t_2}-TF)]\frac{1}{\Delta}d\beta_{t_2} \quad (82)
$$

$$
= \begin{cases} \frac{(\Delta-2TF-3)^2(\Delta+2MF-2TF-4)}{16\Delta^2} & MF\geq TF+\frac{1}{2} \vee 2MF>1 \vee \Delta+2MF>2TF+4 \\ 0 & \text{Otherwise} \end{cases}
$$

By substituting (79), (80), (81) and (82) into the welfare function of the Social Planner, (78), we obtain a functional form whose components are subject to different boundary conditions. The following component has the highest total welfare:

$$\max_{\substack{MF,TF\\TF\leq\tau\\MF\leq\tau\\MF+TF\geq 0\\3<\Delta\leq 5}} \sum_{t_z\in\{t_1,t_2\}} \left(W_{t_z}^{INV}(MF,TF)+W_{t_z}^{Ex}(MF,TF)\right) \tag{83}$$

$$= \frac{2\Delta^3-\Delta^2(4MF+6TF+3)+\Delta(MF(8TF+4)+4TF(TF+2)+7)+8(MF-TF-2)(MF+TF)}{16\Delta^2}$$

The optimal fees presented in Table 2 are determined by the boundary conditions of the different parts of the total welfare functional form. By substituting any $3<\Delta\leq 5$ and the optimal MF^* and TF^* in (84) we obtain the total welfare presented in Table 2. In Online Appendix we show the Social Planner's value function for the Taker-Maker case for the different support values ($\Delta\in\{3.5\tau,4\tau,4.5,\tau,5\tau\}$) in Table 2.

Subcase $\Delta\in(3,5]$: Under the Maker-Taker regime, the Social Planner maximizes total welfare from limit buy orders and market sell orders at P_{-1}, as well as limit buy order and market sell orders at P_1. As before, we present the welfare of the buy side of the market - the sell side being symmetric:

$$\max_{\substack{MF,TF\\MF\leq\tau\\TF\leq\tau\\MF+TF\geq 0\\3<\Delta\leq 5}} \sum_{t_z\in\{t_1,t_2\}} \left(W_{t_z}^{INV}(MF,TF)+W_{t_z}^{Ex}(MF,TF)\right) \tag{84}$$

$$= W_{t_1}^{INV}(x_{-1,t_1}^{LB}|\Xi,L_{t_0})+W_{t_1}^{INV}(x_{1,t_1}^{LB}|\Xi,L_{t_0})+W_{t_2}^{INV}(x_{-1,t_2}^{MS}|\Xi,L_{t_0})+W_{t_2}^{INV}(x_{1,t_2}^{MS}|\Xi,L_{t_0})$$

$$+\left[Pr(x_{-1,t_1}^{LB},\theta_{t_1}^{x_{-1}^{LB}}|\Xi)+Pr(x_{1,t_1}^{LB},\theta_{t_1}^{x_1^{LB}}|\Xi)\right](MF+TF)$$

Where the welfare from a limit buy at P_{-1} and from a limit buy at P_1 with $3<\Delta\leq 5$ are respectively:

$$W_{t_1}^{INV}(x_{-1,t_1}^{LB}|\Xi,L_{t_0}) = \int_{\beta_{t_1}\in B_{t_1}(x_{-1,t_1}^{LB},\Xi,L_{t_0})} [\beta_{t_1}-P(x_{-1,t_1}^{LB})-MF)]\frac{1}{\Delta}d\beta_{t_1}\times Pr(\theta_{t_1}^{x_{-1}^{LB}}|\Xi,L_{t_0}) \tag{85}$$

$$= \begin{cases} -\frac{(-\Delta+2TF+1)(-\Delta-2MF+2TF)(2(MF+TF-1)-\Delta)}{16\Delta^2} & (MF+\frac{3}{2}<TF\wedge MF\geq -1\wedge TF\leq 1\wedge((\Delta>3\wedge\Delta+2MF<2TF)\vee \\ & (\Delta+2MF>2TF\wedge\Delta\leq 5)))\vee(3<\Delta\leq 5\wedge((MF+\frac{3}{2}\geq TF\wedge \\ & TF\geq 0\wedge -1\leq MF\leq -\frac{1}{2})\vee(-\frac{1}{2}<MF\leq 0\wedge 0\leq TF\leq 1))) \\ 0 & \text{Otherwise} \end{cases}$$

$$W_{t_1}^{INV}(x_{1,t_1}^{LB}|\Xi, L_{t_0}) = \int_{\beta_{t_1} \in B_{t_1}(x_{1,t_1}^{LB}, \Xi, L_{t_0})} [\beta_{t_1} - P(x_{1,t_1}^{LB}) - MF)] \frac{1}{\Delta} d\beta_{t_1} \times Pr(\theta_{t_1}^{x_1^{LB}}|\Xi, L_{t_0}) \tag{86}$$

$$= \begin{cases} -\frac{(MF-TF)(-\Delta+2TF-1)(-\Delta+MF+TF+1)}{4\Delta^2} & 0 \le TF \le 1 \wedge 3 < \Delta \le 5 \wedge -1 \le MF < 0 \\ \frac{TF\left(\Delta^2+2TF^2-3\Delta TF+TF-1\right)}{4\Delta^2} & MF = 0 \wedge 0 < TF \le 1 \wedge 3 < \Delta \le 5 \end{cases}$$

Whereas the welfare from a market sell order at P_{-1} and from a market sell order at P_1 when $3 < \Delta \le 5$ are respectively:

$$W_{t_2}^{INV}(x_{-1,t_2}^{MS}|\Xi, L_{t_0}) = Pr(x_{-1,t_1}^{LB}|\Xi, L_{t_0}) \times \int_{\beta_{t_2} \in B_{t_2}(x_{-1,t_2}^{MS}, \Xi, L_{t_1})} [P(x_{-1,t_1}^{LB}) - \beta_{t_2} - TF)] \frac{1}{\Delta} d\beta_{t_2} \tag{87}$$

$$= \begin{cases} \frac{(\Delta-2TF-1)^2(\Delta+2MF-2TF)}{16\Delta^2} & MF+\frac{3}{2} \ge TF \vee MF > -\frac{1}{2} \vee \Delta+2MF > 2TF \\ 0 & \text{Otherwise} \end{cases}$$

$$W_{t_2}^{INV}(x_{1,t_2}^{MS}|\Xi, L_{t_0}) = Pr(x_{1,t_1}^{LB}|\Xi, L_{t_0}) \times \int_{\beta_{t_2} \in B_{t_2}(x_{1,t_2}^{MS}, \Xi, L_{t_1})} [P(x_{1,t_1}^{LB}) - \beta_{t_2} - TF)] \frac{1}{\Delta} d\beta_{t_2} \tag{88}$$

$$= \frac{(\Delta-2TF+1)^2}{8\Delta} \left(\begin{array}{cc} \frac{1}{2} & 2MF+\Delta \le 2TF \wedge MF+\frac{3}{2} < TF \wedge MF \le -\frac{1}{2} \\ \frac{TF}{\Delta} & MF = 0 \wedge TF > 0 \\ \frac{TF-MF}{\Delta} & MF < 0 \wedge \left(2MF+1 > 0 \vee MF+\frac{3}{2} \ge TF \vee 2MF+\Delta > 2TF\right) \end{array} \right)$$

By substituting (85), (86), (87) and (88) into the welfare function of the Social Planner, (84), we obtain a functional form whose components are subject to different boundary conditions. The following component has the highest total welfare:

$$\max_{\substack{MF,TF \\ MF \le \tau \\ TF \le \tau \\ MF+TF \ge 0 \\ 3<\Delta\le 5}} \sum_{t_z \in \{t_1,t_2\}} \left(W_{t_z}^{INV}(MF,TF) + W_{t_z}^{Ex}(MF,TF) \right)$$

$$= \frac{(\Delta-1)\Delta(2\Delta+1)+8MF^2-4\Delta MF(\Delta-2TF+1)+4(\Delta-2)TF^2+2(4-3\Delta)\Delta TF}{16\Delta^2} \tag{89}$$

The optimal fees presented in Table 2 are determined by the boundary conditions of the different parts of the total welfare functional form. By substituting any $3 < \Delta \leq 5$ and the optimal MF^* and TF^* in (89) we obtain the total welfare presented in Table 2. In the Online Appendix we show the Social Planner's value function for the Maker-Taker case for the different values of the support ($\Delta \in \{3.5, 4, 4.5, , 5\}$) in Table 2.