Uncertainty, Contracting, and Beliefs in Organizations*

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Abstract

We examine a multidivisional firm with headquarters exposed to moral hazard by division managers under uncertainty. We show the aggregation and linearity properties of Holmström and Milgrom (1987) hold under IID ambiguity of Chen and Epstein (2002). Due to uncertainty aversion, agents' beliefs depend endogenously on their exposure to uncertainty, either for their position in the organization (hierarchical exposure) or contracts (contractual exposure). Incentive contracts, by loading primarily on division cash-flow, lead division managers to be more conservative than headquarters, aggravating moral-hazard. By hedging uncertainty, headquarters design contracts that reduce disagreement, lower incentive provision costs, promoting effort. Because hedging uncertainty interacts with hedging risk, optimal contracts differ from those in standard principal-agent models. Our model helps explain the prevalence of equity-based incentive contracts and the rarity of relative-performance compensation.

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The provision of incentives in organizations is essential for economic efficiency. A key question is to determine appropriate performance measures for incentive pay. Managerial contracts often are a combination of base-pay, based on narrowly defined division-specific performance measures ("pay-for-performance"), plus a component linked to overall firm profitability (i.e. bonuses, equity-based pay, and other "aggregate" performance measures).[1] The distinction between equity-based and division-specific pay is particularly important for lower-level managers. The case for equity-based incentive contracts for top managers is rather strong as they are responsible for the performance of the overall firm. Absent inter-dependencies across divisions, the use of equity-based pay for division managers and rank-and-file employees is more puzzling. For lower-level employees, equity-based compensation reduces responsiveness of pay to actions, weakening incentives at the cost of increasing their overall risk exposure. In addition, when cash-flows are positively correlated across divisions, to reduce harmful risk bearing incentive contracts should display a relative-performance component, a feature more rarely observed in practice.

We study the impact of uncertainty (or "ambiguity") aversion on the design of incentive contracts in organizations.[2] The key feature of our approach is to acknowledge that most corporate decisions are taken without full knowledge of the probability distributions involved, a situation characterized as uncertainty (Knight, 1921). We model uncertainty aversion by adopting the Minimum Expected Utility approach of Gilboa and Schmeidler (1989), within the continuous time framework with stationary IID uncertainty of Chen and Epstein (2002).

We consider a multi-division firm with headquarters, HQ (the principal), and (two) division managers (the agents). Division cash-flows depend on unobservable

[1]The use of aggregate performance measures, such as bonuses, has been documented in the accounting literature (see e.g., Bushman et al., 1995, and more recently Bouwens and Van Lent, 2007, and Labro and Omartian, 2021). See Murphy (1999, 2013), Frydman and Jenter (2010), Oyer and Schaefer (2011), and Edmans, Gabaix and Jenter (2017) for extensive surveys.

[2]The importance of ambiguity, and the aversion to it, in affecting individual decision making has been shown in both experimental and empirical studies (see, for example, Bossaerts et al., 2010, Hong et al., 2017, Anderson et al., 2009, Ju and Miao, 2012, Jeong et al, 2015; Epstein and Schneider, 2008, and Machina and Siniscalchi, 2014, offer comprehensive reviews).

effort exerted by division managers, and can be (positively or negatively) correlated. Building on Holmström and Milgrom (1987), division managers exert a continuous level of effort and consume only at the end of a finite horizon. To isolate the effect of uncertainty on incentive pay, we rule out synergies or other inter-dependencies across divisions (as, for example, in Holmström, 1982).

Traditional principal-agent theory (Holmström, 1979, 1982*)* suggests that, in this situation, to limit risk exposure, incentive contracts should depend only on performance measures that are informative on actions (the "informativeness principle," Holmström, 2017).[3] The implication is that contracts should hedge division managers' risk by giving a negative (positive) exposure to variables that are positively (negatively) correlated to division cash-flow's residual risk.

These predictions change substantially in the presence of uncertainty aversion. We begin by showing that the aggregation and linearity property of Holmström and Milgrom (1987) hold in our environment with stationary uncertainty. Next, we argue that uncertainty aversion creates the potential for a divergence between division managers' and HQ beliefs. Such disagreement is endogenous, and has two adverse effects. First, traditional incentive contracts, by loading primarily on division cash-flows, lead division managers to hold more conservative estimates than HQ on the productivity of their own division, with a negative impact on incentives to exert effort. More conservative beliefs are due to division managers' greater exposure to uncertainty on their own division than HQ, who instead have exposure to the overall firm. The implication is that HQ must increase pay-for-performance sensitivity to elicit any desired level of effort. Second, disagreement with HQ leads division managers to value compensation contracts at a discount with respect to the value attributed by the (more confident) HQ, increasing the cost of incentive provision. We denote this discount as the (Knightian) "disagreement discount."

HQ can reduce the negative impact of disagreement by managing individual exposure to uncertainty through contracts, with beneficial effects on incentives. The role of contracts in managing beliefs is novel in the theory of contract design. It is a direct consequence of uncertainty aversion and the property that beliefs, and the

[3]Responsiveness of CEO pay to risk factors not informative on their actions ("pay-for-luck") has beed documented by several studies (see, e.g., Bertrand and Mullainathan, 2001, and, more recently, Choi, Gipper and Shi, 2020).

extent of disagreement, are determined endogenously and depend on the exposure of agents to the sources of uncertainty. Differential exposure to uncertainty may be due to their position in the organization (hierarchical exposure) or to the contractual relationships that bind agents (contractual exposure). Hierarchical and contractual exposure concur together to determine the prevailing structure of beliefs in an organization. We show that, by design of incentive contracts, HQ can affect agents beliefs with a positive impact on incentives. An implication is that equity-based incentive contracts can be used to realign internal beliefs, generating consensus by promoting a "shared view" in the organization.[4] The presence of a shared view can reinforce the beneficial effect of equity in fostering internal cooperation (Holmström, 1982).

The key economic driver in our paper is that uncertainty-averse agents hold (weakly) more favorable expectations and, thus, are more confident when they are exposed to multiple sources of uncertainty. This feature is a direct consequence of the benefits of uncertainty hedging that stem from the "uncertainty aversion" axiom of Gilboa and Schmeidler (1989). By being exposed to multiple sources of uncertainty, agents can lower their exposure to each source of uncertainty and, thus, hold more "optimistic" beliefs overall. We interpret "beliefs" broadly, as the probability measure that agents adopt (the "effective beliefs") to assess random variables and consequences of actions.

Optimal contracts depend on the level of uncertainty faced by division managers and HQ. In the simpler case where HQ are uncertainty neutral, optimal contracts depend on the extent of division managers' exposure to uncertainty and on the sign of the correlation between division cash-flows. When division managers face low uncertainty, incentive contracts have the same qualitative features as with no uncertainty: they have a component that depends on the performance of a manager's own division, the pay-for-performance part, plus a second component, the risk-hedging component, that depends on the cash-flow of the other division. When division cash-flows are

[4]The role of equity-based compensation to promote consensus in organizations is examined in Organization Behavior literature, such as Klein (1987), Pearsall, Christian, and Ellis (2010), and Blasi, Freeman, and Kruse (2016), among others. The importance of promoting a shared view is discussed in Zohar and Hofmann (2012). Advantages and disadvantages of disagreement in organizations has been studied in several papers: Dessein and Santos (2006); Landier, Sraer, and Thesmar (2009); Bolton, Brunnermeier, and Veldkamp (2013); and Van den Steen (2005) and (2010).

positively correlated, incentive contracts display relative-performance compensation; when they are negatively correlated, incentive contracts have cross-pay, that is, an equity component. With respect to the no-uncertainty case, uncertainty increases the cost of incentive provision, with the effect of decreasing pay-for-performance sensitivity and cross-division exposure.

When uncertainty faced by division managers is sufficiently large, uncertainty aversion creates the potential for a significant divergence between beliefs held by division managers and HQ. In this case, HQ find it desirable to hedge division managers' uncertainty by offering compensation contracts with greater cross-division exposure, at the cost of greater risk. By hedging division managers' exposure to uncertainty, HQ induce them to hold more favorable expectations on their divisions, with a positive impact on effort. Improvement of division managers beliefs also lowers the disagreement discount and, thus, the cost of incentive provision. Optimal compensation contracts will be determined by trading off costs and benefits of risk and uncertainty hedging. Interestingly, optimal contracts have cross-division exposure (with either equity or relative performance) even in the case of uncorrelated cash-flows. This property is in sharp contrast with the informativeness principle in traditional principal-agent problems with no uncertainty.

When HQ are uncertainty averse as well, their beliefs are also determined endogenously. HQ uncertainty aversion introduces an additional source of disagreement with division managers making it costlier for HQ to offer incentive contracts with relative performance. This happens because relative performance essentially involves division managers holding a "short" position in the other division, while HQ hold a "long" position in both divisions, exacerbating the disagreement discounts. The overall effect is to make contracts with equity-based pay more desirable. Interestingly, pure equity-based contracts are optimal when uncertainty is sufficiently large, irrespective of the correlation between divisional cash-flows.[5]

Our paper is linked to several streams of literature. The first one is the traditional principal-agent theory and the theory of optimal contract design within organizations. Contract theory builds on the seminal work by Mirrlees (1975), (1999) and (1976),

[5]DeMarzo and Kaniel (2017) argue that relative-performance compensation is not desirable when division managers have "keep-up-with-the-Joneses" preferences.

Holmström (1979), (1982), Shavell (1979), and Grossman and Hart (1983). One of the key results of the early stages of this literature is that compensation should be a function of all and only observable variables that are informative on the action selected by the agent. Incentive contracts more directly tailored to shareholder value, such as equity, are shown to be optimal when agents can choose their hidden action from rich sets of possible action-profiles (see, for example, Diamond, 1998, and Chassang, 2013). Oyer (2004) suggests that equity-based compensation (for example, through stock-option plans) have the advantage of directly adjusting employees' compensation to their outside options (which may be correlated to firm value), facilitating satisfaction of the participation constraints.

The second stream is the emerging literature on contract theory under uncertainty. In the spirit of Innes (1990), Lee and Rajan (2020) study the optimal incentive contract between a principal and a single agent where both parties are risk-neutral but uncertainty-averse and the source of uncertainty is the exact probability distribution of the random cash-flow. The paper shows that, contrary to basic case of uncertainty-neutrality of Innes (1990), the optimal contract has equity-like components. Szydlowski and Yoon (2021) considers a dynamic contracting model where an uncertainty-averse principal designs an optimal (dynamic) contract for an uncertainty-neutral agent, and the source of uncertainty is the agent's cost of effort. Different from our paper, uncertainty leads principals to increase pay-for-performance sensitivity (to preserve incentives under the worst-case scenario). Miao and Rivera (2016) consider the optimal contract between uncertainty-averse principal and an uncertainty- and risk-neutral agent, and show that the principal's preference for robustness can cause the incentive-compatibility constraint to be lax.[6] The main feature of these papers is to consider principal-agents problems in isolation. In contrast, in our paper we consider the problem of incentive contracting within organizations, where the principal design contracts with multiple agents who are exposed to distinct sources of uncertainty. In addition, in our paper agents are both risk- and uncertainty averse, creating a new tension between hedging risk and hedging uncertainty.

Closer to our paper, Kellner (2015) examines a principal-agent model with multiple agents and moral hazard, where the principal is risk and uncertainty neutral;

[6]Lee and Rivera (2021) consider optimal liquidity management under ambiguity.

agents can be risk and uncertainty averse and uncertainty is modeled as smooth ambiguity (Klibanoff et al., 2005). Agents are exposed to the same source of uncertainty about (mutually independent) probabilities measures over outcomes. The paper shows that, in this case, for sufficient large uncertainty aversion, optimal contracts have tournament-like features, an extreme form of relative-performance.

In Carroll (2015) a risk-neutral principal, who is uncertain about the set of actions available to a risk- and uncertainty-neutral agent, optimally grants the agent a linear contract that aligns their payoffs. Linear (or affine) contracts are optimal robust contracts under very weak assumptions on the source of uncertainty characterizing the set of technologies available to the agent.[7] In the spirit of Holmström (1982), Dai and Toikka (2018) examines a moral hazard in teams problem, where a risk-neutral principal designs contracts that are robust to uncertainty regarding the underlying game played by uncertainty-neutral agents. The paper shows that optimal robust contracts must have the property that agents' compensation covaries positively, and provides conditions under which optimal robust contracts are linear (or affine). Finally, Walton and Carroll (2019) show that, under mild conditions, optimal contracts are linear within several possible configurations of the organization structure, when principal are risk neutral and agents are risk and uncertainty neutral.

Our paper differs from these in several important ways. A common theme of these papers is to show that linear (or affine) contracts emerge as optimal robust contracts in situations where linearity would not otherwise be obtained in absence of uncertainty. In our paper we take the opposite tack, and we start from a situation similar to Holmström and Milgrom (1987), where optimal contracts are indeed linear without uncertainty.[8] We first show that the linearity property is preserved under stationary and IID uncertainty. We then characterize optimal linear contracts when

[7]Carroll and Meng (2016) provides a microfoundation of uncertainty resulting in linear contracts.

[8]In an earlier version of this paper, we study a discrete-time version of our model, and we show that the main results of this paper hold in a discrete-time framework where contracts are restricted to be linear in division outputs. The connection between optimal contracts in discrete and continuous time has been investigated by Hellwig and Schmidt (2002), Biais et al. (2007), and Sadzik and Stacchetti (2015), among others. By explicitly and directly focusing on continuous-time model, we ignore this important issue, which we leave for future research.

principals are risk neutral, agents are risk averse, and they can both be uncertainty averse. This approach allows us to isolate the specific effect of uncertainty aversion on optimal contract design: when agents are both risk and uncertainty averse, hedging uncertainty can interact with hedging risk, and the two goals can conflict with each other. When uncertainty is sufficiently large, the uncertainty-hedging motive can overcome the risk-hedging motive, reversing important properties of optimal incentive contracts absent uncertainty concerns.

Finally, our paper is related to recent literature on disagreement and heterogenous priors.[9] We argue that the presence of uncertainty, and the aversion to it, can generate differences of beliefs among agents, even in cases where agents are ex-ante identical and share the same set of "core beliefs." Disagreement in our economy emerges endogenously as the consequence of agents' differential exposure to uncertainty.

The paper is organized as follows. We describe the general contracting problem in Section 1. We show that, similar to Holmström and Milgrom (1987), the aggregation and linearity properties of optimal compensation contracts hold under stationary and IID ambiguity in Section 2. Section 3 examines the impact of incentive contracts on beliefs and effort under uncertainty. We study optimal incentive contracts offered by uncertainty-neutral HQ in Section 4, and by uncertainty-averse HQ in Section 5. Finally, we discuss the impact of uncertainty aversion on organizational beliefs in Section 6. Section 7 concludes with the model's implications and directions for further research. With the exception of the proof of Theorem 1, which is included in the body of the paper, all remaining proofs are the Technical Appendix.

1 Uncertainty and Contracting

1.1 The Basic Model

We consider a firm composed by two divisions (or business units): $d \in \{A, B\}$. Each division is run by a division manager supervised by HQ. At each instant $t \in [0, 1]$ each division manager continually chooses a level of effort, $a_{d,t} \in \mathbb{R}_+$, affecting the probability distribution of divisional cash-flows. We assume that cash-flows of both divisions, $Y_t \equiv (Y_{A,t}, Y_{B,t})$, follow the (joint) process

$$dY_t = \mu_t dt + \Gamma dW_t, \tag{1}$$

[9] Boot et al. (2006) and (2008), and Bayar, Chemmanur, and Liu (2011).

where $W_t = (W_{A,t}, W_{B,t}) \in \mathbb{R}^2$ is a standard bivariate Brownian motion defined on the filtered probability space $\left(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, P^a\right)$, with $Y_{A,0} = Y_{B,0} = 0$. Note (Y, W, P^a) is a weak solution to the stochastic differential equations in (1); all processes are progressively measurable with respect to the filtration $(\mathcal{F}_t)_{t\geq 0}$. Following Chen and Epstein (2002), P^a represents the "reference probability," assumed to be common for both division managers and HQ.[10]

Following Holmström and Milgrom (1987), we assume that division manager efforts affect only the drift of its own division with no externalities (or synergies) across divisions, and we set $\mu_t \equiv (\mu_{A,t}, \mu_{B,t})'$ with $\mu_{d,t} = a_{d,t}q_d$, where q_d represents the productivity of division d under the reference probability, P^a. We will refer to division managers' action profile as $a_t = (a_{A,t}, a_{B,t})'$. Division cash-flows are homoskedastic, with constant variance σ^2, and division cash-flows may be (positively or negatively) correlated, with correlation coefficient ρ. Further, we assume effort does not affect the variance-covariance matrix, Σ.[11] Thus, Γ is assumed to be the symmetric square root of the variance-covariance matrix, $\Sigma = \Gamma'\Gamma$, giving

$$\Sigma = \begin{bmatrix} \sigma^2 & \rho\sigma^2 \\ \rho\sigma^2 & \sigma^2 \end{bmatrix}, \Gamma \equiv \begin{bmatrix} \frac{\sigma}{2}\left(\sqrt{1+\rho} + \sqrt{1-\rho}\right) & \frac{\sigma}{2}\left(\sqrt{1+\rho} - \sqrt{1-\rho}\right) \\ \frac{\sigma}{2}\left(\sqrt{1+\rho} - \sqrt{1-\rho}\right) & \frac{\sigma}{2}\left(\sqrt{1+\rho} + \sqrt{1-\rho}\right) \end{bmatrix}. \quad (2)$$

Exerting effort is costly: each division manager suffers an instantaneous monetary cost $c_d(a_{d,t})\,dt$, where $c_d : \mathbb{R}_+ \to \mathbb{R}_+$ is a continuously differentiable, increasing and convex function. For analytical tractability, we set $c_d(a_{d,t}) = \frac{1}{2Z_d}a_{d,t}^2$, where Z_d characterizes effort efficiency of division managers. Following Holmström and Milgrom (1987), division managers and HQ exhibit preferences with constant absolute risk aversion (CARA), and are paid and consume only at the end of the period, $t = 1$.[12]

Effort exerted by each division manager is not observable by either HQ or the other division manager, creating moral hazard. HQ promote effort by offering division managers incentive contracts, $\{w_d\}_{d\in\{A,B\}}$, as follows. We assume that output from each division, $Y_{d,t}$, is publicly observable, and we let $h_t = \{Y_s | s \leq t\}$ represent the

[10]Hansen et al. (2006) refer to the measure P^a as the "approximating model."
[11]Hemmer (2017) and Ball et al. (2020) study contracts when effort affects Σ.
[12]By restricting pay and consumption to occur only at the end, we avoid two complications: intertemporal consumption smoothing and private savings. These issues are examined, for example, in He et al. (2017) who study a dynamic agency problem in a setting without Knightian uncertainty.

entire history of cash-flows from both divisions at each point in time t. HQ can condition compensation to each division manager on the entire history, that is $w_d(h_1)$. We impose the usual square-integrable condition that $E^{P^a}[w_d(h_1)]^2 < \infty$.

Given an incentive contract $w_d(h_1)$ and effort level process $a_d \equiv \{a_{d,t}\}_{t\in[0,1]}$, division manager $d \in \{A, B\}$ earns an end-of-period payoff

$$U_d(h_1) \equiv u\left(w_d(h_1) - \int_0^1 c_d(a_{d,t})\, dt\right), \tag{3}$$

where $u(w) = -e^{-rw}$, and r represents the coefficient of absolute risk aversion for both divisional managers. Similarly, HQ earn end-of-period payoff equal to

$$\Pi(h_1) \equiv \pi(Y_{A,1} + Y_{B,1} - w_A(h_1) - w_B(h_1)), \tag{4}$$

where $\pi(X) = -e^{-RX}$, and R represents the coefficient of absolute risk aversion for company HQ. Because processes are in L^2, they both have finite expectation.

The differential game unfolds as follows. At the beginning of the period, $t = 0$, HQ choose incentive contracts $w_d(h_1)$ for each division manager $d \in \{A, B\}$. We assume that HQ can commit to contracts $\{w_d(h_1)\}_{d\in\{A,B\}}$, which are observable to both managers. After incentive contracts are offered and accepted, division managers continuously and simultaneously choose their level of effort, $a_{d,t}$, after observing history h_t. At the end of the period, $t = 1$, division managers are compensated according to the realized history, h_1, and consumption takes place.

1.2 Uncertainty aversion

We model uncertainty aversion by adopting the minimum expected utility (MEU) approach of Chen and Epstein (2002), a dynamic extension of Gilboa and Schmeidler (1989). We assume that both HQ and division managers are not sure about (i.e., they are ambiguous on) the probability measure P^a. Following Chen and Epstein (2002), we consider beliefs distortions that are mutually absolutely continuous measures with respect to P^a, allowing us to use Girsanov's Theorem.[13] Define a density generator to be a $\mathbb{R}^2$-valued $\mathcal{F}_t$-predictable process θ_t satisfying the Novikov condition,

[13] Miao and Rivera (2016) and Szydlowski and Yoon (2021) use a similar approach.

$E^{P^a}\left[\exp\left(\frac{1}{2}\int_0^1 \theta_s \cdot \theta_s ds\right)\right] < \infty$, so that the process

$$z_t^\theta \equiv \exp\left\{-\frac{1}{2}\int_0^t \theta_s \cdot \theta_s ds - \int_0^t \theta_s dW_s\right\} \tag{5}$$

is a $(P^a, \mathcal{F}_t)$ martingale. By Girsanov's Theorem, θ_t generates an equivalent probability measure $\tilde{P}^{a,\theta}$ on $(\Omega, \mathcal{F})$ such that

$$\frac{d\tilde{P}^{a,\theta}}{dP^a}|_{\mathcal{F}_t} = z_t^\theta, \tag{6}$$

where z_t^θ is the Radon-Nikodym derivative of $\tilde{P}^{a,\theta}$ with respect to P^a when restricted to $\mathcal{F}_t$. Note that, from Girsanov's Theorem, the process

$$W_t^\theta = W_t + \int_0^t \theta_s ds, \tag{7}$$

is a standard Brownian motion under the new measure $\tilde{P}^{a,\theta}$.

Under the measure $\tilde{P}^{a,\theta}$, divisional cash-flows Y^θ follow the process

$$dY_t^\theta = Qa_t dt + \Gamma\left(dW_t^\theta - \theta_t dt\right) = \mu^\theta(a_t)dt + \Gamma dW_t^\theta, \tag{8}$$

where

$$Q \equiv \begin{bmatrix} q_A & 0 \\ 0 & q_B \end{bmatrix} \quad \text{and} \quad \mu^\theta(a_t) \equiv Qa_t - \Gamma\theta_t. \tag{9}$$

Thus, the density generator process θ_t describes decision makers' ("distorted") beliefs on the instantaneous productivity of both divisions.

Following Chen and Epstein (2002), we assume that uncertainty is IID. We allow for the possibility that HQ and division managers may be exposed to different degrees of uncertainty, as follows. For division managers and HQ, we let $\theta_{\delta,t} \in K_\delta(a_t)$, $\delta \in \{HQ, A, B\}$, for all $t \in [0,1]$, where $K_\delta \in \mathbb{R}^2$ is set-continuous (both upper- and lower-hemicontinous) with $K_\delta(a_t)$ a convex set for all a_t. Let

$$\mathcal{P}_\delta^\Theta(a_t) = \left\{\tilde{P}^{a,\theta} | \theta_t \in K_\delta(a_t), \forall t\right\} \tag{10}$$

be the set of admissible priors for division managers and HQ. Note that $\tilde{P} \in P_\delta^\Theta(a_t)$ if and only if there is a $\theta_{\delta,t}$ such that $\tilde{P} = \tilde{P}^{a,\theta}$ and $\theta_{\delta,t} \in K_\delta(a_t)$ for all t. At times, we will assume that divisions are symmetric, that is, we set

$$(S): \qquad Z_A = Z_B \equiv Z, \quad q_A = q_B \equiv q, \quad \text{and} \quad K_A(a_t) = K_B(a_t) \equiv K(a_t). \tag{11}$$

IID uncertainty can be interpreted as nature drawing independent increments $dW_t^{a,\theta}$ of the process $W_t^{a,\theta}$ from different urns at each point in time. These assumptions imply that a division's past cash-flow realizations are not informative on future cash-flows, thus excluding learning (similar to Chen and Epstein, 2002). Importantly, they ensure that divisional managers and HQ face stationary uncertainty. Note that the core of beliefs is rectangular over time, as required for time consistency by Chen and Epstein (2002). This is because the set of priors does not vary over time. However, the set K may not be a rectangle. Indeed, similar to Equation (3.12) of Chen and Epstein (2002), we will consider strictly convex ("round") sets K. This approach implies time-invariant uncertainty hedging.

2 Aggregation and Linearity under Uncertainty

At the beginning of the game, $t = 0$, HQ offer division managers a pair of contracts, $w(h_1) \equiv \{w_d(h_1)\}_{d\in\{A,B\}}$, and a set of (history-dependent) instructions $a \equiv \{a_d\}_{d\in\{A,B\}}$ to maximize expected payoff, that is to solve

$$\max_{\{w,a\}} \min_{\tilde{P}\in\mathcal{P}_{HQ}^{\Theta}(\{a_d,a_{d'}\})} E^{\tilde{P}}\pi\left(Y_{A,1}+Y_{B,1}-w_A(h_1)-w_B(h_1)\right) \tag{12}$$

subject to the constraints that (i) each division managers choose an effort process, a_d, given the other division manager's action profile, to solve

$$\max_{\tilde{a}_d} \min_{\tilde{P}\in\mathcal{P}_{d}^{\Theta}(\{\tilde{a}_d,a_{d'}\})} E_t^{\tilde{P}}u\left(w_d(h_1)-\int_0^1 c_d(\tilde{a}_{d,t})\,dt\right), \tag{13}$$

and (ii) the pairs $\{a_d, w_d(h_1)\}_{d\in\{A,B\}}$ satisfy their participation constraints

$$\min_{\tilde{P}\in\mathcal{P}_{d}^{\Theta}(\{a_d,a_{d'}\})} E_0^{\tilde{P}}u\left(w_d(h_1)-\int_0^1 c_d(a_{d,t})\,dt\right) \geq u_0 = 0 \tag{14}$$

for $d, d' \in \{A, B\}$, and $d \neq d'$, where u_0 is a division manager's reservation utility, which is normalized to zero (without loss of generality). Note that in problem (12) - (14) a division manager's uncertainty exposure is endogenous and is determined by the incentive contract, $w_d(h_1)$, offered by HQ. Contractual exposure concurs to determine a division manager's effective beliefs, $\tilde{P}_d$. Given their higher-level position in the firm hierarchy, HQ exposure to uncertainty is determined by their residual claim in firm cash-flow, given incentive contracts offered to both division managers in the firm.[14] HQ hierarchical exposure concurs to determine HQ effective beliefs, $\tilde{P}_{HQ}$. The triplet $\{\tilde{P}_{HQ}, \tilde{P}_A, \tilde{P}_B\}$ determines the belief structure prevalent in the firm.

Definition 1 *An equilibrium is a set of contracts , $w(h_1) \equiv \{w_d(h_1)\}_{d\in\{A,B\}}$, and action processes $\{a_A, a_B\}$, such that:*

- *(i) Given incentive contracts $w(h_1)$, for every history h_t each division manager selects effort, a_d, optimally, solving (13), given the other division manager's action process, $a_{d'}$ for $d' \neq d$;*
- *(ii) HQ offer contracts $w(h_1)$ that maximizes (12) subject to (13) - (14).*

The following theorem establishes that the aggregation and linearity property of Holmström and Milgrom (1987) holds in the case of stationary (IID) uncertainty with two division managers.

Theorem 1 *The optimal contract between HQ and division managers is linear in cash-flows, $w_d(h_1) = s_d + \beta_d Y_{d,1} + \gamma_d Y_{d',1}$, with constant s_d, β_d, γ_d, for $d, d' \in \{A, B\}$, and $d \neq d'$. The optimal contract induces constant effort, $a_{d,t} = a_d$, and constant beliefs, $\tilde{P}^{a,\theta}$, with constant distortions, $\theta_{d,t} = \theta_d$ and $\theta_{HQ,t} = \theta_{HQ}$, for all t.*

Proof of Theorem 1. Each division manager selects a_t to maximize

$$U_{d,t} \equiv \min_{\tilde{P}\in\mathcal{P}_d^{\Theta}(\{\tilde{a}_d, a_{d'}\})} E_t^{\tilde{P}} u\left(w_d(h_1) - \int_0^1 c_d(\tilde{a}_{d,t})\, dt\right).$$

Given worst-case scenario process, θ_d^*, by the Law of Iterated Expectations, $U_{d,t}$ is a martingale adapted to Y^θ. By the martingale representation theorem, $U_{d,t}$ is an Itô Process adapted to Y^θ with zero drift (Theorem 4.33 of Jacod and Shiryaev, 1987). Define $\tilde{w}_{d,t}$ as the certainty equivalent pay, given history: $u\left(\tilde{w}_{d,t} - \int_0^1 c_d\left(a_{d,t}^*\right) dt\right) = U_{d,t}$. As a twice-continuously differentiable function of an Itô Process, $\tilde{w}_{d,t}$ is an Itô Process: $d\tilde{w}_{d,t} = A_{d,t}dt + B'_{d,t}dY^\theta$ for predictable processes $A_{d,t} \in \mathbb{R}$, $B_{d,t} \in \mathbb{R}^2$. Because $u = -e^{-rw}$, $u_t = 0$, $u_w = re^{-rw}$, and $u_{ww} = -r^2e^{-rw}$, so by Itô's Lemma,

$$dU_d = re^{-r\tilde{w}_{d,t}}\left[A_{d,t} + B'_{d,t}\left(Qa_t - \Gamma\theta_t\right) - \frac{r}{2}B'_{d,t}\Sigma B_{d,t}\right]dt + re^{-r\tilde{w}_{d,t}}B'_{d,t}\Gamma dW_t^\theta.$$

[14] Note that, for simplicity, we assume that HQ are full residual claimants in firm cash-flow. More generally, HQ themselves act in the context of incentive contracts set-up by a compensation committee, exposing them to contractual exposure as well.

This is the evolution of expected utility along the equilibrium path, with optimal effort process $a^*_{d,t}$, and worst-case scenario process $\theta^*_{d,t}$. Because U_d is a martingale, the drift is zero: $A_{d,t} = \frac{r}{2} B'_{d,t} \Sigma B_{d,t} - B'_{d,t} \left(Q a^*_t - \Gamma \theta^*_t \right)$.

Off-equilibrium, the agent could deviate from optimal effort $a^*_{d,t}$ to $\tilde{a}_{d,t}$ from time t to $t + \Delta$. This would give them utility $\hat{U}_d = E_{\tilde{a}_{d,t}} \left[u \left(w_d (h_1) - \int_0^1 c_d (\tilde{a}_{d,t}) \, dt \right) \right]$. Because u is CARA, we can express this as $\hat{U}_d = U_d C_d$, where U_d is the equilibrium utility and $C_d = \exp \left[r \int_t^{t+\Delta} c_d (\tilde{a}_{d,t}) - c_d \left(a^*_{d,t} \right) dt \right]$. By the product rule, $d\hat{U}_d = dU_d \cdot C_d + U_d \cdot dC_d$. Note $\frac{dC_d}{C_d} = r \left(c_d (\tilde{a}_{d,t}) - c_d \left(a^*_{d,t} \right) \right)$. Substituting in,

$$
\begin{aligned}
d\hat{U}_d &= r e^{-r \tilde{w}_{d,t}} C_d \left[A_{d,t} + B'_{d,t} \left(Q \tilde{a}_t - \Gamma \theta^* (\tilde{a}_t) \right) - \frac{r}{2} B'_{d,t} \Sigma B_{d,t} - c_d (\tilde{a}_{d,t}) + c_d \left(a^*_{d,t} \right) \right] dt \\
&\quad + r e^{-r \tilde{w}_{d,t}} C_d B'_{d,t} \Gamma dW^\theta_t .
\end{aligned}
$$

Note minimizing (maximizing) expected utility is equivalent to minimizing (maximizing) the drift of the process expected utility. Because effort and beliefs do not affect $A_{d,t}$ or $\frac{r}{2} B'_{d,t} \Sigma B_{d,t}$, effort and beliefs solve $\max_a \min_\theta \Upsilon$, where

$$
\Upsilon \equiv B'_{d,t} \left(Q \tilde{a} - \Gamma \theta \right) - c_{d,t} (\tilde{a}_t) .
$$

Define the monetary payoff to the principal as $X \equiv Y_A + Y_B - w_A - w_B$, so

$$
dX = \left[(1 - B_{A,t} - B_{B,t})' (Qa - \Gamma \theta) - A_{A,t} - A_{B,t} \right] dt + (1 - B_{A,t} - B_{B,t})' \Gamma dW^\theta_t .
$$

Note $\Pi(t) = \min_{\tilde{P}^\theta \in \mathcal{P}^\Theta_{HQ}} E_t \Pi(1)$, so $\Pi(t)$ is an Itô Process. Because HQ utility is CARA, $\pi = -e^{-RX}$, so $\pi_t = 0$, $\pi_x = R \, e^{-RX}$, and $\pi_{xx} = -R^2 e^{-RX}$. Applying Itô's Lemma and substituting in for $A_{d,t}$,

$$
d\Pi = R \, e^{-RX} P dt + R \, e^{-RX} (1 - B_{A,t} - B_{B,\tau})' \Gamma dW^\theta_t
$$

where

$$
\begin{aligned}
P \equiv\; & (1 - B_{A,t} - B_{B,t})' \left(Q a^* - \Gamma \theta^{HQ*} \right) + B'_{A,t} \left(Q a^* - \Gamma \theta^{A*} \right) + B'_{B,t} \left(Q a^* - \Gamma \theta^{B*} \right) \\
& - \frac{r}{2} B'_{A,t} \Sigma B_{A,t} - \frac{r}{2} B'_{B,t} \Sigma B_{B,t} - \frac{R}{2} (1 - B_{A,t} - B_{B,\tau})' \Sigma (1 - B_{A,t} - B_{B,\tau})
\end{aligned}
$$

HQ solve $\max_B \min_{\theta \in K_{HQ}} P$ and division managers solve $\max_a \min_{\theta \in K_d} \Upsilon$. Neither of these depend on t, w, or X. Therefore, the optimal $B_{d,t} = B_d$, $a_{d,t} = a_d$, $\theta_{d,t} = \theta_d$, $\theta_{HQ,t} = \theta_{HQ}$ for all $t \in [0, 1]$: linear contracts are optimal. ■

The process $w(h_t)$ is progressively measurable with respect to the filtration $(\mathcal{F}_t)_{t\geq 0}$ generated by the bivariate Brownian motion W_t. Thus, similar to Holmström and Milgrom (1987), the martingale representation theorem ensures that it can be represented as an Itô process, which guarantees instantaneous linearity of incentive contracts. Translation invariance of CARA utility (precluding wealth effects) and IID uncertainty ensure that HQ and division managers face the same instantaneous optimization problem at every point in the tree, ensuring overall linearity. In the optimal contract, HQ grant division managers a constant share β_d of their own division, a constant exposure γ_d to the other division, inducing constant effort, $a_d = a_{d,t}$, and constant belief distortions, (θ_d, θ_{HQ}), for all t, for all $t \in [0,1]$. The coefficient β_d determines the pay-for-performance sensitivity of compensation, while the coefficient $|\gamma_d|$ determines its cross-division exposure. Equity-based compensation can be achieved by setting $\gamma_d > 0$, and relative-performance compensation by setting $\gamma_d < 0$.

Theorem 1 implies that the solution to the dynamic model is equivalent to the solution of a corresponding static problem where HQ offer only affine contracts that depend on division cash-flows. The static problem that corresponds to the dynamic model can be written in certainty equivalent form, as follows. Letting $b_A \equiv (\beta_A, \gamma_A)'$, $b_B \equiv (\gamma_B, \beta_B)'$, $\phi \equiv (\phi_A, \phi_B) = (\mathbf{1} - b_A - b_B)'$, where $\mathbf{1} = (1,1)'$, HQ choose a pair of incentive contracts and action profiles, $\{w_d, a_d\}_{d\in\{A,B\}}$, that maximize their (instantaneous) certainty equivalent objective function, solving

$$\max_{\{w,a\}} \quad \min_{\theta^{HQ}\in K_{HQ}(a)} \pi^{\theta} \equiv \phi' \mu^{\theta_{HQ}}(a) - \frac{R}{2}\phi'\Sigma\phi - s_A - s_B, \tag{15}$$

subject to the constraint that division managers maximize the certainty equivalent of their objective function

$$\max_{\tilde{a}_d} \min_{\theta\in K_d(a)} u_d^{\theta} \equiv s_d + b_d'\mu^{\theta_d}(\tilde{a}_d, a_{d'}) - \frac{r}{2} b_d'\Sigma b_d - c_d(a_d), \tag{16}$$

and to the participation constraints

$$\min_{\theta_d\in K_d(a)} s_d + b_d'\mu^{\theta_d}(a) - \frac{r}{2} b_d'\Sigma b_d - c_d(a_d) \geq 0 \tag{17}$$

for $d \in \{A, B\}$. Note that, absent uncertainty, $K_{HQ}(a) = K_d(a) = \{0\}$ and problem (15) - (17) collapses to the corresponding static problem of Holmström and Milgrom (1987). Further, Theorem 1 shows that the optimal contract implements constant effort, a_t, and constant distortions θ_t, so is sufficient to consider $\theta \in K_d(a)$.

The main trade-offs faced by HQ in problem (15) - (17) can be decomposed as follows. Because of translation invariance of CARA, the fixed component of incentive contracts, s_d, is set to make the participation constraint (17) bind in optimal contracts. After substitution into the objective function, we obtain

$$\pi^{\theta} = \mathbf{1}'\mu^{\hat{\theta}_{HQ}}(a) - \frac{R}{2}\phi'\Sigma\phi - \sum_{d\in\{A,B\}}\{\frac{r}{2}b_d'\Sigma b_d + c_d(a_d) + b_d'[\mu^{\hat{\theta}_{HQ}}(a) - \mu^{\hat{\theta}_d}(a)]\}, \quad (18)$$

where $\hat{\theta}_{HQ}$ and $\hat{\theta}_d$ are, respectively, the beliefs held by HQ and division manager d, for $d \in \{A, B\}$. HQ payoff consists of four components. The first one is the expected value of the two divisions, $\mathbf{1}'\mu^{\hat{\theta}_{HQ}}(a)$, which depends on effort exerted by division managers; the second one is given by the required risk-premia for HQ and division managers, $\frac{R}{2}\phi'\Sigma\phi$ and $\frac{r}{2}b_d'\Sigma b_d$; the third one is the cost of providing effort by division managers, $c_d(a_d)$. These components are common to the traditional problem without uncertainty aversion.

The last component is due to uncertainty aversion, which affects HQ in three separate ways. First, HQ valuation of both divisions, $\mu^{\hat{\theta}_{HQ}}(a)$, is based on beliefs held by HQ, $\hat{\theta}_{HQ}$, which are endogenous. Second, from the incentive constraint (16), effort exerted by division managers depends on their worst-case scenario, $\hat{\theta}_d$, depressing incentives. This implies that HQ must increase the pay-for-performance sensitivity, β_d, to elicit any desired level of effort, increasing the cost of incentive provision. The worst-case scenario, $\hat{\theta}_d$, however, is itself endogenous, and is determined by a division manager's overall exposure to uncertainty through incentive contracts. By hedging uncertainty through contracts, HQ can improve a division manager's assessment of her division productivity, $\mu^{\hat{\theta}_d}(a)$, promoting effort.

The third effect of uncertainty aversion, given by the last term in (18), is to create a divergence between HQ and division managers on the valuation of compensation contracts, $b_d'[\mu^{\hat{\theta}_{HQ}}(a) - \mu^{\hat{\theta}_d}(a)]$. This terms acts through division managers' participation constraints (17), and reflects the fact that HQ value compensation contracts at their own worst-case scenario, $\hat{\theta}_{HQ}$, while division managers value contracts at theirs, $\hat{\theta}_d$, creating a disagreement on the assessment of incentive contracts valuations. If HQ are more confident than division managers on division productivity, $\mu^{\hat{\theta}_{HQ}}(a) > \mu^{\hat{\theta}_d}(a)$, division managers discount the value of their compensation contracts, relative to the

HQ valuation, making it more costly (from HQ point of view) to satisfy their incentive and participation constraint, (16) - (17), increasing the cost of incentive provision. We denote this additional cost of incentive-based pay as a "disagreement discount," which represents the "Knightian" cost of disagreement.

In its generality, Theorem 1 precludes derivation of closed-form expressions for optimal contracts. To derive explicit solutions of optimal contracts, we will introduce parametric specifications of the core-beliefs sets $K_\delta(a)$, for $\delta \in \{HQ, A, B\}$, and will assume that HQ are risk neutral, $R = 0$. We will consider two possible configurations of HQ beliefs: uncertainty neutrality first, and then uncertainty aversion.[15]

3 Uncertainty and Incentive Contracts

As a benchmark, we start by characterizing the solution to the optimal contracting problem for our two-division firm without uncertainty, a setting similar to Holmström and Milgrom (1987).

3.1 The No-Uncertainty Benchmark

Absent uncertainty concerns, with $K_{HQ}(a_t) = K_d(a_t) = \{0\}$, HQ and division managers share the same beliefs and agree on the reference probability measure P^a.

Theorem 2 *(Holmström and Milgrom) Let HQ be risk neutral: optimal contracts are linear functions of the end-of-period cash-flows of both divisions:* $w_d(h_1) = s_d + \beta_d Y_{d,1} + \gamma_d Y_{d',1}$, *for all* t *and* $d \in \{A, B\}$, *with*

$$\beta_d = \frac{1}{1 + r\sigma^2(1-\rho^2)/(Z_d q_d^2)}, \quad \gamma_d = -\rho\beta_d, \tag{19}$$

and induce optimal effort

$$a_d = \beta_d Z_d q_d = \frac{Z_d q_d}{1 + r\sigma^2(1-\rho^2)/(Z_d q_d^2)}. \tag{20}$$

Optimal contract $w(h_1)$ is linear in the end-of-period cash-flow of both divisions, and depends on the correlation between divisional cash-flows. When cash-flows are

[15] Because uncertainty-neutral HQ hold firm beliefs on division productivity, we can denote this case as one of a "visionary leadership." In contrast, uncertainty-averse HQ pragmatically adapt (in equilibrium) their beliefs to firm characteristics, we can denote this case as one of "pragmatic leadership." The impact of leadership styles is examined in Rotemberg and Saloner (1993) and (2000).

uncorrelated, optimal contracts have no exposure to the other division performance, $\gamma_d = 0$. If cash-flows are correlated, it is optimal for HQ to hedge division manager risk exposure by making compensation contingent on performance from both divisions $|\gamma_d| > 0$. With positive correlation, HQ set $\gamma_d < 0$ and contracts display "relative-performance" compensation; with negative correlation, HQ set $\gamma_d > 0$ and incentive contracts display an equity component through cross-pay. The benefit is that hedging a division manager risk exposure reduces the cost of incentive provision and allows HQ to increase pay-for-performance sensitivity, improving incentives. Finally, when division managers are risk-neutral, $r = 0$, HQ set $\beta_d = 1$, making them full residual claimant; in this case, cross-pay γ_d is indeterminate because side bets are irrelevant for risk-neutral agents.

3.2 Incentive contracts and beliefs

Under MEU preferences of Gilboa and Schmeidler (1989) division managers' beliefs are endogenous and solve the inner problem of (16). The key property that we exploit is that division managers may become relatively more confident about the prospects of their own division if they have exposure also to the other division's source of uncertainty, a feature that is the direct of consequence of uncertainty hedging. This means that, by proper design of incentive contracts, HQ can affect the probability measure used by division managers to assess the productivity of their division, and thus mitigate the adverse effect of uncertainty on effort.

Our results hold when the sets $K_\delta(a)$, $\delta \in \{HQ, A, B\}$, are strictly convex with smooth boundaries, allowing uncertainty hedging to affect beliefs. For tractability, and to generate closed-form solutions, we assume that both HQ and division managers consider deviations, $\mathcal{P}_d^\theta$, that are in a neighborhood of the reference probability, P^a, as follows. We assume that

$$K_\delta(a) = \left\{\theta \middle| \left[-\ln\left(1 - \frac{|D\theta_A + N\theta_B|}{a_A q_A}\right) - \ln\left(1 - \frac{|N\theta_A + D\theta_B|}{a_B q_B}\right)\right] \leq \kappa_\delta\right\} \quad (21)$$

for $\delta \in \{HQ, A, B\}$, where, from (2),

$$D \equiv \frac{\sigma}{2}\left(\sqrt{1+\rho} + \sqrt{1-\rho}\right), \quad N \equiv \frac{\sigma}{2}\left(\sqrt{1+\rho} - \sqrt{1-\rho}\right). \quad (22)$$

Note that κ_δ reflects the degree of confidence in the reference probability P^a held by agent δ, for $\delta \in \{HQ, A, B\}$, where $\kappa_\delta = 0$ indicates full confidence, and increasing uncertainty is characterized by greater κ_δ. In the special case of uncorrelated cash-flows, (21) simplifies to

$$K_\delta(a) = \left\{\theta \middle| \left[-\ln\left(1 - \frac{\sigma|\theta_A|}{a_A q_A}\right) - \ln\left(1 - \frac{\sigma|\theta_B|}{a_B q_B}\right)\right] \leq \kappa_\delta\right\}. \tag{23}$$

Note these expressions are not history dependent and provide a special case of IID uncertainty, as in Chen and Epstein (2002).

An important implication of the specification of the core beliefs set based on (21) and (22) is that decision makers behave as if divisional productivity itself, q_d, is uncertain, as follows. From (16), division managers beliefs are determined by minimizing their objective function u_d^θ where, from (9), the cash-flow process Y has drift $\mu^\theta(a) \equiv Qa - \Gamma\theta$. Consider the alternative representation of division managers' and HQ objective functions

$$\hat{u}_d \equiv s_d + b_d' \hat{Q}^d a - \frac{r}{2} b_d' \Gamma' \Gamma b_d - c_d(a_d), \tag{24}$$

$$\hat{\pi} \equiv (\mathbf{1} - b_A - b_B)' \hat{Q}^{HQ} a - s_A - s_B, \tag{25}$$

with $\hat{Q}^\delta \equiv \begin{bmatrix} \hat{q}_A^\delta & 0 \\ 0 & \hat{q}_B^\delta \end{bmatrix}$, where $\hat{q}_{d'}^\delta$ represents the belief of agent $\delta \in \{A, B, HQ\}$ on the productivity of $d' = \{A, B\}$ and, thus, the drift of the cash-flow process, $\mu(a) \equiv \hat{Q}^\delta a$.

Lemma 1 *The following two problems are equivalent:*

$$\min_{\theta \in K_d(a)} u_d^\theta = \min_{\hat{q}^d \in C_d} \hat{u}_d, \tag{26}$$

where $\hat{q}^d \equiv (\hat{q}_A^d, \hat{q}_B^d)$ *and*

$$C_\delta \equiv \left\{\hat{q}^\delta \middle| \ln\left(\frac{1}{1 - \left|\frac{\hat{q}_A^\delta}{q_A} - 1\right|}\right) + \ln\left(\frac{1}{1 - \left|\frac{\hat{q}_B^\delta}{q_B} - 1\right|}\right) \leq \kappa_\delta\right\}, \tag{27}$$

for $\delta \in \{A, B\}$*. Similarly, for HQ* $\min_{\theta \in K_{HQ}(a)} \pi^\theta$ *is equivalent to* $\min_{\hat{q}^{HQ} \in C_{HQ}} \hat{\pi}$.

Lemma 1 implies that the characterization of uncertainty with density generators in (7) can be interpreted as modeling uncertainty on productivity of both divisions, as specified in (24). The advantage of focusing on the latter problem is that the specification for distance of probability measures in (27) allows us to obtain closed form solutions for optimal contracts. Figure 1 provides a numerical example of the core belief set (27).[16]

We start with the characterization of division managers' belief assessments. Division managers' assessment of the productivity of both divisions depend on the pair of incentive contracts offered by HQ. From (24), given incentive contract $w(h_1) = \{w_d(h_1)\}_{d\in\{A,B\}}$, division managers beliefs $\hat{q}^d(a,w)$ are obtained as

$$\arg\min_{\hat{q}_d} \ \hat{u}_d = s_d + b_d'\hat{Q}^d a - \frac{r}{2}b_d'\Sigma b_d - c_d(a_d), \tag{28}$$

$$s.t. \quad \ln\left(\frac{1}{1-\left|\frac{\hat{q}_A}{q_A}-1\right|}\right) + \ln\left(\frac{1}{1-\left|\frac{\hat{q}_B}{q_B}-1\right|}\right) \le \kappa_d.$$

Note that incentive contracts offered by HQ will have $\beta_d > 0$, so that division managers will exert strictly positive effort, $a_d > 0$.

Lemma 2 *Let $\beta_d a_d > 0$ and*

$$H_d \equiv \frac{\gamma_d a_{d'} q_{d'}}{\beta_d a_d q_d}. \tag{29}$$

A division manager's assessment of the productivity of both divisions, $\hat{q}^d = \{\hat{q}^d_d, \hat{q}^d_{d'}\}$, for $d, d' \in \{A,B\}$, and $d \neq d'$, is equal to:

i) $\hat{q}^d_d = q_d$, *and* $\hat{q}^d_{d'} = e^{-\kappa_d}q_{d'}$ *for* $H_d \ge e^{\kappa_d}$

ii) $\hat{q}^d_d = (e^{-\kappa_d}H_d)^{\frac{1}{2}}q_d$ *and* $\hat{q}^d_{d'} = \left(e^{-\kappa_d}\frac{1}{H_d}\right)^{\frac{1}{2}}q_{d'}$ *for* $H_d \in (e^{-\kappa_d}, e^{\kappa_d})$

iii) $\hat{q}^d_d = e^{-\kappa_d}q_d$ *and* $\hat{q}^d_{d'} = q_{d'}$ *for* $H_d \in [-e^{-\kappa_d}, e^{-\kappa_d}]$

iv) $\hat{q}^d_d = (e^{-\kappa_d}|H_d|)^{\frac{1}{2}}q_d$ *and* $\hat{q}^d_{d'} = \left[2-\left(e^{-\kappa_d}\frac{1}{|H_d|}\right)^{\frac{1}{2}}\right]q_{d'}$ *for* $H_d \in (-e^{\kappa_d}, -e^{-\kappa_d})$

v) $\hat{q}^d_d = q_d$ *and* $\hat{q}^d_{d'} = (2-e^{-\kappa_d})q_{d'}$ *for* $H_d \le -e^{\kappa_d}$

Division managers beliefs toward a division productivity depend on the relative exposure to the cash-flow from each division, measured by H_d, as affected by incentive contract w_d. Because H_d affects a division manager relative exposure to the uncertainty of the two divisions, we refer to H_d as the "uncertainty hedging" ratio. Note that $sign(H_d) = sign(\gamma_d)$ and that H_d is an increasing function of γ_d.

[16] The core-belief set (27) represents a (tractable) approximation of a corresponding core-belief set based on Eq. 3.12 in Chen and Epstein (2002), which is displayed, for illustration, as the dashed line in Figure 1. An important difference from their specification is that we allow the source-dependent ambiguity aversion to scale in effort, a_d. A similar approximating approach is adopted in Dicks and Fulghieri (2019), (2021), and Lee and Rivera (2021).

Several features emerge from Lemma 2. First, when HQ grant pay-for-performance only, that is $\gamma_d = 0 = H_d$, or a small exposure to the other division cash-flow, as in case (iii), division managers will assess the prospects of their own division conservatively, with $\hat{q}_d^d = e^{-\kappa} q_d$. In this case, division managers will be less confident on their own division productivity, disincentivizing effort.

Division manager assessments of productivity of their own division, $\hat{q}_d^d$ is however an increasing function of their exposure to the other division, $|\gamma_d|$. Thus, incentive contracts that offer progressively increasing exposure to the other division, as in case (ii) and (iv), induce division managers to become more confident on their own division, $\hat{q}_d^d$. Finally, if incentive contracts offer substantial increase of the exposure to other division, a large value of $|\gamma_d|$, as in case (i) and (v), division managers will become very confident on their own division, setting $\hat{q}_d^d = q_d$. This beneficial effect on division manager beliefs can be obtained by either giving a division manager cross-pay, $\gamma_d > 0$, or with relative-performance compensation, $\gamma_d < 0$.

The impact of $|\gamma_d|$ on a division manager's assessment of the productivity of the other division depends on the sign of γ_d. If the incentive contract includes cross-pay, $\gamma_d > 0$, increasing exposure to the other division progressively worsens the assessment of that other division productivity, as in cases (ii) and (i). If the incentive contract includes relative performance, $\gamma_d < 0$, increasing exposure to the other division (lower γ_d) progressively improves the assessment of its productivity, as in cases (iv) and (v), where in both cases $\hat{q}_{d'}^d > q_{d'}$. The more optimistic assessment reflects the fact that, when $\gamma_d < 0$, better performance in the other division reduces a division manager's compensation, which is a concern to the division manager.

3.3 Incentive contracts and effort

We now determine Nash equilibrium of the dynamic effort selection by division managers, given a pair of incentive contracts, $w(h_1) = \{w_d(h_1)\}_{d \in \{A,B\}}$. Given division managers' beliefs, characterized in Lemma (2), effort is determined by solving

$$\max_{\tilde{a}_d} \ \hat{u}_d(a, \hat{q}^d(a, w)) = s_d + b_d' \hat{Q}^d(\tilde{a}_d, a_{d'})' - \frac{r}{2} b_d' \Sigma b_d - c_d(\tilde{a}_d), \tag{30}$$

for $d \in \{A, B\}$ and $d \neq d'$. We have the following.

Lemma 3 *Given a pair of incentive contracts, $\{w_d = (\beta_d, \gamma_d)\}_{d \in \{A,B\}}$, there is a unique Nash equilibrium effort exerted by division managers, $\{a_A, a_B\}$, equal to $a_d =$*

$\beta_d Z_d \hat{q}^d_d$, where division manager beliefs, $\hat{q}^d_d$, are as in Lemma 2. Equilibrium effort a_d is increasing in pay-performance sensitivity, β_d, exposure to the other division, $|\gamma_d|$, efficiency of effort, Z_d, and decreasing in uncertainty κ_d. Further, if $|H_d| \in (e^{-\kappa_d}, e^{\kappa_d})$, a_d is also increasing in $\beta_{d'}$, $|\gamma_{d'}|$, and $Z_{d'}$, and decreasing in $\kappa_{d'}$.

Lemma 2 and 3 together imply that incentive contracts affect division manager effort through two distinct channels. The first one is the traditional effect of inducing effort by rewarding division managers on the basis of direct performance measures. The second channel is through the impact of incentive contracts on managerial assessment of the success probability of their projects. Specifically, incentive contracts can be used by HQ to lead uncertainty-averse division managers to hold more favorable assessment of the productivity of their own division, with a positive effect on effort. This is a new channel and the key driver of our paper.

If division managers are uncertainty neutral, their optimal level of effort in (20), a_d, is an increasing function of her own division-based pay, β_d, but is not affected by either their cross-division pay, γ_d, nor the action of the other division manager, $a_{d'}$. The only effect of cross-division exposure is to hedge a division manager's risk exposure, reducing the cost of incentive provision.

Uncertainty aversion introduces a link across division managers' effort levels. From Lemma 2, exposure to the other division, $|\gamma_d| > 0$, makes effort exerted by a division manager, a_d, an increasing function of effort of the other division manager, $a_{d'}$. This is because greater effort from the other manager decreases the relative exposure of a division manager to uncertainty on her own division, leading to more favorable beliefs and greater effort. This new source of (positive) externality is due to the (beneficial) effect of uncertainty hedging, and is driven solely by beliefs.

4 Uncertainty-Neutral Principal

Consider first the case where HQ are uncertainty neutral and hold beliefs $\hat{q}^{HQ}_d = q_d$ for both divisions, while division managers are uncertainty averse. Given Lemma 1, problem (12) - (14) becomes

$$\max_{\{w_d, a_d\}_{d\in\{A,B\}}} \hat{\pi} = (\mathbf{1} - b_A - b_B)' \, Qa - s_A - s_B \tag{31}$$

subject to the constraint that a prescribed action a_d solves

$$\max_{\tilde{a}_d} \min_{\hat{q}_d} \ \hat{u}_d = s_d + b_d' \hat{Q}^d (\tilde{a}_d, a_{d'})' - \frac{r}{2} b_d' \Sigma b_d - c_d (\tilde{a}_d), \tag{32}$$

and the participation constraint

$$\min_{\hat{q}_d} \ s_d + b_d' \hat{Q}^d a - \frac{r}{2} b_d' \Sigma b_d - c_d (a_d) \geq 0, \tag{33}$$

for $d \in \{A, B\}$ and $d \neq d'$. We consider first the easier case in which division managers are uncertainty averse but risk-neutral.

Theorem 3 *If HQ are risk- and uncertainty-neutral and division managers are uncertainty averse but risk neutral, optimal incentive contracts have*

$$|H_d| = \frac{|\gamma_d| a_{d'} q_{d'}}{\beta_d a_d q_d} = 1,$$

which induce division managers beliefs $(\hat{q}_d^d, \hat{q}_{d'}^d)$ *to be equal to*

$$\begin{aligned} \hat{q}_d^d &= e^{-\frac{\kappa_d}{2}} q_d < q_d, \\ \hat{q}_{d'}^d &= e^{-\frac{\kappa_d}{2}} q_{d'} < q_{d'} \text{ for } \gamma > 0 \quad \text{and } \hat{q}_{d'}^d = \left(2 - e^{-\frac{\kappa_d}{2}}\right) q_{d'} > q_{d'} > \hat{q}_d^d \text{ for } \gamma < 0 \end{aligned} \tag{34}$$

for $d, d' \in \{A, B\}$, *and* $d \neq d'$. *Optimal contracts set*

$$\beta_d = \frac{1}{1 + 3\left(1 - \hat{q}_d^d / q_d\right)} < 1, \text{ and } |\gamma_d| = \xi_d \beta_d, \tag{35}$$

$$\text{where } \xi_d \equiv \frac{a_d q_d}{a_{d'} q_{d'}} = \frac{1 - 3\left(1 - \hat{q}_{d'}^{d'} / q_{d'}\right)}{1 - 3\left(1 - \hat{q}_d^d / q_d\right)} \frac{\hat{q}_d^d / q_d}{\hat{q}_{d'}^{d'} q_{d'}} \frac{Z_d q_d^2}{Z_{d'} q_{d'}^2}. \tag{36}$$

Pay-for-performance sensitivity, β_d*, and effort,* a_d*, are both decreasing in uncertainty,* κ_d*. If condition (S) holds, equity is optimal,* $\beta_d = \gamma_d$*, and* $\hat{q}_d^d = \hat{q}_{d'}^d = e^{-\frac{\kappa}{2}} q < q$.

If division managers are uncertainty averse but risk neutral, hedging their risk is not a concern for HQ. The presence of uncertainty, by making division managers less confident than HQ on the productivity of their own division, has two adverse effects. First, it has the detrimental effect on the incentives to exert effort by their managers. This implies that HQ must increase pay-for-performance sensitivity to elicit any desired level of effort. Second, more conservative beliefs reduce the value of the incentive contract, w_d, as assessed by division managers, relative to the value assessed by the more confident HQ (the disagreement discount). The combined effect is to make it costlier for HQ to induce effort, giving (35). Note that the pay-for-performance sensitivity, β_d, and thus effort, a_d, are both decreasing functions of the extent of the disagreement between a division manager and HQ, given by $\hat{q}_d^d/q_d$.

The role of cross-division exposure, $|\gamma_d|$, is to improve division managers' beliefs by hedging their uncertainty. From Lemma 2, an increase of cross-division exposure (partially) offsets the negative effect of uncertainty on beliefs, with beneficial effect on effort. Absent risk-aversion considerations, the optimal contract equates a division manager's overall exposure to cash-flow uncertainty from both divisions to be the same, setting their uncertainty hedge ratio $|H_d| = 1$.

Note that HQ are indifferent between granting compensation with cross-pay, $\gamma_d > 0$, or relative-performance, $\gamma_d < 0$, as the optimal contracts depends only on the size of the cross-division exposure, $|\gamma_d|$. This property reflects the fact that division managers' beliefs are only affected by the absolute value of their exposure to the other division, $|\gamma_d|$, and not by its sign. The extent of cross-division exposure, $|\gamma_d|$, is still proportional to the pay-for-performance sensitivity parameter, with $|\gamma_d| = \xi_d \beta_d$, where the proportionality factor ξ_d depends on the relative exposure to uncertainty of the two division managers, affecting the term $\hat{q}_d^d/q$, and the relative size of the two divisions, captured by the term $Z_d q_d^2 / Z_{d'} q_{d'}^2$. This implies that cross division exposure is greater for (relatively) less confident division managers and for larger divisions.

If divisions are symmetric, the uncertainty hedge ratio can be set to unity with pure equity contracts: $\beta = \gamma < 1$. Interestingly, in this case, both division managers hold the same beliefs on their own as well as the other division, $\hat{q}_d^d = \hat{q}_{d'}^d = e^{-\frac{\kappa}{2}} q$, leading to consensus (that is, a "shared view") in the organization. In addition, HQ hold (endogenously) more optimistic beliefs than those of division managers for their own divisions, $\hat{q}_d^d = e^{-\frac{\kappa_d}{2}} q < q$, making HQ to appear as "visionary" in the organization. Finally, absent risk aversion, a contract with extreme relative performance, with $\gamma = -\beta$, is also optimal. In this case, from (34), we have $\hat{q}_d^d < q < \hat{q}_{d'}^d$, and division managers are more confident on the other division that they are on their own, creating envy and discord in the organization, a potentially undesirable configuration of internal beliefs.

An important implication of Theorem 3 is that the optimal contract (35) differs from the corresponding case of risk-neutral division managers with no uncertainty of Theorem 2, where division managers become full residual claimants in their own division, with $\beta_d = 1$, and with no role for cross-pay γ_d. The reason is that, when uncertainty is a concern, making division managers full residual claimant exacerbates pessimism toward their division, depressing effort. In this case, HQ find it optimal to reduce pay-for-performance sensitivity, $\beta_d < 1$, and to hedge division manager uncertainty by offering exposure to the other division's uncertainty, setting $|\gamma_d| > 0$.

The presence of risk-aversion affects optimal contracts because hedging uncertainty creates a risk exposure, which is costly for risk-averse division managers. The optimal contract in this case depends on the relative importance of the risk-hedging and the uncertainty-hedging motives. For tractability, with risk-averse division managers we focus on the symmetric case under condition (S).

Theorem 4 *Let condition (S) hold. There is a threshold $\bar{\kappa}(r, \rho)$ (defined in Appendix), with $\bar{\kappa}(0, \rho) = 0$, such that for $d, d' \in \{A, B\}$, and $d \neq d'$:*
1. If $\kappa \leq \bar{\kappa}$, optimal incentive contracts induce division managers beliefs $\hat{q}_d^d = e^{-\kappa} q$ and $\hat{q}_{d'}^d = q$, by setting

$$\beta = \frac{1}{1 + \left(1 - \hat{q}_d^d / q\right) + r\sigma^2 \left(1 - \rho^2\right) / (Zq\hat{q}_d^d)} > 0, \quad \gamma = -\rho\beta. \tag{37}$$

Pay-for-performance sensitivity, β, and Nash equilibrium effort, a, are both decreasing in uncertainty, κ; the threshold $\bar{\kappa}(r, \rho)$ is increasing in both r and $|\rho|$.
2. If $\kappa > \bar{\kappa}$, optimal incentive contracts induce division managers to hold the same beliefs as in (34) of Theorem 3 by setting

$$\beta = \frac{1}{1 + 3\left(1 - \hat{q}_d^d / q\right) + 2r\sigma^2 \left(1 - |\rho|\right) / (Zq\hat{q}_d^d)} > 0, \quad |\gamma| = \beta \tag{38}$$

with $sign(\gamma) = -sign(\rho)$. When $\rho = 0$, HQ are indifferent between setting $\gamma = \pm\beta$.

When division managers face low levels of uncertainty, $\kappa \leq \bar{\kappa}$, uncertainty aversion does not significantly affect beliefs and, thus, their incentives to exert effort. At these low levels of uncertainty, the disagreement between division managers and HQ is relatively small, with $\hat{q}_d^d = e^{-\kappa} q < q$, corresponding to case (iii) in Lemma 2. The presence of uncertainty is again to increase the cost of incentive provision, leading to a decrease of the pay-for-performance sensitivity, β. The optimal cross-division exposure, $|\gamma|$, is still proportional to $|\rho|$, and is set to limit a division manager's overall risk exposure, with a corresponding reduction of required risk-premium, as in the benchmark case. Overall, optimal incentive contracts mirror those in Theorem 2. The main difference is that the presence of uncertainty, by increasing the cost of incentive provision, reduces both pay-for-performance sensitivity and effort.

When division managers are sufficiently exposed to uncertainty on division productivity, $\kappa > \bar{\kappa}$, HQ find it optimal to hedge uncertainty and offer incentive contracts with greater cross-division exposure, $|\gamma| = \beta > |\rho| \beta$, at the cost of greater risk exposure. The presence of such uncertainty, if left unchallenged, would significantly depress effort. By granting greater cross-division exposure, HQ limit pessimism held by division managers, promoting effort. Optimal contracts grant division managers a sufficient share of the other division to induce them to hold beliefs that are more closely aligned with those held by HQ, with $\hat{q} = e^{-\frac{\kappa}{2}} q > e^{-\kappa} q$ (corresponding to cases (ii) and (iv) in Lemma 2). To hedge division-manger risk exposure, the sign of the cross-division exposure, γ, is again the opposite to the sign of the correlation coefficient, with $sign(\gamma) = -sign(\rho)$. When the cash-flows of the two divisions are uncorrelated, cross division exposure does not produce any risk-hedging benefit (but only uncertainty hedging), and HQ are again indifferent between setting $\gamma = \pm\beta$ (as in Theorem 3).

Interestingly, the optimal cross-division exposure is set to a greater level, $|\gamma| = \beta$, than the one absent uncertainty, $|\gamma| = |\rho|\beta$ in Theorem 2. Deviations from optimal risk hedging, however, are costly and occur only when the benefits from uncertainty hedging are sufficiently large, generating a discrete jump in cross-division exposure, from $|\gamma| = |\rho| \beta$ to $|\gamma| = \beta > |\rho| \beta$. The discontinuity is due to the fact that, with low uncertainty, $\kappa \leq \bar{\kappa}$, division managers beliefs are in case (iii). In this situation, small deviations from optimal risk-sharing have no impact on division managers beliefs, while negatively affecting their welfare. Deviations from optimal risk hedging are optimal only when they lead to sufficiently large uncertainty-hedging benefits, due to improvements of division managers beliefs, leading HQ to set the uncertainty hedging ratio again at $|H_d| = 1$.

Optimality of "pure-equity" compensation, $|\gamma| = \beta$, in Theorem 4 is the consequence of division symmetry, leading HQ to grant equal exposure to both two divisions. If divisions are not symmetric, and HQ wishes to implement interior beliefs, as in case (ii) and (iv) of Lemma 2, optimal contracts still involve cross-division exposure, $|\gamma_d| > 0$. However, the composition of pay-for-performance sensitivity, β_d, and cross-division exposure, $|\gamma_d|$, will now depend on the relative size the two divisions (which affects division managers' uncertainty exposure) and their relative risk-exposure.

Corollary 1 *Let the optimal contract be such that both division managers have interior beliefs, $|H_d| \in (e^{-\kappa_d}, e^{\kappa_d})$, and let $a_d q_d > a_{d'} q_{d'}$, for $d \neq d'$. Then the optimal contract $\{\beta_d, \gamma_d\}_{d \in \{A,B\}}$ has*

$$\beta_d a_d q_d + r\sigma^2 \beta_d^2 = |\gamma_d|\, a_{d'} q_{d'} + r\sigma^2 \gamma_d^2, \tag{39}$$

with $|\gamma_{d'}| > \xi_{d'} \beta_{d'}$ and $|\gamma_d| < \xi_d \beta_d$.

If the two divisions are of differing size, and the optimal contract induces beliefs that are either in case (ii) or case (iv) of Lemma 2, then the optimal contracts equates the total (expected) cost to HQ of a division manager's exposure to the two divisions. This cost is the sum of two components: for their own division, it is the sum of the (expected) pay-for-performance component, $\beta_d a_d q_d$, and of the corresponding risk premium, $r\sigma^2 \beta_d^2$, and for the other division is the sum of cross-pay, $|\gamma_d|\, a_{d'} q_{d'}$, and of the corresponding risk premium, $r\sigma^2 \gamma_d^2$. In addition, and with respect to the optimal contract in Theorem 3, the presence of risk aversion has the effect increasing cross-division exposure for the relatively smaller division, $|\gamma_{d'}| > \beta_{d'} \xi_{d'}$, and to decrease such exposure for the larger division, $|\gamma_d| < \xi_d \beta_d$.

In summary, an important implication of Theorem 4 and Corollary 1 is that optimal incentive contracts have positive cross exposure, $|\gamma| > 0$, even when division managers are risk averse and division cash-flows are not correlated, a clear contrast with the "informativeness principle." This means that the presence of (sufficiently large) uncertainty leads to incentive contracts that would not otherwise be optimal under risk aversion alone.

5 Uncertainty-Averse Principal

Different from the case of uncertainty-neutral principal, beliefs held by uncertainty-averse HQ are not fixed but, rather, are determined endogenously as well. Since the properties of Lemma 1 applies also to HQ, their beliefs $\{\hat{q}_A^{HQ}, \hat{q}_B^{HQ}\}$ are determined by solving

$$\min_{\{\hat{q}_A^{HQ},\hat{q}_B^{HQ}\}\in C_{HQ}} \hat{\pi} = \sum_{d\in\{A,B\}} \left(\phi_d \hat{q}_d^{HQ} a_d - s_d\right), \tag{40}$$

where $\phi_d = 1 - \beta_d - \gamma_{d'} > 0$, for $d, d' \in \{A, B\}$, $d \neq d'$, and

$$C_{HQ} \equiv \left\{ \hat{q}^{HQ} | \ln\left(\frac{1}{1-\left|\frac{\hat{q}_A^{HQ}}{q_A}-1\right|}\right) + \ln\left(\frac{1}{1-\left|\frac{\hat{q}_B^{HQ}}{q_B}-1\right|}\right) \leq \kappa_{HQ} \right\}. \tag{41}$$

The following lemma characterizes HQ beliefs for the case in which HQ have positive residual exposure in either division, $\beta_d + \gamma_{d'} < 1$ (which will be the relevant case in subsequent analysis).

Lemma 4 *Let $\phi_d > 0$, $d \in \{A, B\}$ with $d' \neq d$, and*

$$H_d^{HQ} \equiv \frac{\phi_{d'} a_{d'} q_{d'}}{\phi_d a_d q_d}, \tag{42}$$

Headquarters assessment of both divisions, $(\hat{q}_A^{HQ}, \hat{q}_B^{HQ})$, is equal to:

i) $\hat{q}_d^{HQ} = q_d$ *and* $\hat{q}_{d'}^{HQ} = e^{-\kappa_{HQ}} q_{d'}$ *for* $H_d^{HQ} > e^{\kappa_{HQ}}$

ii) $\hat{q}_d^{HQ} = \left[e^{-\kappa_{HQ}} H_d^{HQ}\right]^{\frac{1}{2}} q_d,$ *for* $H_d^{HQ} \in [e^{-\kappa_{HQ}}, e^{\kappa_{HQ}}]$

iii) $\hat{q}_d^{HQ} = e^{-\kappa_{HQ}} q_d$ *and* $\hat{q}_{d'}^{HQ} = q_{d'}$ *for* $H_d^{HQ} < e^{-\kappa_{HQ}}$

Similar to Lemma 2, HQ beliefs depend on their relative exposure to the two divisions, as measured by the corresponding uncertainty ratio H_d^{HQ} (note that $H_{d'}^{HQ} = 1/H_d^{HQ}$). When HQ have moderate exposure to both divisions, as in case (ii) with $H_d^{HQ} \in [e^{-\kappa_{HQ}}, e^{\kappa_{HQ}}]$, they have conservative beliefs toward each division, $\hat{q}_d^{HQ} < q_d$, and become less confident toward a division when relative exposure to that division increases. When HQ have a sufficiently large exposure to a division, as in cases (i) and (iii) with $H_d^{HQ} > e^{\kappa_{HQ}}$ or $H_d^{HQ} < e^{-\kappa_{HQ}}$, they will be even less confident toward that division, $\hat{q}_d^{HQ} = e^{-\kappa} q_d$, and correspondingly more confident on the other division, $\hat{q}_{d'}^{HQ} = q_{d'}$.

Optimal contracts depend on the extent of uncertainty faced by HQ relative to division managers. We start again with the simpler case where division managers are uncertainty averse but risk neutral. Beliefs for division managers are still given in Lemma 2, and effort in Lemma 3. For expositional simplicity, we focus on the case in which division managers are exposed to the same uncertainty: $\kappa_A = \kappa_B = \kappa$.[17]

Theorem 5 *Let both HQ and division managers be uncertainty averse but risk neutral. If divisions are not too dissimilar, with* $\eta_d \equiv (Z_{d'}/Z_d)^{1/2} q_{d'}/q_d \in (e^{-\kappa_{HQ}}, e^{\kappa_{HQ}})$ *and the uncertainty faced by HQ is positive but not too large relative to that faced by division managers,* $\kappa_{HQ} < \kappa - 2\ln\frac{3}{2}$*, optimal incentive contracts have* $H_d^{HQ} = H_d = \eta_d$*, and HQ align division managers' beliefs with theirs*

$$\begin{aligned} \hat{q}_d^d &= \hat{q}_d^{d'} = e^{-\frac{\kappa-\kappa_{HQ}}{2}} \hat{q}_d^{HQ} = e^{-\frac{\kappa}{2}} q_d \eta_d^{\frac{1}{2}}, \quad \text{and} && (43) \\ \hat{q}_d^{HQ} &= e^{-\frac{\kappa_{HQ}}{2}} q_d \eta_d^{\frac{1}{2}}, && (44) \end{aligned}$$

for $d, d' \in \{A, B\}$*, and* $d \neq d'$*. Optimal incentive contracts offer pure equity, with*

$$\beta_d = \gamma_d = \frac{1}{1 + 3(1 - \hat{q}_d^d/\hat{q}_d^{HQ})} < 1. \tag{45}$$

When divisions are not too dissimilar and HQ are not too uncertainty averse relative to division managers (which ensures that HQ has a positive exposure to both divisions, $1 - \beta_d - \gamma_{d'} > 0$, and that their beliefs fall in case (ii) of Lemma 4), optimal incentive contracts are pure equity, $\beta_d = \gamma_d$. Beliefs, pay-for-performance sensitivity and effort levels mimic those in Theorem 3, with the difference that now HQ beliefs are endogenous and equal $\hat{q}_d^{HQ}$ rather than q_d. Absent risk-aversion, in optimal contracts HQ equate their uncertainty-hedging ratio with respect to each division to the uncertainty hedging ratio of its division manager by setting $H_d^{HQ} = H_d$.

Pay-for-performance sensitivity, β_d, cross-pay, γ_d, and effort level, a_d, now depend on the difference between the uncertainty faced by HQ and division managers, $\kappa_{HQ} - \kappa < 0$. In particular, an increase of the uncertainty faced by HQ, for given uncertainty faced division managers, increases pay-for-performance sensitivity, cross-pay, and effort. This happens because a smaller difference in uncertainty faced by HQ and division managers reduces the disagreement discount. A smaller discount lowers the cost of incentive provisions and induce HQ to offer contracts with larger pay-for-performance sensitivity, leading to greater effort. Greater pay-for-performance

[17] It is possible, although rather messy, to extend the analysis to the case in which division managers are exposed to different levels of uncertainty, $\kappa_A \neq \kappa_B$. The optimal contract in Theorem 5 is still equity, $\beta_d = \gamma_d$, but division managers receive different equity shares: $\beta_A \neq \beta_B$.

sensitivity, however, increases a division manager's exposure to uncertainty, which is offset by a corresponding increase of cross-pay. Beliefs held by HQ and division managers are aligned in the sense that they both hold the same assessment on the relative productivity of both divisions, $\hat{q}_d^{HQ}/\hat{q}_{d'}^{HQ} = \hat{q}_d^d/\hat{q}_{d'}^d$.

Optimal contracts with risk-averse division managers are characterized in the following theorems. For tractability, we focus on the symmetric case, condition (S).

Theorem 6 *Let condition (S) hold. There are thresholds $(\hat{\kappa}, \hat{\kappa}^{HQ})$ (defined in the Appendix) such that for $d, d' \in \{A, B\}$ and $d \neq d'$:*
1. if $\kappa \leq \hat{\kappa}$ and $\kappa_{HQ} \leq \hat{\kappa}^{HQ}$, optimal incentive contracts induce beliefs for division managers and HQ equal to $\hat{q}_d^d = e^{-\kappa} q < \hat{q}_{d'}^d = q$ and $\hat{q}_d^{HQ} = e^{-\frac{\kappa_{HQ}}{2}} q < q$ by setting

$$\beta = \frac{1}{1 + 2(\rho - \bar{\rho})\left(\frac{\hat{q}_{d'}^d}{\hat{q}_{d'}^{HQ}} - 1\right) + \left(1 - \frac{\hat{q}_d^d}{\hat{q}_d^{HQ}}\right) + \frac{r\sigma^2(1-\rho^2+\bar{\rho}^2)}{Z\hat{q}_d^{HQ}\hat{q}_d^d}}, \quad \gamma = -(\rho - \bar{\rho})\beta, \qquad (46)$$

where $\bar{\rho} \equiv \hat{q}_d^d \left(\hat{q}_{d'}^d - \hat{q}_{d'}^{HQ}\right) \frac{Z}{r\sigma^2} = \frac{e^{-\kappa} q^2 Z}{r\sigma^2}\left(1 - e^{-\frac{\kappa_{HQ}}{2}}\right) > 0$.
2. If $\kappa > \hat{\kappa}$ or $\kappa_{HQ} > \hat{\kappa}^{HQ}$ and $\rho \leq 0$ optimal incentive contracts induce beliefs for division managers and HQ equal to $\hat{q}_d^d = \hat{q}_{d'}^d = e^{-\frac{\kappa}{2}} q$ and $\hat{q}_d^{HQ} = e^{-\frac{\kappa_{HQ}}{2}} q$ by setting

$$\beta = \gamma = \hat{\beta} \equiv \frac{1}{1 + 3\left(1 - \hat{q}_d^d/\hat{q}_d^{HQ}\right) + \frac{2r\sigma^2(1+\rho)}{Z\hat{q}_d^{HQ}\hat{q}_d^d}}. \qquad (47)$$

When both HQ and division managers are uncertainty averse, and division managers are risk averse, optimal incentive contracts depend on the extent of their exposure to uncertainty and on the correlation between divisional cash-flows. When overall exposure to uncertainty is sufficiently low, Case 1, optimal contracts mirror again those absent uncertainty of Theorem 2. The effect of uncertainty is again to reduce pay-for-performance sensitivity, β.

Interestingly, relative-performance compensation, $\gamma < 0$, is now optimal only if correlation is sufficiently large, $\rho > \bar{\rho} \geq 0$ (note that $\bar{\rho} = 0$ when $\kappa_{HQ} = 0$). The reason is that HQ uncertainty aversion increases the disagreement discount, raising the cost of hedging division manager risk with relative performance compensation. This happens because relative-performance compensation for division manager d generates a "short" exposure to the other division, d', while HQ still have a "long" position in that division. From Lemma 4, when HQ are uncertainty averse and hold a long position in d', they are more pessimistic than the reference probability, $\hat{q}_{d'}^{HQ} < q$. In contrast, from Lemma 2, division managers are more confident on the other division d' than the reference probability, $\hat{q}_{d'}^{d} \geq q$. The combined effect is that HQ and division managers now hold more divergent views on the productivity of that division, increasing the disagreement discount and, thus, the cost of hedging risk exposure.

The implication is that relative-performance compensation is optimal only when the risk-hedging benefits are sufficiently large, that is, when $\rho > \bar{\rho}$. The threshold $\bar{\rho}$ is a decreasing function of a division's risk, and of division managers' risk aversion, and is an increasing function of division size (which increases HQ exposure to a division's uncertainty, exacerbating the disagreement discount). If division cash-flows are moderately positively correlated, $0 \leq \rho < \bar{\rho}$, optimal contracts have an equity component, $\gamma > 0$, different from the benchmark case. Finally, HQ and division managers are pessimistic on both divisions, and their assessment of division productivity depends on their relative degree of uncertainty, with $\hat{q}_d^d \gtrless \hat{q}_d^{HQ}$ as $\frac{\kappa_{HQ}}{2} \gtrless \kappa$.

When uncertainty faced by either HQ or division managers is sufficiently large, Case 2, optimal incentive contracts depend on the sign of the correlation coefficient between division cash-flows. When division cash-flows are negatively correlated, $\rho \leq 0$, optimal contracts are pure equity again, with $\beta_d = \gamma_d$. Furthermore, in this case, division managers have the same beliefs on the productivity of both divisions, with $\hat{q}_d^d = \hat{q}_{d'}^d = e^{-\frac{\kappa}{2}}q$, for $d, d' \in \{A, B\}$ and $d \neq d'$, and again $\hat{q}_d^d \gtrless \hat{q}_d^{HQ}$ as $\kappa_{HQ} \gtrless \kappa$. If $\kappa_{HQ} = \kappa$, HQ and division managers share the same vision in the firm, reaching consensus in the organization.

The case of large uncertainty for either HQ or division managers and positive correlation of division cash-flow is examined in the following theorems.

Theorem 7 *Let condition (S) holds. There is a $\hat{\kappa}_1^{HQ}$ and $\hat{\xi} \in (e^{-\kappa}, 1)$ (both defined in the Appendix), with $\hat{\xi} = 1$ when $\kappa_{HQ} = 0$, such that, if $\kappa > \hat{\kappa}$, $\kappa_{HQ} \leq \hat{\kappa}_1^{HQ}$, and $\rho > 0$, optimal incentive contracts for $d, d' \in \{A, B\}$, and $d \neq d'$ induce beliefs equal to $\hat{q}_d^d = e^{-\frac{\kappa}{2}}\hat{\xi}^{\frac{1}{2}}q$ and $\hat{q}_{d'}^d > q$, and $\hat{q}_d^{HQ} = e^{-\frac{\kappa_{HQ}}{2}}q < q$, by setting*

$$\beta = \frac{1}{1 + \left(\frac{\hat{q}_{d'}^d}{\hat{q}_{d'}^{HQ}} - 1\right)\hat{\xi} + 2(1 - \frac{\hat{q}_d^d}{\hat{q}_d^{HQ}}) + \frac{2r\sigma^2\left(1-\rho\hat{\xi}\right)}{Z\hat{q}_d^{HQ}\hat{q}_d^d}}; \quad \gamma = -\hat{\xi}\beta < 0, \tag{48}$$

where $\hat{\xi}$ is increasing in r, σ, κ, and decreasing in Z, q, κ_{HQ}.

Optimal incentive contracts with positively correlated cash-flows depend critically on the degree of uncertainty affecting HQ. When division cash-flows are positively correlated and HQ are exposed to low levels of uncertainty, $\kappa_{HQ} \leq \hat{\kappa}_1^{HQ}$, while division managers are exposed to large uncertainty, $\kappa > \hat{\kappa}$, optimal contracts have a relative-performance component, with $\gamma < 0$. Cross-division exposure is again proportional to pay-for-performance sensitivity by a factor $\hat{\xi}$, which depends on the level of division managers' risk aversion and their exposure to uncertainty, relative to the uncertainty faced by HQ. Greater managerial risk aversion and cash-flow risk increase the importance of hedging division manager's risk, leading to more cross-division exposure (bigger $\hat{\xi}$). Similarly, greater uncertainty aversion by division managers increases the importance of uncertainty hedging, leading again to more cross-division exposure. In contrast, greater uncertainty aversion by HQ and exposure to a division uncertainty (larger values of Z and q), by exacerbating the disagreement discount, increase the cost of both risk- and uncertainty-hedging. The effect is to reduce optimal cross-division exposure, worsening division managers' confidence in their own division: $\hat{q}_d^d = e^{-\frac{\kappa}{2}}\hat{\xi}^{\frac{1}{2}}q$ (where $\hat{\xi} < 1$).

When uncertainty faced by HQ is sufficiently large, it becomes optimal to grant a division manager positive exposure to the other division, leading to the following.

Theorem 8 *Let condition (S) holds. There is a $\hat{\kappa}_2^{HQ}$ (defined in the Appendix) such that, for $\kappa_{HQ} > \hat{\kappa}_2^{HQ}$ optimal incentive contracts for $d, d' \in \{A, B\}$, and $d \neq d'$, induce beliefs for division managers equal to $\hat{q}_d^d = \hat{q}_{d'}^d = e^{-\frac{\kappa}{2}}q$ and for HQ equal to $\hat{q}_d^{HQ} = e^{-\frac{\kappa_{HQ}}{2}}q < q$ by setting $\beta = \gamma = \hat{\beta}$.*

When HQ are exposed to sufficiently large uncertainty, $\kappa_{HQ} > \hat{\kappa}_2^{HQ}$, optimal incentive contracts are again pure equity with $\beta = \gamma$, with no relative-performance compensation even when division cash-flows are positively correlated. The reason is that large uncertainty exacerbates disagreement on relative-performance compensation and results into a more significant cost of hedging division-manager risk. In this situation, hedging risk can conflict with hedging uncertainty. With sufficiently large uncertainty, the uncertainty-hedging motive overcomes the risk-hedging motive, and HQ forego altogether the risk-hedging benefits of relative-performance. Rather, they offer pure-equity contracts that better aligns division managers beliefs with theirs, lowering the cost of incentive provision and promoting effort. This case is an important reversal of the predictions of the standard optimal contracting problem with no uncertainty of Theorem 2.

Finally, note that equity compensation when HQ are uncertainty averse is optimal even in the case of heterogenous divisions.

Corollary 2 *Let the optimal contract be such that HQ granting positive exposure to both divisions, $\beta_d, \gamma_d > 0$, and both division managers, as well as HQ have beliefs as in case (ii) of Lemma (2) and (4), with $H_d \in (e^{-\kappa_d}, e^{\kappa_d})$ and $H_d^{HQ} \in (e^{-\kappa_{HQ}}, e^{\kappa_{HQ}})$. Then the optimal contract $\{\beta_d, \gamma_d\}_{d\in\{A,B\}}$ has*

$$\beta_d a_d \hat{q}_d^{HQ} + r\sigma^2 \beta_d^2 = \gamma_d a_{d'} \hat{q}_{d'}^{HQ} + r\sigma^2 \gamma_d^2. \tag{49}$$

In addition, $\phi_d a_d \hat{q}_d^{HQ} = \phi_{d'} a_{d'} \hat{q}_{d'}^{HQ}$, and HQ optimally grants both divisions equity compensation: $\beta_d = \gamma_d$.

Similar to Corollary (1), the optimal contract with interior beliefs for both HQ and division managers equates the total (expected) cost to HQ of a division manager's exposure to both division, giving (49). Different from Corollary 1, however, $\hat{q}^{HQ}$ is now endogenous. From Lemma 4, when HQ has interior beliefs, HQ equate expected exposure to each division, $\phi_d a_d \hat{q}_d^{HQ} = \phi_{d'} a_{d'} \hat{q}_{d'}^{HQ}$, which implies that $\beta_d = \gamma_d$. Corollary 2 shows that, when HQ are uncertainty averse, optimality of equity compensation is the outcome of HQ desire to align division managers beliefs with theirs.

6 Uncertainty and Beliefs in Organizations

We develop a novel theory of belief formation in organizations based on uncertainty aversion. We argue that the presence of uncertainty, and the aversion to it, can generate belief heterogeneity even in cases where agents share the same set of "core beliefs." Belief heterogeneity emerges endogenously as the consequence of agents' differential exposure to the sources of uncertainty in the organization.

Individual exposure to uncertainty can be determined first by the position occupied by an agent in the organization. Top executives are exposed to all the uncertainty factors that affect a firm, either directly, or through the relevant economic environment surrounding their firm. In contrast, division managers are disproportionally exposed to uncertainty factors affecting their own division. Exposure to division uncertainty may derive, for example, from the impact of firm performance on division managers' human capital, affecting career opportunities within the firm or their outside options. We refer to this exposure to uncertainty as hierarchical exposure, because it depends on an agent's position in the hierarchy of the organization.

The second form of exposure depends on the contractual arrangements in the organization. Division managers make choices in the context of a web of contracts and rules (organizational protocols) that govern firms. We refer to this exposure to uncertainty as contractual exposure, because it depends on all the (implicit or explicit) contractual arrangements that surround agents.

Hierarchical exposure and contractual exposure together concur to the determination of the belief structure in an organization. The structure of beliefs that emerges in equilibrium is endogenous and depends on both its hierarchical configuration and the contractual relationships that bind agents together. An implication of our paper is that internal beliefs can be managed by both organization design and contract design. In this paper, we focus on the latter. We argue that, by proper design of incentive contracts, HQ can affect beliefs within the organization and induce a more favorable belief system, promoting efficiency.

We show that disagreement emerges as an equilibrium outcome that determines the belief structure in an organization. For example, in our model managers in the upper levels of the hierarchy can (endogenously) be more confident about their firm's future performance than lower-level employees. This implies that rank-and-file managers perceive members of the top management team of a firm (such as CEOs and CFOs) as overconfident and unrealistically confident.

We also argue that the extent of internal disagreement depends on the level of uncertainty that characterize different layers in the organization. When the upper levels in the hierarchy are relatively less concerned about uncertainty than lower-levels, uncertainty concerns deeper down in the hierarchy can generate significant disagreement in the organization. HQ can respond by designing contracts with greater cross-division exposure, through either a more significant equity-based compensation (when division cash-flows are positively correlated) or enhanced relative-performance provisions (with negatively correlated cash-flows).

Our model provides a theoretical foundation of the links between compensation structure and beliefs systems in organizations.[18] The effect of equity-based compensation is to realign internal beliefs, promoting a shared view and internal consensus. In contrast, relative-performance compensation has two divergent effects on internal beliefs. First, it improves and realigns a division managers' beliefs on their division with those of HQ, with beneficial effect on effort provision. The disadvantage of relative-performance compensation is that it may lead division managers to be more confident on the other divisions in the firm, relative to theirs, creating envy and discord. Such discord may interfere with overall management and performance of the organization, for example by affecting the internal allocation of resources.

Finally, a large exposure to uncertainty by top levels in the organization increases the cost of relative-performance compensation. In this situation, HQ may prefer to forego the risk-hedging benefits of relative-performance and, rather, offer cheaper equity-based contracts. Such equity-based contracts provide uncertainty-hedging and promote effort, with the additional benefit of fostering consensus.

7 Conclusions and Future Research

We examine the impact of uncertainty aversion on the design of optimal incentive contracts in an organization. We studied the problem faced by a multidivisional firm, for simplicity with two divisions, where agents may be uncertainty averse. Divisional managers exert unobservable effort that affects the productivity of their division, creating moral hazard. The contracting problem is complicated by the fact that division managers are uncertainty averse, making them unduly conservative in the eyes of their HQ. Such disagreement is endogenous, and is the outcome of the risk-exposure created in the incentive contracts to promote effort.

We showed that the structure of optimal incentive depends on the level of uncertainty that affects firms. For firms with low uncertainty, incentive contracts still exhibit pay-for-performance compensation when division cash-flows are negatively correlated, and relative-performance compensation when division cash-flows are positively correlated, but less than the no-uncertainty case. For firms characterized by high levels of uncertainty, optimal incentive contracts are more likely to have cross-pay compensation or straight-equity.

[18] Links between pay and sentiment is shown in several papers, such as Bergman and Jenter (2007), Heaton (2002), and Oyer and Schaefer (2005), among others.

Our paper can explain how young firms surrounded by greater uncertainty offer equity compensation to their employees, with little scope for relative-performance measures. As they mature, resolving much of the earlier uncertainty, firms then switch to compensation with more pronounced relative-performance features. Our paper can also explain the common use of aggregate measures of performance, such as bonuses geared to the overall performance of an organization. Such reward schemes play a role similar to the equity-based compensation we examine in our paper.

The analysis in our paper can be extended in several ways. First, it would be interesting to examine moral multi-tasking situations, as discussed in Holmström and Milgrom (1991). Our paper suggests an important aspect of uncertainty hedging and its impact on task assignment and optimal compensation. An additional avenue of research is to determine the impact of uncertainty on organization design. For example, it is plausible to expect that organizations in highly uncertain environments have a relatively flat structure, to promote uncertainty hedging. Our paper is also essentially a partial equilibrium model. An interesting question is to examine the impact of labor market forces in a process where heterogenous agents are matched with heterogenous firms. We leave these important questions for future research.

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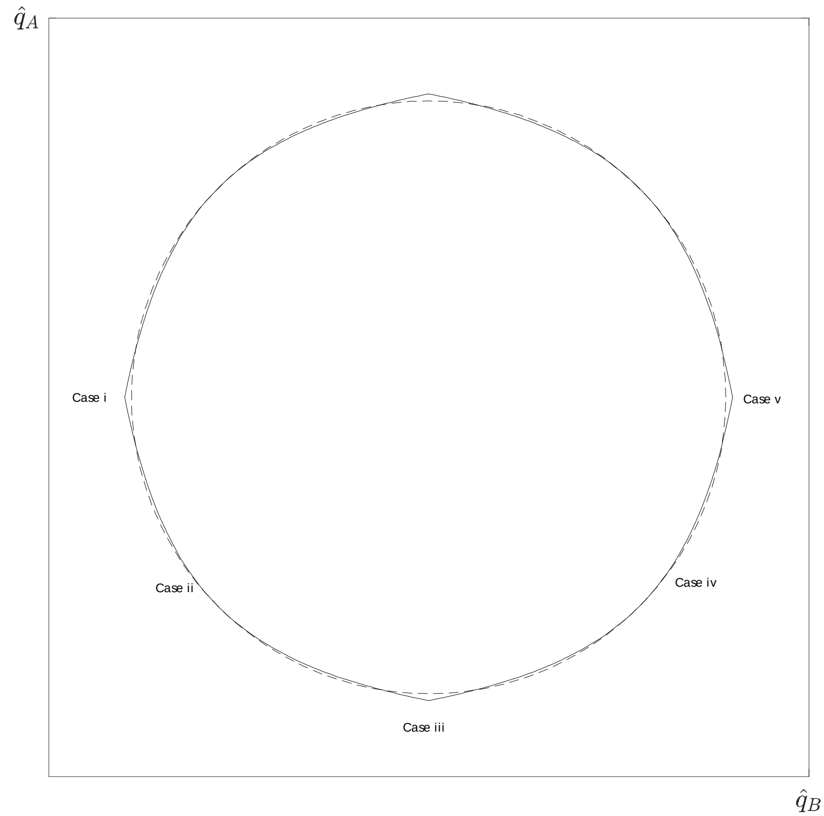

Figure 1: Core of Beliefs

The figure displays the core-belief set, Equation (27), and the 5 cases of Lemma 1 for $d = A$ under parameter values $q_A = q_B = 100$ and $\kappa_A = \ln(5)$. In Case (i), $H_A > e^{\kappa_A}$, and the division manager holds the reference beliefs toward her own division, $\hat{q}_A = q_A$, and extreme pessimism toward the other division, $\hat{q}_B << q_B$. In Case (ii), $H_A \in (e^{-\kappa_A}, e^{\kappa_A})$, leads to moderate pessimism toward both divisions, $\hat{q}_d < q_d$, $d \in \{A, B\}$. In Case (iii), $H_A \in (-e^{-\kappa_A}, e^{\kappa_A})$, leads to extreme pessimism toward her own division, $\hat{q}_A << q_A$, and to reference beliefs toward the other division, $\hat{q}_B = q_B$. In Case (iv), $H_A \in (-e^{\kappa_A}, -e^{-\kappa_A})$, leads to moderate pessimism toward her division, $\hat{q}_A < q_A$, and to optimism toward the other division, $\hat{q}_B > q_B$. In Case (v), $H_A < -e^{\kappa_A}$, leads again to hold the reference beliefs toward her own division, $\hat{q}_A = q_A$, and to be very confident toward the other division, $\hat{q}_B >> q_B$. The dotted line represents the core of beliefs from Equation (3.12) of Chen and Epstein (2002), with $(\hat{q}_A - q_A)^2 + (\hat{q}_B - q_B)^2 \leq k_A$.

Technical Appendix for
"Uncertainty, Contracting, and Beliefs in Organizations"

by David L. Dicks and Paolo Fulghieri

Proof of Theorem 2. Linearity follows from Theorem 1, by setting $K_A = K_B = K_{HQ} = \{0\}$; thus compensation contract to division manager d is $w_d = s_d + \beta_d Y_{d,1} + \gamma_d Y_{d',1}$. Substituting for μ^θ and Σ in (13), division manager d selects a_d to solve

$$\max_{a_d} \; u_d = s_d + \beta_d q_d a_d + \gamma_d q_{d'} a_{d'} - \frac{r\sigma^2}{2}\left(\beta_d^2 + 2\rho\beta_d\gamma_d + \gamma_d^2\right) - c_d\left(a_d\right).$$

Because u_d is strictly concave, the incentive constraint is fully characterized by the first-order condition and the unique maximizer is $a_d = \beta_d Z_d q_d$. Because of translation invariance of u_d, (14) always binds at an optimum, giving

$$s_d = \frac{r\sigma^2}{2}\left(\beta_d^2 + 2\rho\beta_d\gamma_d + \gamma_d^2\right) + c_d\left(a_d\right) - \beta_d q_d a_d - \gamma_d q_{d'} a_{d'}.$$

Substituting for s_d into HQ objective, (12), we obtain

$$\hat{\pi} = \sum_{d\in\{A,B\}} \left[q_d a_d - \frac{r\sigma^2}{2}\left(\beta_d^2 + 2\rho\beta_d\gamma_d + \gamma_d^2\right) - c_d\left(a_d\right)\right],$$

Substituting for $a_d = \beta_d Z_d q_d$ in $\hat{\pi}$ and differentiating we obtain that

$$\beta_d = \frac{1}{1 + r\sigma^2\left(1-\rho^2\right)/\left(Z_d q_d^2\right)}, \quad \text{and} \quad \gamma_d = -\rho\beta_d.$$

Second order conditions are satisfied by concavity of (12). ∎

Proof of Lemma 1. Consider deviations $\theta \in K_\delta(a)$, where $K_\delta(a)$ is defined in (21). By Girsanov's Theorem, deviation θ sets drift $\mu = Qa - \Gamma\theta$, where Γ is defined in (2) and D, N are defined in (22). Thus, $\mu_A = q_A a_A - D\theta_A - N\theta_B = \hat{q}_A a_A$ where $q_A - \hat{q}_A = \frac{1}{a_A}\left(D\theta_A + N\theta_B\right)$. Similarly, $\mu_B = \hat{q}_B a_B$, where $q_B - \hat{q}_B = \frac{1}{a_B}\left(N\theta_A + D\theta_B\right)$. Thus, $|D\theta_A + N\theta_B| = a_A\left|\hat{q}_A - q_A\right|$ and $|N\theta_A + D\theta_B| = a_B\left|\hat{q}_B - q_B\right|$. Substituting in (21), we obtain that $\theta \in K_\delta(a)$ if and only if $\hat{q}^\delta \in C_\delta$, giving (27). ∎

Proof of Lemma 2. Division managers determine $\left(\hat{q}_d^d, \hat{q}_{d'}^d\right)$ in (28). We will focus on two cases: we start with the case where $\gamma_d \geq 0$, and then we consider the case $\gamma_d < 0$. Consider $\tilde{q}_d^d = q_d + \delta$, for $\delta > 0$. Switching to $\tilde{q}_d^{d-} = q_d - \delta$ lowers $\hat{u}_d$ by $2\beta_d a_d \delta$ while leaving the constraint unchanged. Therefore, it must be that $\hat{q}_d^d \leq q_d$.

Similarly, switching from $\tilde{q}_{d'}^{d} = q_{d'} + \delta$, for $\delta > 0$ to $\tilde{q}_{d'}^{d-} = q_{d'} - \delta$ lowers $\hat{u}_d$ by $2\gamma_d a_{d'} \delta$, leaving the constraint unchanged. Therefore, it must also be that $\hat{q}_{d'}^{d} \leq q_{d'}$. Thus, we can express the Lagrangian as

$$\mathcal{L} \equiv -\hat{u}_d - \lambda \left[g_c - \kappa_d \right] - \tau_d \left(\hat{q}_d^d - q_d \right) - \tau_{d'} \left(\hat{q}_{d'}^d - q_{d'} \right)$$

where $g_c \equiv \ln \frac{q_d}{\hat{q}_d^d} + \ln \frac{q_{d'}}{\hat{q}_{d'}^d}$. Because problem (28) admits corner solutions, we characterize its solution by use of the full Kuhn-Tucker conditions:

$$\begin{aligned}
\frac{\partial \mathcal{L}}{\partial \hat{q}_d^d} &= -\frac{\partial \hat{u}_d}{\partial \hat{q}_d^d} - \lambda \frac{\partial g_c}{\partial \hat{q}_d^d} - \tau_d = -\beta_d a_d + \frac{\lambda}{\hat{q}_d^d} - \tau_d = 0, \\
\frac{\partial \mathcal{L}}{\partial \hat{q}_{d'}^d} &= -\frac{\partial \hat{u}_d}{\partial \hat{q}_{d'}^d} - \lambda \frac{\partial g_c}{\partial \hat{q}_{d'}^d} - \tau_{d'} = -\gamma_d a_{d'} + \frac{\lambda}{\hat{q}_{d'}^d} - \tau_{d'} = 0, \\
& \lambda \left(g_c - \kappa_d \right) + \tau_d \left(\hat{q}_d^d - q_d \right) + \tau_{d'} \left(\hat{q}_{d'}^d - q_{d'} \right) = 0, \\
\lambda &\geq 0, \ \tau_{d'} \geq 0, \ \tau_d \geq 0, \ \kappa_d - g_c \geq 0, \ q_d - \hat{q}_d^d \geq 0, \ q_{d'} - \hat{q}_{d'}^d \geq 0.
\end{aligned}$$

Note first that, from the definition of g_c, to satisfy the constraint $\kappa_d - g_c \geq 0$ it must be $\hat{q}_d^d > 0$ and $\hat{q}_{d'} > 0$, which implies that $\frac{\partial \mathcal{L}}{\partial \hat{q}_d^d} = \frac{\partial \mathcal{L}}{\partial \hat{q}_{d'}^d} = 0$. Note also that $\beta_d a_d > 0$ implies that $\lambda > 0$, and thus that $g_c - \kappa_d = 0$. In addition, it cannot be that both $\tau_d > 0$ and $\tau_{d'} > 0$ because, if so, then $\hat{q}_d^d = q_d$ and $\hat{q}_{d'}^d = q_{d'}$, which would imply that $g_c = 0 < \kappa_d$, which contradicts $\lambda > 0$. This leaves us with three types of solutions: $\tau_d = \tau_{d'} = 0$, $\tau_d > 0 = \tau_{d'}$, and $\tau_d = 0 < \tau_{d'}$.

If $\tau_d = \tau_{d'} = 0$, then $\frac{\partial \mathcal{L}}{\partial \hat{q}_d^d} = \frac{\partial \mathcal{L}}{\partial \hat{q}_{d'}^d} = 0$ together imply that $\lambda = \beta_d a_d \hat{q}_d^d$ and $\lambda = \gamma_d a_{d'} \hat{q}_{d'}^d$, giving $\beta_d a_d \hat{q}_d^d = \gamma_d a_{d'} \hat{q}_{d'}^d$. Because $g_c = \kappa_d$ implies that $\hat{q}_d^d \hat{q}_{d'}^d = e^{-\kappa_d} q_d q_{d'}$, after substitution this implies that $\frac{\beta_d a_d}{\gamma_d a_{d'}} \left(\hat{q}_d^d \right)^2 = e^{-\kappa_d} q_d q_{d'}$, or equivalently, $\hat{q}_d^d = \left[e^{-\kappa_d} H_d \right]^{\frac{1}{2}} q_d$, where $H_d = \frac{\gamma_d a_{d'} q_{d'}}{\beta_d a_d q_d}$. Similarly, $\hat{q}_{d'}^d = \left[e^{-\kappa_d} \frac{1}{H_d} \right]^{\frac{1}{2}} q_{d'}$. In order for this to be feasible, however, it must be that $\hat{q}_d^d \leq q_d$, or equivalently, $H_d \leq e^{\kappa_d}$, and $\hat{q}_{d'}^d \leq q_{d'}$, or equivalently, $H_d \geq e^{-\kappa_d}$, giving case (ii). If $\tau_d > 0 = \tau_{d'}$, then $\hat{q}_d^d = q_d$ and, from $g_c = \kappa_d$, also $\hat{q}_{d'}^d = e^{-\kappa_d} q_{d'}$. Note that $\frac{\partial \mathcal{L}}{\partial \hat{q}_{d'}^d} = 0$ implies that $\lambda = \gamma_d a_{d'} e^{-\kappa_d} q_{d'}$ and, from $\frac{\partial \mathcal{L}}{\partial \hat{q}_d^d} = 0$, we have that

$$\tau_d = -\beta_d a_d + \frac{\gamma_d a_{d'} e^{-\kappa_d} q_{d'}}{q_d} = \beta_d a_d \left(H_d e^{-\kappa_d} - 1 \right) > 0,$$

which requires $H_d > e^{\kappa_d}$, giving case (i). Finally, if $\tau_d = 0 < \tau_{d'}$, then $\hat{q}_{d'}^d = q_{d'}$ and, from $g_c = \kappa_d$, also $\hat{q}_d^d = e^{-\kappa_d} q_d$. Note that now $\frac{\partial \mathcal{L}}{\partial \hat{q}_d^d} = 0$ implies that $\lambda = \beta_d a_d e^{-\kappa_d} q_d$, and, from $\frac{\partial \mathcal{L}}{\partial \hat{q}^d_{d'}} = 0$, we have that

$$\tau_{d'} = -\gamma_d a_{d'} + \frac{\beta_d a_d e^{-\kappa_d} q_d}{q_{d'}} = \gamma_d a_{d'} \left(H_d^{-1} e^{-\kappa_d} - 1 \right) \geq 0,$$

which requires $0 \leq H_d < e^{-\kappa_d}$, giving part of case (iii).

The case with $\gamma_d < 0$ proceeds in a similar way, giving cases (iv), (v) and the remainder of case (iii), and is omitted. Note that in the case of interior beliefs, case (iv), for $H_d \in (-e^{\kappa_d}, -e^{-\kappa_d})$ we have

$$\hat{q}^d_d = \left[e^{-\kappa_d} |H_d| \right]^{\frac{1}{2}} q_d, \text{ and } \hat{q}^d_{d'} = \left(2 - \left[e^{-\kappa_d} |H_d|^{-1} \right]^{\frac{1}{2}} \right) q_{d'}.$$

Finally, in case (v) we have $\hat{q}^d_{d'} = q_{d'}$ and $\hat{q}^d_{d'} = (2 - e^{-\kappa_d}) q_{d'}$ for $H_d \leq -e^{\kappa_d}$. ■

Proof of Lemma 3. The lemma is shown in two steps. First, we obtain division managers' best response functions, $a_d = Z_d \beta_d \hat{q}^d_d$, as function of their beliefs, as in Lemma 2. Second, because $\hat{q}^d_d$ is positive, continuous, and increasing in $a_{d'}$, we characterize the Nash equilibrium in terms of $\log(a_d)$ and we apply the contraction mapping theorem, proving uniqueness.

Division manager $d \in \{A, B\}$ chooses effort level a_d to solve (30) by setting

$$\frac{d}{da_d} \hat{u}_d(a, \hat{q}^d_d(a, w)) = \frac{\partial \hat{u}_d}{\partial a_d} + \frac{\partial \hat{u}_d}{\partial \hat{q}^d_d} \frac{\partial \hat{q}^d_d}{\partial a_d} + \frac{\partial \hat{u}_d}{\partial \hat{q}^d_{d'}} \frac{\partial \hat{q}^d_{d'}}{\partial a_d} = \frac{\partial \hat{u}_d}{\partial a_d} = 0,$$

where the second equality holds by the envelope theorem, as follows. For cases (ii) and (iv) of Lemma 2, we have that $\frac{\partial \hat{u}_d}{\partial \hat{q}^d_d} = \lambda \frac{\partial g}{\partial \hat{q}_d}$ and $\frac{\partial \hat{u}_d}{\partial \hat{q}^d_{d'}} = \lambda \frac{\partial g}{\partial \hat{q}_{d'}}$, giving

$$\frac{\partial \hat{u}_d}{\partial \hat{q}^d_d} \frac{\partial \hat{q}^d_d}{\partial a_d} + \frac{\partial \hat{u}_d}{\partial \hat{q}^d_{d'}} \frac{\partial \hat{q}^d_{d'}}{\partial a_d} = \lambda \left(\frac{\partial g}{\partial \hat{q}_d} \frac{\partial \hat{q}^d_d}{\partial a_d} + \frac{\partial g}{\partial \hat{q}_{d'}} \frac{\partial \hat{q}^d_{d'}}{\partial a_d} \right) = \lambda \frac{dg}{da_d} = 0$$

because $g = e^{-\kappa_d}$. In cases (i)-(iii)-(v), $\hat{q}^d_d$ and $\hat{q}^d_{d'}$ do not depend on a_d, and $\frac{\partial \hat{q}^d_d}{\partial a_d} = \frac{\partial \hat{q}^d_{d'}}{\partial a_d} = 0$, giving $\frac{d\hat{u}_d}{da_d} = \frac{\partial \hat{u}_d}{\partial a_d} = \beta_d \hat{q}^d_d - \frac{a_d}{Z_d} = 0$.

Thus, the best response functions are $a_d = Z_d \beta_d \hat{q}^d_d$, where beliefs $\hat{q}^d_d$ are from Lemma 2. If $\gamma_d = 0$, we have that $H_d = 0$, giving $a_d = Z_d \beta_d e^{-\kappa_d} q_d$. If $\gamma_d \neq 0$, the best response depends on the effort by the other division manager, $a_{d'}$. If the other division manager, $d' \neq d$, exerts low effort $a_{d'} < a^L_{d'} \equiv \frac{Z_d \beta_d^2 e^{-2\kappa_d} q_d^2}{|\gamma_d| q_{d'}}$, we have that $|H_d| < e^{-\kappa_d}$ and division manager d holds pessimistic belief as in case (iii) of Lemma 2, $\hat{q}^d_d = e^{-\kappa_d} q_d$, giving $a_d = a^{1*}_d \equiv Z_d \beta_d e^{-\kappa_d} q_d$. If division manager d' exerts moderate level of effort, $a^L_{d'} \leq a_{d'} < a^H_{d'} \equiv \frac{Z_d \beta_d^2 e^{\kappa_d} q_d^2}{|\gamma_d| q_{d'}}$, division manager d hold beliefs as in case

(ii) of Lemma 2, if $\gamma_d > 0$, and as in case (iv), if $\gamma_d < 0$; thus $|H_d| \in [e^{-\kappa_d}, e^{\kappa_d}]$ and $a_d = [Z_d^2 |\gamma_d| a_{d'} \beta_d e^{-\kappa_d} q_{d'} q_d]^{\frac{1}{3}}$. Finally, if division manager d' exerts a high level of effort, $a_{d'} > a_{d'}^H$, division manager d hold beliefs as in case (i) of Lemma 2, if $\gamma_d > 0$, and as in case (v), if $\gamma_d < 0$; thus $|H_d| > e^{\kappa_d}$ and $a_d = Z_d \beta_d q_d$. The best response function for DM d is therefore given by

$$a_d^*(a_{d'}) = \begin{cases} a_d^{1*} \equiv Z_d \beta_d e^{-\kappa_d} q_d & a_{d'} < a_{d'}^L \\ \tilde{a}_d^*(a_{d'}) \equiv [Z_d^2 |\gamma_d| a_{d'} \beta_d e^{-\kappa_d} q_{d'} q_d]^{\frac{1}{3}} & a_{d'}^L \leq a_{d'} \leq a_{d'}^H \\ a_d^{2*} \equiv Z_d \beta_d q_d & a_{d'} > a_{d'}^H \end{cases}.$$

A Nash equilibrium is a pair $\{a_A, a_B\}$ such that $a_d = a_d^*(a_{d'})$, $d \in \{A, B\}$, $d \neq d'$. Note that $a_d^*(a_{d'})$ is a positive, continuous, and increasing function of $a_{d'}$. Expressing the best response in logs, we obtain

$$\ln a_d^*(\ln a_{d'}) = \begin{cases} \ln Z_d \beta_d e^{-\kappa_d} q_d & \ln a_{d'} < \ln a_{d'}^L \\ \ln [Z_d^2 |\gamma_d| \beta_d e^{-\kappa_d} q_{d'} q_d]^{\frac{1}{3}} + \frac{1}{3} \ln (a_{d'}) & \ln a_{d'}^L \leq \ln a_{d'} \leq \ln a_{d'}^H \\ \ln Z_d \beta_d q_d & \ln a_{d'} > \ln a_{d'}^H \end{cases}.$$

Further, note $\frac{d \ln a_d^*}{d \ln a_{d'}} = 0$ for $a_{d'} < a_{d'}^L$ and $a_{d'} > a_{d'}^H$, while $\frac{d \ln a_d^*}{d \ln a_{d'}} = \frac{1}{3}$ for $a_{d'}^L < a_{d'} < a_{d'}^H$. Define $F : \mathbb{R}^2 \to \mathbb{R}^2$ so that $F \equiv (\ln a_A^*(\ln a_B), \ln a_B^*(\ln a_A))'$, and let $d(x, y)$ be the Euclidean distance. For $x, y \in \mathbb{R}^2$, define $\tilde{x}_d \equiv \max\{\ln a_d^L, \min\{x_d, \ln a_d^H\}\}$ and $\tilde{y}_d \equiv \max\{\ln a_d^L, \min\{y_d, \ln a_d^H\}\}$, we have

$$\begin{aligned} d(F(x), F(y)) &= \sqrt{(\ln a_A^*(x_B) - \ln a_A^*(y_B))^2 + (\ln a_B^*(x_A) - \ln a_B^*(y_A))^2} \\ &= \sqrt{(\ln a_A^*(\tilde{x}_B) - \ln a_A^*(\tilde{y}_B))^2 + (\ln a_B^*(\tilde{x}_A) - \ln a_B^*(\tilde{y}_A))^2} \\ &= \sqrt{\left[\frac{1}{3}(\tilde{x}_B - \tilde{y}_B)\right]^2 + \left[\frac{1}{3}(\tilde{x}_A - \tilde{y}_A)\right]^2} = \frac{1}{3} d(\tilde{x}, \tilde{y}) \leq \frac{1}{3} d(x, y), \end{aligned}$$

which implies that $0 \leq d(F(x), F(y)) \leq \frac{1}{3} d(x, y)$ for all $x, y \in \mathbb{R}^2$. Thus, F is a contraction mapping and the Nash Equilibrium exists and is unique.

Because the best-response function is constant if d' exerts low effort, $a_{d'} < a_{d'}^L$, and if d' exerts high effort, $a_{d'} > a_{d'}^H$, the Nash Equilibrium is fully determined. All that remains to be determined is the Nash Equilibrium effort for d when $a_{d'}^L \leq a_{d'} \leq a_{d'}^H$. There are three possible cases:

(1) If $a_{d'} = a_{d'}^{1*} > a_{d'}^{L}$, so that $|H_{d'}| \leq e^{-\kappa_{d'}}$, then

$$a_d = \tilde{a}_d^*\left(a_{d'}^{1*}\right) = \left[Z_d^2 Z_{d'} e^{-(\kappa_d+\kappa_{d'})} |\gamma_d| \beta_{d'} \beta_d q_{d'}^2 q_d\right]^{\frac{1}{3}};$$

(2) If $a_{d'} = a_{d'}^{2*} < a_{d'}^{H}$, so that $|H_{d'}| \geq e^{\kappa_{d'}}$, then

$$a_d = \tilde{a}_d^*\left(a_{d'}^{2*}\right) = \left[Z_d^2 Z_{d'} e^{-\kappa_d} |\gamma_d| \beta_{d'} \beta_d q_{d'}^2 q_d\right]^{\frac{1}{3}};$$

(3) if $a_{d'}^{1*} < a_{d'} < a_{d'}^{2*}$, so that $|H_{d'}| \in (e^{-\kappa_{d'}}, e^{\kappa_{d'}})$, then setting $a_d = \tilde{a}_d^*(a_{d'})$ and $a_{d'} = \tilde{a}_{d'}^*(a_d)$, after solving we obtain

$$a_d = \breve{a}_d \equiv \left[e^{-\kappa_d} Z_d^2 \beta_d |\gamma_d|\right]^{\frac{3}{8}} \left[e^{-\kappa_{d'}} Z_{d'}^2 \beta_{d'} |\gamma_{d'}|\right]^{\frac{1}{8}} \left[q_d q_{d'}\right]^{\frac{1}{2}}. \tag{A1}$$

Comparative statics follow by direct differentiation. ■

Proof of Theorem 3. Because (33) binds and $r = 0$, HQ payoff $\hat{\pi}$ is now equal to

$$\hat{\pi} = \sum_{\substack{d,d' \in \{A,B\},\\ d' \neq d}} \left(q_d a_d - \beta_d a_d \left(q_d - \hat{q}_d^d\right) - \gamma_d a_{d'} \left(q_{d'} - \hat{q}_{d'}^d\right) - \frac{a_d^2}{2Z_d} \right),$$

where a_d are the Nash-equilibrium effort levels of Lemma 3. The proof is in two steps. First, we show that $\hat{\pi}$ is symmetric in γ_d around zero; in the second step, we find the optimal contract under the restriction that $\gamma_d \geq 0$.

Note that, from Lemma 2, $\hat{q}_d^d$ depends on γ_d only through its absolute value, $|\gamma_d|$. Thus, from Lemma 3, equilibrium action $a_d = \beta_d Z_d \hat{q}_d^d$ also depends on $|\gamma_d|$ only. This implies the first term of the disagreement discount, $\beta_d a_d \left(q_d - \hat{q}_d^d\right)$, depends only on $|\gamma_d|$. We next show that, if $\gamma_d < 0$, the second term of the disagreement discount, $\gamma_d a_{d'} \left(q_{d'} - \hat{q}_{d'}^d\right)$, is unchanged by offering cross pay, $|\gamma_d|$, rather than relative performance evaluation, $\gamma_d < 0$. From Lemma 2, let $\hat{q}_{d'}^{d+}$ be the belief held by the DM when receiving $|\gamma_d|$ instead of $\gamma_d < 0$. We will show $\gamma_d a_{d'} \left(q_{d'} - \hat{q}_{d'}^d\right) = |\gamma_d| a_{d'} \left(q_{d'} - \hat{q}_{d'}^{d+}\right)$. Consider in turn cases (iii), (iv) and (v) in Lemma 2.

First, in case (v) we have that $H_d < -e^{\kappa_d}$ and $\hat{q}_{d'}^d = (2 - e^{-\kappa_d}) q_{d'}$. This implies that replacing γ_d with $|\gamma_d|$ gives that $|H_d| > e^{\kappa_d}$ and beliefs will be as in case (i). Thus, setting $\hat{q}_{d'}^{d+} = e^{-\kappa_d} q_{d'}$ we obtain

$$|\gamma_d| a_{d'} \left(q_{d'} - \hat{q}_{d'}^{d+}\right) = |\gamma_d| a_{d'} \left(1 - e^{-\kappa_d}\right) q_{d'} = \gamma_d a_{d'} \left(e^{-\kappa_d} - 1\right) q_{d'} = \gamma_d a_{d'} \left(q_{d'} - \hat{q}_{d'}^d\right).$$

In case (iii), we have that $|H_d| < e^{-\kappa_d}$. This implies that $\hat{q}^{d+}_{d'} = \hat{q}^d_{d'} = q_{d'}$, so

$$|\gamma_d|\, a_{d'}\left(q_{d'} - \hat{q}^{d+}_{d'}\right) = \gamma_d a_{d'}\left(q_{d'} - \hat{q}^{d}_{d'}\right) = 0.$$

In case (iv), $H_d \in (-e^{\kappa_d}, -e^{-\kappa_d})$ and $\hat{q}^d_{d'} = \left(2 - \left[e^{-\kappa_d}\frac{\beta_d a_d q_d}{|\gamma_d| a_{d'} q_{d'}}\right]^{\frac{1}{2}}\right) q_{d'}$, giving

$$\gamma_d a_{d'}\left(q_{d'} - \hat{q}^d_{d'}\right) = \gamma_d a_d \left(\left[\frac{e^{-\kappa_d}\beta_d a_d q_d}{|\gamma_d|\, a_{d'} q_{d'}}\right]^{\frac{1}{2}} - 1\right) q_{d'} = |\gamma_d|\, a_{d'}\left(1 - \left[\frac{e^{-\kappa_d}\beta_d a_d q_d}{|\gamma_d|\, a_{d'} q_{d'}}\right]^{\frac{1}{2}}\right) q_{d'}.$$

This implies that replacing γ_d with $|\gamma_d|$, beliefs will be as in case (ii). Thus, setting $\hat{q}^{d+}_{d'} = \left[e^{-\kappa_d}\frac{\beta_d a_d q_d}{|\gamma_d| a_{d'} q_{d'}}\right]^{\frac{1}{2}} q_{d'}$ we obtain

$$|\gamma_d|\, a_{d'}\left(q_{d'} - \hat{q}^{d+}_{d'}\right) = \gamma_d a_{d'}\left(q_{d'} - \hat{q}^{d}_{d'}\right).$$

Therefore, $\hat{\pi}(\gamma_d) = \hat{\pi}(|\gamma_d|)$ and $\hat{\pi}$ is symmetric in γ_d around zero.

Because HQ is indifferent between $|\gamma_d|$ and γ_d, it is sufficient to consider $\gamma_d \geq 0$. If $\gamma_d > e^{\kappa_d}\frac{\beta_d a_d q_d}{a_{d'} q_{d'}}$, division manager beliefs are in case (i) of Lemma 2, with $\hat{q}^d_d = q_d$ and $\hat{q}^d_{d'} = e^{-\kappa_d} q_{d'}$, giving $a_d = \beta_d Z_d q_d$. Thus, $\frac{\partial \hat{\pi}}{\partial \gamma_d} = -a_{d'} q_{d'}\left(1 - e^{-\kappa_d}\right) < 0$, and setting $\gamma_d > e^{\kappa_d}\frac{\beta_d a_d q_d}{a_{d'} q_{d'}}$ is not optimal. Similarly, if $\gamma_d < e^{-\kappa_d}\frac{\beta_d a_d q_d}{a_{d'} q_{d'}}$, division manager beliefs are in case (iii) of Lemma 2, with $\hat{q}^d_d = e^{-\kappa_d} q_d$ and $\hat{q}^d_{d'} = q_{d'}$, giving $a_d = \beta_d Z_d e^{-\kappa_d} q_d$. In addition, $\hat{q}^d_{d'} = q_{d'}$ and $\hat{q}^d_d = e^{-\kappa_d} q_d$ together imply that $\frac{\partial \hat{\pi}}{\partial \gamma_d} = 0$ and it is weakly optimal to set $\gamma_d \geq e^{-\kappa_d}\frac{\beta_d a_d q_d}{a_{d'} q_{d'}}$. This implies that HQ set $e^{-\kappa_d}\frac{\beta_d a_d q_d}{a_{d'} q_{d'}} \leq \gamma_d \leq e^{\kappa_d}\frac{\beta_d a_d q_d}{a_{d'} q_{d'}}$ and induce beliefs that are in case (ii) of 2, with $H_d \in [e^{-\kappa_d}, e^{\kappa_d}]$.

Because the participation constraint binds, HQ objective function becomes

$$\hat{\pi} = (\mathbf{1} - b_A - b_B)' Q \check{a}_d + \left(\hat{u}_A(\check{a}_A, \hat{q}^A) - s_A\right) + \left(\hat{u}_B(\check{a}_B, \hat{q}^B) - s_B\right).$$

where $\hat{u}_d(\check{a}_d, \hat{q}^d) = \min_{\hat{q}^d \in C^d} \hat{u}_d$, with $\hat{u}_d = s_d + \beta_d \check{a}_d \hat{q}^d_d + \gamma_d \check{a}_{d'} \hat{q}^d_{d'} - \frac{\check{a}_d^2}{2Z_d} = 0$ and where $\check{a}_d$ is the Nash equilibrium given by (A1) in the proof of Lemma 3. This implies that

$$\begin{aligned}\frac{d\hat{\pi}}{d\beta_d} &= -q_d \check{a}_d + \left(1 - \beta_d - \gamma_{d'}\right) q_d \frac{\partial \check{a}_d}{\partial \beta_d} + \left(1 - \beta_{d'} - \gamma_d\right) q_{d'} \frac{\partial \check{a}_{d'}}{\partial \beta_d} \\ &\quad + \frac{d\hat{u}_d(\check{a}_d, \hat{q}^d(\check{a}_d, w_d))}{d\beta_d} + \frac{d\hat{u}_{d'}(\check{a}_{d'}, \hat{q}^{d'}(\check{a}_{d'}, w_{d'}))}{d\beta_d}.\end{aligned}$$

Because $\frac{\partial \hat{u}_d}{\partial \beta_d} = \check{a}_d \hat{q}^d_d$, $\frac{\partial \hat{u}_d}{\partial \check{a}_{d'}} = \gamma_d \hat{q}^d_{d'}$, and $\frac{\partial \check{a}_{d'}}{\partial \beta_d} = \frac{\check{a}_{d'}}{8\beta_d}$, by applying the envelope theorem on $\hat{u}_d(\breve{a}_d, \hat{q}^d)$, we obtain that

$$\frac{d\hat{u}_d(\breve{a}_d, \hat{q}^d(\breve{a}_d, w_d))}{d\beta_d} = \frac{\partial \hat{u}_d}{\partial \beta_d} + \frac{\partial \hat{u}_d}{\partial \breve{a}_{d'}} \frac{\partial \breve{a}_{d'}}{\partial \beta_d} = \breve{a}_d \hat{q}^d_d + \gamma_d \hat{q}^d_{d'} \frac{\breve{a}_{d'}}{8\beta_d}. \tag{A2}$$

Similarly, because $\frac{\partial \hat{u}_{d'}}{\partial \beta_d} = 0$, $\frac{\partial \hat{u}_{d'}}{\partial \breve{a}_d} = \gamma_{d'} \hat{q}^{d'}_d$, and $\frac{\partial \breve{a}_d}{\partial \beta_d} = \frac{3\breve{a}_d}{8\beta_d}$, by applying the envelope theorem on $\hat{u}_{d'}(\breve{a}_{d'}, \hat{q}^{d'})$, we obtain that

$$\frac{d\hat{u}_{d'}(\breve{a}_{d'}, \hat{q}^{d'}(\breve{a}_{d'}, w_{d'}))}{d\beta_d} = \frac{\partial \hat{u}_{d'}}{\partial \beta_d} + \frac{\partial \hat{u}_{d'}}{\partial \breve{a}_d} \frac{\partial \breve{a}_d}{\partial \beta_d} = \gamma_{d'} \hat{q}^{d'}_d \frac{3\breve{a}_d}{8\beta_d}. \tag{A3}$$

Together, (A2) and (A3) give that

$$\begin{aligned} \frac{d\hat{\pi}}{d\beta_d} &= -\breve{a}_d \left(q_d - \hat{q}^d_d\right) + (1 - \beta_d - \gamma_{d'})\, q_d \frac{3\breve{a}_d}{8\beta_d} + (1 - \beta_{d'} - \gamma_d)\, q_{d'} \frac{\breve{a}_{d'}}{8\beta_d} \\ &\quad + \gamma_d \hat{q}^d_{d'} \frac{\breve{a}_{d'}}{8\beta_d} + \gamma_{d'} \hat{q}^{d'}_d \frac{3\breve{a}_d}{8\beta_d}. \end{aligned} \tag{A4}$$

Consider now γ_d. We have that

$$\begin{aligned} \frac{d\hat{\pi}}{d\gamma_d} &= -q_{d'} \breve{a}_{d'} + (1 - \beta_d - \gamma_{d'})\, q_d \frac{\partial \breve{a}_d}{\partial \gamma_d} + (1 - \beta_{d'} - \gamma_d)\, q_{d'} \frac{\partial \breve{a}_{d'}}{\partial \gamma_d} \\ &\quad + \frac{d\hat{u}_d(\breve{a}_d, \hat{q}^d(\breve{a}_d, w_d))}{d\gamma_d} + \frac{d\hat{u}_{d'}(\breve{a}_{d'}, \hat{q}^{d'}(\breve{a}_{d'}, w_{d'}))}{d\gamma_d}. \end{aligned}$$

Because $\frac{\partial \hat{u}_d}{\partial \gamma_d} = \breve{a}_{d'} \hat{q}^d_{d'}$, $\frac{\partial \hat{u}_d}{\partial \breve{a}_{d'}} = \gamma_d \hat{q}^d_{d'}$, and $\frac{\partial \breve{a}_{d'}}{\partial \gamma_d} = \frac{\breve{a}_{d'}}{8\gamma_d}$, by applying the envelope theorem on $\hat{u}_d(\breve{a}_d, \hat{q}^d)$, we obtain that

$$\frac{d\hat{u}_d(\breve{a}_d, \hat{q}^d(a_d, w_d))}{d\gamma_d} = \frac{\partial \hat{u}_d}{\partial \gamma_d} + \frac{\partial \hat{u}_d}{\partial \breve{a}_{d'}} \frac{\partial \breve{a}_{d'}}{\partial \gamma_d} = \breve{a}_{d'} \hat{q}^d_{d'} + \gamma_d \hat{q}^d_{d'} \frac{\breve{a}_{d'}}{8\gamma_d}. \tag{A5}$$

Similarly, because $\frac{\partial \hat{u}_{d'}}{\partial \gamma_d} = 0$, $\frac{\partial \hat{u}_{d'}}{\partial \breve{a}_d} = \gamma_{d'} \hat{q}^{d'}_d$, and $\frac{\partial \breve{a}_d}{\partial \gamma_d} = \frac{3\breve{a}_d}{8\gamma_d}$, by applying the envelope theorem on $\hat{u}_{d'}(\breve{a}_{d'}, \hat{q}^{d'})$, we obtain that

$$\frac{d\hat{u}_{d'}(\breve{a}_{d'}, \hat{q}^{d'}(\breve{a}_{d'}, w_{d'}))}{d\gamma_d} = \frac{\partial \hat{u}_{d'}}{\partial \gamma_d} + \frac{\partial \hat{u}_{d'}}{\partial \breve{a}_d} \frac{\partial \breve{a}_d}{\partial \gamma_d} = \gamma_{d'} \hat{q}^{d'}_d \frac{3\breve{a}_d}{8\gamma_d}. \tag{A6}$$

Together, (A5) and (A6) give that

$$\begin{aligned} \frac{d\hat{\pi}}{d\gamma_d} &= -\breve{a}_{d'} \left(q_{d'} - \hat{q}^d_{d'}\right) + (1 - \beta_d - \gamma_{d'})\, q_d \frac{3\breve{a}_d}{8\gamma_d} + (1 - \beta_{d'} - \gamma_d)\, q_{d'} \frac{\breve{a}_{d'}}{8\gamma_d} \\ &\quad + \gamma_d \hat{q}^d_{d'} \frac{\breve{a}_{d'}}{8\gamma_d} + \gamma_{d'} \hat{q}^{d'}_d \frac{3\breve{a}_d}{8\gamma_d}. \end{aligned} \tag{A7}$$

Thus, from (A4) and (A7) we obtain the first-order conditions:

$$\frac{d\hat{\pi}}{d\beta_d} = -\breve{a}_d\left(q_d - \hat{q}_d^d\right) + \frac{\Delta_d}{\beta_d} = 0; \quad \frac{d\hat{\pi}}{d\gamma_d} = -\breve{a}_{d'}\left(q_{d'} - \hat{q}_{d'}^d\right) + \frac{\Delta_d}{\gamma_d} = 0,$$

where $\Delta_d \equiv \left(1 - \beta_d - \gamma_{d'}\right) q_d \frac{3\breve{a}_d}{8} + \left(1 - \beta_{d'} - \gamma_d\right) q_{d'} \frac{\breve{a}_{d'}}{8} + \gamma_d \hat{q}_{d'}^d \frac{\breve{a}_{d'}}{8} + \gamma_{d'} \hat{q}_d^{d'} \frac{3\breve{a}_d}{8}$, giving

$$\beta_d \breve{a}_d\left(q_d - \hat{q}_d^d\right) = \gamma_{d'}\breve{a}_{d'}\left(q_{d'} - \hat{q}_{d'}^d\right). \tag{A8}$$

Because, from Lemma 2, $\beta_d \breve{a}_d \hat{q}_d^d = \gamma_d \breve{a}_{d'} \hat{q}_{d'}^d$, we have that (A8) implies that $\beta_d \breve{a}_d q_d = \gamma_d \breve{a}_{d'} q_{d'}$ and thus that $H_d = 1$, leading to $\hat{q}_d^d = \hat{q}_{d'}^d = e^{-\frac{\kappa_d}{2}} q_d$ and $\breve{a}_d = e^{-\frac{\kappa_d}{2}} \beta_d Z_d q_d$. Substituting the values of γ_d and $\breve{a}_d$ into HQ objective, we obtain

$$\hat{\pi} = \sum_{\substack{d,d' \in \{A,B\}, \\ d' \neq d}} \left[\beta_d Z_d q_d \hat{q}_d^d - 2\beta_d^2 Z_d \hat{q}_d^d\left(q_d - \hat{q}_d^d\right) - \frac{\beta_d^2 Z_d \left(\hat{q}_d^d\right)^2}{2}\right],$$

Differentiating, we obtain

$$\frac{d\hat{\pi}}{d\beta_d} = Z_d q_d \hat{q}_d^d - 4\beta_d Z_d \hat{q}_d^d\left(q_d - \hat{q}_d^d\right) - \beta_d Z_d\left(\hat{q}_d^d\right)^2 = 0,$$

giving

$$\beta_d = \frac{1}{1 + 3\left(1 - \hat{q}_d^d / q_d\right)}.$$

Finally, setting $H_d = 1$ gives

$$\gamma_d = \frac{\breve{a}_d q_d}{\breve{a}_{d'} q_{d'}} \beta_d = \xi_d \beta_d, \text{ where } \xi_d \equiv \frac{\breve{a}_d q_d}{\breve{a}_{d'} q_{d'}}.$$

Substituting for the values of $\breve{a}_d$ and $\breve{a}_{d'}$, given the expression for beliefs in Lemma 2, we obtain

$$\xi_d = \frac{1 - 3\left(1 - \hat{q}_{d'}^{d'} / q_{d'}\right)}{1 - 3\left(1 - \hat{q}_d^d / q_d\right)} \frac{\hat{q}_d^d / q_d}{\hat{q}_{d'}^{d'} q_{d'}} \frac{Z_d q_d^2}{Z_{d'} q_{d'}^2}.$$

If HQ implement the symmetric contract, with $\gamma_d = -\frac{\breve{a}_d q_d^d}{\breve{a}_{d'} q_{d'}^d} \beta_d$, we obtain that $\hat{q}_{d'}^d = \left(2 - e^{-\frac{\kappa_d}{2}}\right) q_{d'}$. Thus $|\gamma_d| = \xi_d \beta_d$. If divisions are symmetric, and condition (S) holds, $\xi_d = 1$. Comparative statics follow by direct differentiation. ■

Proof of Theorem 4. Because the participation constraint (33) binds, HQ payoff,

$\hat{\pi}$, now is equal to

$$\sum_{\substack{d,d'\in\{A,B\}\\ d'\neq d}}\left[\left(1-\beta_d-\gamma_{d'}\right)q_d a_d+\beta_d a_d \hat{q}_d^d+\gamma_d a_{d'}\hat{q}_{d'}^d-\frac{a_d^2}{2Z_d}-\frac{r\sigma^2\left(\beta_d^2+2\beta_d\gamma_d\rho+\gamma_d^2\right)}{2}\right]$$

where $\{a_A, a_B\}$ are the Nash equilibrium effort levels of Lemma 3.

Different from the case of Theorem 3, because of the presence of the last term, HQ objective function $\hat{\pi}$ admits multiple strict local maxima. The proof therefore proceeds in two steps. First, we consider candidate optimal contracts that induce division managers to hold one of four possible configurations of beliefs (implied by Lemma 2). Specifically, we consider contracts as follows. Case (A): a small exposure to the other division leading to $|H_d| < e^{-\kappa_d}$, corresponding to case (iii) of Lemma 2; Case (B): a moderate positive exposure to the other division, leading to $H_d \in (e^{-\kappa_d}, e^{\kappa_d})$, corresponding to case (ii) of Lemma 2; Case (C): a moderate negative exposure to the other division, leading to $H_d \in (-e^{\kappa_d}, -e^{-\kappa_d})$, corresponding to case (iv) of Lemma 2; Case (D): a large (negative or positive) exposure to the other division, leading to $|H_d| > e^{\kappa_d}$ corresponding to cases (i) and (v) of Lemma 2. Second, we compare payoffs to HQ from optimal contracts in these regions and we determine the globally optimal contract.

Case (A): If $|H_d| < e^{-\kappa_d}$, have $\hat{q}_d^d = e^{-\kappa_d} q_d$ and $\hat{q}_{d'}^d = q_{d'}$, which do not depend on γ_d. Similarly, by Lemma 3, $a_d = \beta_d Z_d e^{-\kappa_d} q_d$, which does not depend on γ_d as well. Therefore, setting

$$\frac{\partial \hat{\pi}}{\partial \gamma_d} = -r\sigma^2\left(\rho\beta_d + \gamma_d\right) = 0$$

gives $\gamma_d = -\rho\beta_d$ and γ_d is set to hedge risk with no effect on incentives. Substituting in $\hat{\pi}$ and differentiating we obtain

$$\frac{\partial \hat{\pi}}{\partial \beta_d} = \left(1-2\beta_d\right)Z_d q\hat{q}_d^d + \beta_d Z_d\left(\hat{q}_d^d\right)^2 - r\sigma^2\beta_d\left(1-\rho^2\right)$$

Therefore

$$\beta_d^1 \equiv \frac{1}{1+\left(1-\hat{q}_d^d/q\right)+r\sigma^2\left(1-\rho^2\right)/(Zq\hat{q}_d^d)}.$$

After substitution, this gives HQ payoff under condition (S)

$$\hat{\pi}^1 \equiv \frac{\left[e^{-\kappa}Zq^2\right]^2}{\left(2-e^{-\kappa}\right)e^{-\kappa}Zq^2+r\sigma^2\left(1-\rho^2\right)}.$$

Case (B): If $H_d \in (e^{-\kappa}, e^{\kappa})$, we can express the payoff to HQ as

$$\hat{\pi} = (\mathbf{1} - b_A - b_B)' Qa + \left(\hat{u}_A(a_A, \hat{q}^A(a_A, w_A)) - s_A\right) + \left(\hat{u}_B(a_B, \hat{q}^B(a_B, w_B)) - s_B\right),$$

where $\hat{u}_d(a_d, \hat{q}^d(a_d, w_d)) = \min_{\hat{q}^d \in C^d} \hat{u}_d$, with

$$\hat{u}_d(a_d, \hat{q}^d(a_d, w_d)) = \beta_d a_d \hat{q}^d_d + \gamma_d a_{d'} \hat{q}^d_{d'} - \frac{r\sigma^2}{2}\left(\beta_d^2 + 2\rho\beta_d\gamma_d + \gamma_d^2\right) - \frac{a_d^2}{2Z_d} = 0,$$

and where $\check{a}_d$ is the Nash equilibrium given by (A1). Because $\hat{u}_d$ is strictly concave and the minimum operator is concave, $\hat{u}_d(a_d, \hat{q}^d(a_d, w_d))$ is strictly concave. Therefore, $\hat{\pi}$ is strictly concave as well. Thus, first-order conditions of optimality are sufficient for a local optimum. Similar to the proof of Theorem 3, we have

$$\begin{aligned} \frac{d\hat{\pi}}{d\beta_d} &= -q_d\check{a}_d + (1 - \beta_d - \gamma_{d'})\, q_d \frac{\partial \check{a}_d}{\partial \beta_d} + (1 - \beta_{d'} - \gamma_d)\, q_{d'} \frac{\partial \check{a}_{d'}}{\partial \beta_d} \\ &\quad + \frac{d\hat{u}_d(\check{a}_d, \hat{q}^d(\check{a}_d, w_d))}{d\beta_d} + \frac{d\hat{u}_{d'}(\check{a}_{d'}, \hat{q}^{d'}(\check{a}_{d'}, w_{d'}))}{d\beta_d}. \end{aligned}$$

In this region, from (A1), we have $\frac{\partial \check{a}_d}{\partial \beta_d} = \frac{3\check{a}_d}{8\beta_d}$ and $\frac{\partial \check{a}_{d'}}{\partial \beta_d} = \frac{\check{a}_{d'}}{8\beta_d}$. Because $\frac{\partial \hat{u}_d}{\partial \check{a}_{d'}} = \gamma_d \hat{q}^d_{d'}$ and $\frac{\partial \hat{u}_d}{\partial \beta_d} = a_d \hat{q}^d_d - r\sigma^2(\beta_d + \rho\gamma_d)$, by applying the envelope theorem on $\hat{u}_d(\check{a}_d, \hat{q}^d)$:

$$\frac{d\hat{u}_d(\check{a}_d, \hat{q}^d(\check{a}_d, w_d))}{d\beta_d} = a_d \hat{q}^d_d - r\sigma^2(\beta_d + \rho\gamma_d) + \gamma_d \hat{q}^d_{d'} \frac{\check{a}_{d'}}{8\beta_d}. \tag{A9}$$

Similarly, because $\frac{\partial \hat{u}_{d'}}{\partial \beta_d} = 0$ and $\frac{\partial \hat{u}_{d'}}{\partial \check{a}_d} = \gamma_{d'} \hat{q}^{d'}_d$, from (A9) and (A3) we obtain

$$\begin{aligned} \frac{d\hat{\pi}}{d\beta_d} &= -a_d\left(q_d - \hat{q}^d_d\right) + (1 - \beta_d - \gamma_{d'})\, q_d \frac{3\check{a}_d}{8\beta_d} + (1 - \beta_{d'} - \gamma_d)\, q_{d'} \frac{\check{a}_{d'}}{8\beta_d} \\ &\quad - r\sigma^2(\beta_d + \rho\gamma_d) + \gamma_d \hat{q}^d_{d'} \frac{\check{a}_{d'}}{8\beta_d} + \gamma_{d'} \hat{q}^{d'}_d \frac{3\check{a}_d}{8\beta_d}. \end{aligned} \tag{A10}$$

Consider now γ_d. We have that

$$\begin{aligned} \frac{d\hat{\pi}}{d\gamma_d} &= -q_{d'}\check{a}_{d'} + (1 - \beta_d - \gamma_{d'})\, q_d \frac{\partial \check{a}_d}{\partial \gamma_d} + (1 - \beta_{d'} - \gamma_d)\, q_{d'} \frac{\partial \check{a}_{d'}}{\partial \gamma_d} \\ &\quad + \frac{d\hat{u}_d(\check{a}_d, \hat{q}^d(\check{a}_d, w_d))}{d\gamma_d} + \frac{d\hat{u}_{d'}(\check{a}_{d'}, \hat{q}^{d'}(\check{a}_{d'}, w_{d'}))}{d\gamma_d}. \end{aligned}$$

Because $\frac{\partial \hat{u}_d}{\partial \gamma_d} = \check{a}_{d'} \hat{q}^d_{d'}$, $\frac{\partial \hat{u}_d}{\partial \check{a}_{d'}} = \gamma_d \hat{q}^d_{d'}$, and $\frac{\partial \check{a}_{d'}}{\partial \gamma_d} = \frac{\check{a}_{d'}}{8\gamma_d}$, by applying the envelope theorem on $\hat{u}_d(\check{a}_d, \hat{q}^d)$, we obtain that

$$\frac{d\hat{u}_d(\check{a}_d, \hat{q}^d(a_d, w_d))}{d\gamma_d} = a_{d'} \hat{q}^d_{d'} - r\sigma^2(\gamma_d + \rho\beta_d) + \gamma_d \hat{q}^d_{d'} \frac{\check{a}_{d'}}{8\gamma_d}. \tag{A11}$$

Similarly, because $\frac{\partial \hat{u}_{d'}}{\partial \gamma_d} = 0$, $\frac{\partial \hat{u}_{d'}}{\partial \check{a}_d} = \gamma_{d'} \hat{q}_d^{d'}$, and $\frac{\partial \check{a}_d}{\partial \gamma_d} = \frac{3\check{a}_d}{8\gamma_d}$, from (A11) and (A6) we obtain

$$\begin{aligned} \frac{d\hat{\pi}}{d\gamma_d} &= -\check{a}_{d'}\left(q_{d'} - \hat{q}_{d'}^d\right) + \left(1 - \beta_d - \gamma_{d'}\right) q_d \frac{3\check{a}_d}{8\gamma_d} + \left(1 - \beta_{d'} - \gamma_d\right) q_{d'} \frac{\check{a}_{d'}}{8\gamma_d} \\ &\quad - r\sigma^2\left(\gamma_d + \rho\beta_d\right) + \gamma_d \hat{q}_{d'}^d \frac{\check{a}_{d'}}{8\gamma_d} + \gamma_{d'} \hat{q}_d^{d'} \frac{3\check{a}_d}{8\gamma_d}. \end{aligned} \tag{A12}$$

Thus, from (A10) and (A12), we obtain the first-order conditions

$$\begin{aligned} \frac{d\hat{\pi}}{d\beta_d} &= -\check{a}_d\left(q_d - \hat{q}_d^d\right) - r\sigma^2\left(\beta_d + \rho\gamma_d\right) + \frac{\Delta_d}{\beta_d} = 0, \\ \frac{d\hat{\pi}}{d\gamma_d} &= -\check{a}_{d'}\left(q_{d'} - \hat{q}_{d'}^d\right) - r\sigma^2\left(\gamma_d + \rho\beta_d\right) + \frac{\Delta_d}{\gamma_d} = 0, \end{aligned}$$

where $\Delta_d \equiv \left(1 - \beta_d - \gamma_{d'}\right) q_d \frac{3\check{a}_d}{8} + \left(1 - \beta_{d'} - \gamma_d\right) q_{d'} \frac{\check{a}_{d'}}{8} + \gamma_d \hat{q}_{d'}^d \frac{\check{a}_{d'}}{8} + \gamma_{d'} \hat{q}_d^{d'} \frac{3\check{a}_d}{8}$, giving

$$\beta_d \check{a}_d\left(q_d - \hat{q}_d^d\right) + r\sigma^2\left(\beta_d^2 + \rho\gamma_d\beta_d\right) = \gamma_d \check{a}_{d'}\left(q_{d'} - \hat{q}_{d'}^d\right) + r\sigma^2\left(\gamma_d^2 + \rho\beta_d\gamma_d\right).$$

By Lemma 2, we have that $\beta_d \check{a}_d \hat{q}_d^d = \gamma_d \check{a}_{d'} \hat{q}_{d'}^d$, which implies that

$$\beta_d \check{a}_d q_d + r\sigma^2 \beta_d^2 = \gamma_d \check{a}_{d'} q_{d'} + r\sigma^2 \gamma_d^2$$

We will guess and verify that, due to the symmetry condition (S), it is optimal to implement symmetric effort, $\check{a}_d = \check{a}_{d'} = \check{a}$, and that $q_d = q$, $\kappa_d = \kappa$, and $Z_d = Z$. Define $f(x) \equiv x\check{a}q + r\sigma^2 x^2$. Note $f'(x) = \check{a}q + 2r\sigma^2 x > 0$ for $x > 0$, so that f is monotonic over positive numbers and $f(\gamma_d) = f(\beta_d)$ if and only if $\gamma_d = \beta_d$. Thus, $\hat{q}_d^d = \hat{q}_{d'}^d = e^{-\frac{\kappa}{2}} q$ and $\check{a}_d = e^{-\frac{\kappa}{2}} Z \beta_d^{\frac{3}{4}} \beta_{d'}^{\frac{1}{4}} q$. In order to optimally implement the same effort, it must be that $\beta_d = \beta_{d'}$, so $\check{a} = e^{-\frac{\kappa}{2}} Z\beta q$. Thus, we obtain the first-order condition

$$\frac{d\hat{\pi}}{d\beta_d} = -Z\beta_d \hat{q}_d^d\left(q - \hat{q}_d^d\right) + \left(1 - 2\beta_d\right) q\hat{q}_d^d \frac{Z}{2} - r\sigma^2 \beta_d (1 + \rho) + \frac{Z\beta_d\left(\hat{q}_d^d\right)^2}{2} = 0.$$

Therefore

$$\beta_d^2 \equiv \frac{1}{1 + 3\left(1 - \hat{q}_d^d/q\right) + 2r\sigma^2\left(1 - |\rho|\right)/(Zq\hat{q}_d^d)}.$$

After substitution, this gives HQ payoff

$$\hat{\pi}^2 \equiv \frac{Z^2 e^{-\kappa} q^4}{Ze^{-\frac{\kappa}{2}} q^2\left(4 - 3e^{-\frac{\kappa}{2}}\right) + 2r\sigma^2 (1 + \rho)}.$$

Because β_d is the same for both divisions, this verifies that a is symmetric. Because

HQ objective $\hat{\pi}$ is strictly concave on this region, there is only one solution on this region, which implies that the symmetric solution is the unique solution.

Case (C): Consider $H_d \in (-e^{\kappa}, -e^{-\kappa})$ with $\beta_d > 0 > \gamma_d$. Following the same process as in case (B) above, we have

$$\begin{aligned}\frac{d\hat{\pi}}{d\beta_d} &= -q_d\check{a}_d + (1-\beta_d-\gamma_{d'})\, q_d\frac{\partial \check{a}_d}{\partial \beta_d} + (1-\beta_{d'}-\gamma_d)\, q_{d'}\frac{\partial \check{a}_{d'}}{\partial \beta_d} \\ &\quad + \frac{d\hat{u}_d(\check{a}_d, \hat{q}^d(\check{a}_d, w_d))}{d\beta_d} + \frac{d\hat{u}_{d'}(\check{a}_{d'}, \hat{q}^{d'}(\check{a}_{d'}, w_{d'}))}{d\beta_d}.\end{aligned}$$

Because in this region $\frac{\partial \check{a}_d}{\partial \beta_d} = \frac{3\check{a}_d}{8\beta_d}$ and $\frac{\partial \check{a}_{d'}}{\partial \beta_d} = \frac{\check{a}_{d'}}{8\beta_d}$, from (A9) and (A3) we obtain that

$$\begin{aligned}\frac{d\hat{\pi}}{d\beta_d} &= -a_d\left(q_d - \hat{q}^d_d\right) + (1-\beta_d-\gamma_{d'})\, q_d\frac{3\check{a}_d}{8\beta_d} + (1-\beta_{d'}-\gamma_d)\, q_{d'}\frac{\check{a}_{d'}}{8\beta_d} \\ &\quad - r\sigma^2(\beta_d+\rho\gamma_d) + \gamma_d\hat{q}^d_{d'}\frac{\check{a}_{d'}}{8\beta_d} + \gamma_{d'}\hat{q}^{d'}_d\frac{3\check{a}_d}{8\beta_d}.\end{aligned} \tag{A13}$$

Consider now γ_d. We have that

$$\begin{aligned}\frac{d\hat{\pi}}{d\gamma_d} &= -q_{d'}\check{a}_{d'} + (1-\beta_d-\gamma_{d'})\, q_d\frac{\partial \check{a}_d}{\partial \gamma_d} + (1-\beta_{d'}-\gamma_d)\, q_{d'}\frac{\partial \check{a}_{d'}}{\partial \gamma_d} \\ &\quad + \frac{d\hat{u}_d(\check{a}_d, \hat{q}^d(\check{a}_d, w_d))}{d\gamma_d} + \frac{d\hat{u}_{d'}(\check{a}_{d'}, \hat{q}^{d'}(\check{a}_{d'}, w_{d'}))}{d\gamma_d}.\end{aligned}$$

Because $\frac{\partial \check{a}_d}{\partial \gamma_d} = \frac{3\check{a}_d}{8\gamma_d}$, $\frac{\partial \check{a}_{d'}}{\partial \gamma_d} = \frac{\check{a}_{d'}}{8\gamma_d}$ and $\frac{\partial \hat{u}_d}{\partial \check{a}_{d'}} = \gamma_d\hat{q}^d_{d'}$, by applying the envelope theorem on $\hat{u}_{d'}(\check{a}_{d'}, \hat{q}^{d'})$, we obtain that

$$\frac{d\hat{u}_d(\check{a}_d, \hat{q}^d(a_d, w_d))}{d\gamma_d} = a_{d'}\hat{q}^d_{d'} - r\sigma^2(\gamma_d+\rho\beta_d) + \hat{q}^d_{d'}\frac{\check{a}_{d'}}{8}. \tag{A14}$$

Similarly, because $\frac{\partial \hat{u}_{d'}}{\partial \gamma_d} = 0$, $\frac{\partial \hat{u}_{d'}}{\partial \check{a}_d} = \gamma_{d'}\hat{q}^{d'}_d$, and $\frac{\partial \check{a}_d}{\partial \gamma_d} = \frac{3\check{a}_d}{8\gamma_d}$, by applying the envelope theorem on $\hat{u}_{d'}(\check{a}_{d'}, \hat{q}^{d'})$, we obtain that

$$\frac{d\hat{u}_{d'}(\check{a}_{d'}, \hat{q}^{d'}(\check{a}_{d'}, w_{d'}))}{d\gamma_d} = \gamma_{d'}\hat{q}^{d'}_d\frac{3\check{a}_d}{8\gamma_d}. \tag{A15}$$

Together (A14) and (A15) give that

$$\begin{aligned}\frac{d\hat{\pi}}{d\gamma_d} &= -\check{a}_{d'}\left(q_{d'} - \hat{q}^d_{d'}\right) + (1-\beta_d-\gamma_{d'})\, q_d\frac{3\check{a}_d}{8\gamma_d} + (1-\beta_{d'}-\gamma_d)\, q_{d'}\frac{\check{a}_{d'}}{8\gamma_d} \\ &\quad - r\sigma^2(\gamma_d+\rho\beta_d) + \hat{q}^d_{d'}\frac{\check{a}_{d'}}{8} + \gamma_{d'}\hat{q}^{d'}_d\frac{3\check{a}_d}{8\gamma_d}.\end{aligned} \tag{A16}$$

Thus, from (A13) and (A16), we obtain the first-order conditions

$$\begin{aligned}\frac{d\hat{\pi}}{d\beta_d} &= -\check{a}_d\left(q_d-\hat{q}_d^d\right)-r\sigma^2\left(\beta_d+\rho\gamma_d\right)+\frac{\Delta_d}{\beta_d}=0,\\ \frac{d\hat{\pi}}{d\gamma_d} &= -\check{a}_{d'}\left(q_{d'}-\hat{q}_{d'}^d\right)-r\sigma^2\left(\gamma_d+\rho\beta_d\right)+\frac{\Delta_d}{\gamma_d}=0,\end{aligned}$$

where $\Delta_d \equiv \left(1-\beta_d-\gamma_{d'}\right)q_d\frac{3\check{a}_d}{8}+\left(1-\beta_{d'}-\gamma_d\right)q_{d'}\frac{\check{a}_{d'}}{8}+\gamma_d\hat{q}_{d'}^d\frac{\check{a}_{d'}}{8}+\gamma_{d'}\hat{q}_d^{d'}\frac{3\check{a}_d}{8}$, giving

$$\beta_d\check{a}_d\left(q_d-\hat{q}_d^d\right)+r\sigma^2\left(\beta_d^2+\rho\gamma_d\beta_d\right)=\gamma_d\check{a}_{d'}\left(q_{d'}-\hat{q}_{d'}^d\right)+r\sigma^2\left(\gamma_d^2+\rho\beta_d\gamma_d\right). \quad \text{(A17)}$$

Again, in this region, $\hat{q}_d^d=\left[e^{-\kappa_d}\left|H_d\right|\right]^{\frac{1}{2}}q_d$, and $\hat{q}_{d'}^d=\left(2-\left[e^{-\kappa_d}\left|H_d\right|^{-1}\right]^{\frac{1}{2}}\right)q_{d'}$, where $H_d=\frac{\gamma_d a_{d'}q_{d'}}{\beta_d a_d q_d}$. Thus,

$$\gamma_d\check{a}_{d'}\left(q_{d'}-\hat{q}_{d'}^d\right)=\gamma_d\check{a}_{d'}q_{d'}\left(e^{-\frac{\kappa_d}{2}}\left|H_d\right|^{-\frac{1}{2}}-1\right)=-\gamma_d\check{a}_{d'}q_{d'}-e^{-\frac{\kappa_d}{2}}\left(\beta_d a_d q_d\left|\gamma_d\right|a_{d'}q_{d'}\right)^{\frac{1}{2}}.$$

Similarly,

$$\beta_d\check{a}_d\hat{q}_d^d=e^{-\frac{\kappa_d}{2}}\left(\beta_d\check{a}_d q_d\left|\gamma_d\right|\check{a}_{d'}q_{d'}\right)^{\frac{1}{2}}$$

Therefore, after substitution, we obtain that (A17) becomes

$$\beta_d\check{a}_d q_d+r\sigma^2\beta_d^2=\left|\gamma_d\right|\check{a}_{d'}q_{d'}+r\sigma^2\gamma_d^2.$$

We guess again that HQ optimally implement the same effort from both divisions, $\check{a}_d=\check{a}_{d'}$, which implies that $f\left(\left|\gamma_d\right|\right)=f\left(\beta_d\right)$, where again $f\left(x\right)\equiv x\check{a}q+r\sigma^2x^2$. This implies that $\left|\gamma_d\right|=\beta_d$, or equivalently, that $\gamma_d=-\beta_d$, so that $H_d=-1$. Thus, $\hat{q}_d^d=e^{-\frac{\kappa}{2}}q$, and $\hat{q}_{d'}^d=\left(2-e^{-\frac{\kappa}{2}}\right)q$. To be consistent with this guess, it must be that $\beta_{d'}=\beta_d$, so that $\check{a}_d=\check{a}_{d'}=e^{-\frac{\kappa}{2}}Z\beta_d q$. Substituting in $\hat{\pi}$ and differentiating we obtain

$$\frac{d\hat{\pi}}{d\beta_d}=-Z\beta_d\hat{q}_d^d\left(q_d-\hat{q}_d^d\right)-r\sigma^2\beta\left(1+\rho\right)+\frac{1}{2}\left(1-2\beta_d\right)Zq\hat{q}_d^d+\frac{1}{2}\beta_d Z\left(\hat{q}_d^d\right)^2$$

$$\beta_d^3\equiv\frac{1}{1+3(1-3\hat{q}_d^d/q)+2r\sigma^2\left(1-\rho\right)/(Zq\hat{q}_d^d)}.$$

After substitution, this gives HQ payoff

$$\hat{\pi}^3\equiv\frac{Z^2e^{-\kappa}q^4}{Ze^{-\frac{\kappa}{2}}q^2\left(4-3e^{-\frac{\kappa}{2}}\right)+2r\sigma^2\left(1-\rho\right)},$$

which verifies the guess that HQ optimally implements symmetric effort. Comparing $\hat{\pi}^2$ and $\hat{\pi}^3$, observe that they differ only for the final term in the denominator. Thus,

$\hat{\pi}^3 \gtreqless \hat{\pi}^2$ as $\rho \gtreqless 0$, and

$$\max\left\{\hat{\pi}^2, \hat{\pi}^3\right\} = \frac{Z^2 e^{-\kappa} q^4}{Z e^{-\frac{\kappa}{2}} q^2 \left(4 - 3e^{-\frac{\kappa}{2}}\right) + 2r\sigma^2 \left(1 - |\rho|\right)}.$$

Case (D): If $\gamma_d > e^{\kappa}\beta_d$, we have that $\hat{q}_d^d = q_d$ and $\hat{q}_{d'}^d = e^{-\kappa} q_{d'}$, so

$$\frac{\partial \hat{\pi}}{\partial \gamma_d} = -a_{d'} q_{d'} \left(1 - e^{-\kappa}\right) - r\sigma^2 \left(\rho\beta_d + \gamma_d\right) < 0,$$

and setting $\gamma_d > e^{\kappa}\beta_d$ is not optimal. Similarly, if $\gamma_d < -e^{\kappa}\beta_d$, we have that $\hat{q}_d^d = q_d$ and $\hat{q}_{d'}^d = \left(2 - e^{-\kappa}\right) q$

$$\frac{\partial \hat{\pi}}{\partial \gamma_d} = a_{d'} q_{d'} \left(1 - e^{-\kappa}\right) + r\sigma^2 \left(|\gamma_d| - \rho\beta_d\right) > 0$$

and setting $\gamma_d < -e^{\kappa}\beta_d$ is not optimal. Thus, under symmetry, $|H_d| \leq e^{\kappa}$.

The second and final step is to compare $\max\left\{\hat{\pi}^2, \hat{\pi}^3\right\}$ and $\hat{\pi}^1$. Let

$$f(\kappa) \equiv 2\left(1 - e^{-\frac{\kappa}{2}}\right)^2 Z q^2 + r\sigma^2 \left(1 - |\rho|\right) \left[e^{\kappa}\left(1 + |\rho|\right) - 2\right],$$

so that $\max\left\{\hat{\pi}^2, \hat{\pi}^3\right\} > \hat{\pi}^1$ if and only if $f > 0$. Note $f(0) = -r\sigma^2 \left(1 - |\rho|\right)^2 < 0$,

$$f'(\kappa) = 2\left(1 - e^{-\frac{\kappa}{2}}\right) e^{-\frac{\kappa}{2}} Z q^2 + r\sigma^2 e^{\kappa}\left(1 - \rho^2\right) > 0$$

and $\lim_{\kappa \to \infty} f(\kappa) = +\infty$, which implies there is a unique $\bar{\kappa}$ such that $\max\left\{\hat{\pi}^2, \hat{\pi}^3\right\} > \hat{\pi}^1$ if and only if $\kappa > \bar{\kappa}$. Thus, for $\kappa \leq \bar{\kappa}$ the optimal contract is in Case (A), with $\beta_d = \beta_d^1$ and $\gamma_d = -\rho\beta_d$, leading to (37), and for $\kappa > \bar{\kappa}$ the optimal contract is in Case (B) for $\rho < 0$, with $\beta_d = \beta_d^2$ and $|\gamma_d| = \beta_d$, or in Case (C) for $\rho > 0$, with $\beta_d = \beta_d^3$ and $|\gamma_d| = \beta_d$, leading to (38).

Finally, note that the first term of f, $2\left(1 - e^{-\frac{\kappa}{2}}\right)^2 Z q^2$, is strictly positive. Because $f(\bar{\kappa}) = 0$, it must be that $r\sigma^2 \left(1 - |\rho|\right) \left[e^{\bar{\kappa}}\left(1 + |\rho|\right) - 2\right] < 0$. This implies that $\frac{\partial f}{\partial r} = \sigma^2 \left(1 - |\rho|\right) \left[e^{\kappa}\left(1 + |\rho|\right) - 2\right] < 0$ in a neighborhood of $\bar{\kappa}$. By the implicit function theorem, we obtain that $\frac{d\bar{\kappa}}{dr} = -\frac{\frac{\partial f}{\partial r}}{f'(\kappa)} > 0$, and $\bar{\kappa}$ is increasing in r. Finally, for $\rho \neq 0$, define $\kappa_\rho \equiv -\ln\left(|\rho|\right)$ and note that

$$f(\kappa_\rho) = 2\left(1 - \sqrt{|\rho|}\right)^2 Z q^2 + r\sigma^2 \frac{\left(1 - |\rho|\right)^2}{|\rho|} > 0$$

which implies that $\bar{\kappa} < \kappa_\rho$. ■

Proof of Corollary 1. In the proof of Theorem 4, we showed that $\beta_d a_d q_d + r\sigma^2 \beta_d^2 =$

$|\gamma_d|\, a_{d'} q_{d'} + r\sigma^2\gamma_d^2$. Define $f(\beta_d, |\gamma_d|) = \beta_d a_d q_d + r\sigma^2\beta_d^2 - |\gamma_d|\, a_{d'} q_{d'} - r\sigma^2\gamma_d^2$, and note that in an optimal contract, $f = 0$. Note also that $f(\beta_d, \beta_d) = \beta_d (a_d q_d - a_{d'} q_{d'}) > 0$ and that

$$f\left(\beta_d, \frac{a_d q_d}{a_{d'} q_{d'}} \beta_d\right) = r\sigma^2\beta_d^2 \left(1 - \frac{a_d^2 q_d^2}{a_{d'}^2 q_{d'}^2}\right) < 0.$$

Thus, $f(\beta_d, |\gamma_d|) = 0$ implies $|\gamma_d| \in (\beta_d, \frac{a_d q_d}{a_{d'} q_{d'}} \beta_d)$ for $\frac{a_d q_d}{a_{d'} q_{d'}} > 1$, and $|\gamma_d| \in (\frac{a_d q_d}{a_{d'} q_{d'}} \beta_d, \beta_d,)$ for $\frac{a_d q_d}{a_{d'} q_{d'}} < 1$. ■

Proof of Lemma 4. Proof is isomorphic to proof for Lemma 2 and is omitted. ■

Proof of Theorem 5. We guess and verify that headquarters have positive exposure to both divisions, $\phi_d = 1 - \beta_d - \gamma_d > 0$, and that beliefs are as in case (ii) of Lemma 4, $H_d^{HQ} \in (e^{-\kappa_{HQ}}, e^{\kappa_{HQ}})$. Because (33) binds and $r = 0$, HQ payoff $\hat{\pi}$ is equal to

$$\sum_{\substack{d,d' \in \{A,B\} \\ d \neq d'}} \left[a_d q_d - (1 - \beta_d - \gamma_{d'})\, a_d \left(q_d - \hat{q}_d^{HQ}\right) - \beta_d a_d \left(q_d - \hat{q}_d^d\right) - \gamma_d a_{d'} \left(q_{d'} - \hat{q}_{d'}^d\right)\right],$$

where $\hat{q}^d = (\hat{q}_d^d, \hat{q}_{d'}^d)$ are division manager beliefs from Lemma 2, a_d are the Nash equilibrium effort levels from Lemma 3, and $\hat{q}^{HQ} = (\hat{q}_d^{HQ}, \hat{q}_{d'}^{HQ})$ are HQ beliefs from Lemma 4. The proof is in two steps and is similar to the proof of Theorem 3. First, we show that $\gamma_d < 0$ is suboptimal; then we find the optimal contract for $\gamma_d \geq 0$.

Similar to Theorem 3, switching from γ_d to $|\gamma_d|$ does not affect $\hat{q}_d^d$, and thus does not affect a_d and $\beta_d a_d \left(q_d - \hat{q}_{d'}^d\right)$. Letting again $\hat{q}_{d'}^{d+}$ be the belief held by a division manager when receiving $|\gamma_d|$ instead of $\gamma_d < 0$, we have that $\gamma_d a_{d'} \left(q_{d'} - \hat{q}_{d'}^d\right) = |\gamma_d|\, a_{d'} \left(q_{d'} - \hat{q}_{d'}^{d+}\right)$ for all $\gamma_d < 0$. This implies that

$$(1 - \beta_{d'} - |\gamma_d|)\, a_{d'} \left(q_{d'} - \hat{q}_{d'}^{HQ}\right) < (1 - \beta_{d'} - \gamma_d)\, a_{d'} \left(q_{d'} - \hat{q}_{d'}^{HQ}\right)$$

for $\gamma_d < 0$ because $\hat{q}_{d'}^{HQ} < q_{d'}$, and thus that setting $\gamma_d < 0$ is dominated by offering its absolute value, $|\gamma_d|$.

Because HQ strictly prefers offering $|\gamma_d| > 0$ to all $\gamma_d < 0$, it is sufficient to consider $\gamma_d \geq 0$. If HQ sets $\gamma_d > e^{\kappa_d} \frac{\beta_d a_d q_d}{a_{d'} q_{d'}}$, division manager beliefs are in case (i) of Lemma 2, with $\hat{q}_d^d = q_d$ and $\hat{q}_{d'}^d = e^{-\kappa} q_{d'}$, giving $a_d = \beta_d Z_d q_d$. Thus, $\frac{\partial \hat{\pi}}{\partial \gamma_d} = -a_{d'} \left(\hat{q}_{d'}^{HQ} - \hat{q}_{d'}^d\right) < 0$ because $\hat{q}_{d'}^{HQ} \in (e^{-\kappa_{HQ}} q_d, q_d)$ and $\kappa_{HQ} < \kappa$, so setting $\gamma_d > e^{\kappa_d} \frac{\beta_d a_d q_d}{a_{d'} q_{d'}}$ is not optimal. Similarly, if $0 < \gamma_d < e^{-\kappa_d} \frac{\beta_d a_d q_d}{a_{d'} q_{d'}}$, division managers beliefs are in case (iii) of Lemma 2, with $\hat{q}_d^d = e^{-\kappa_d} q_d$ and $\hat{q}_{d'}^d = q_d$, giving $a_d = \beta_d Z_d e^{-\kappa_d} q_d$. In addition, $\frac{\partial \hat{\pi}}{\partial \gamma_d} =$

$a_{d'}\left(\hat{q}^{d}_{d'}-\hat{q}^{HQ}_{d'}\right)>0$ because $\hat{q}^{HQ}_{d'}\in(e^{-\kappa_{HQ}}q_d,q_d)$, so setting $\gamma_d<e^{-\kappa_d}\frac{\beta_d a_d q_d}{a_{d'}q_{d'}}$ is not optimal. This implies that HQ set $e^{-\kappa_d}\frac{\beta_d a_d q_d}{a_{d'}q_{d'}}\le\gamma_d\le e^{\kappa_d}\frac{\beta_d a_d q_d}{a_{d'}q_{d'}}$ and induce beliefs that are in case (ii) of Lemma 2, with $H_d\in(e^{-\kappa},e^{\kappa})$.

Similar to the proof of Theorem 3, we can express HQ's objective as

$$\hat{\pi}=\phi_A\check{a}_A\hat{q}^{HQ}_A+\phi_B\check{a}_B\hat{q}^{HQ}_B+\left(\hat{u}_A(a_A,\hat{q}^A(a_A,w_A))-s_A\right)+\left(\hat{u}_B(a_B,\hat{q}^B(a_B,w_B))-s_B\right),$$

where $\phi_d=1-\beta_d-\gamma_{d'}$, $\hat{u}_d(\check{a}_d,\hat{q}^d)=\min_{\hat{q}^d\in C^d}\hat{u}_d$, with $\hat{u}_d=s_d+\beta_d\check{a}_d\hat{q}^d_d+\gamma_d\check{a}_{d'}\hat{q}^d_{d'}-\frac{\check{a}_d^2}{2Z_d}=0$, and $\check{a}_d$ is the Nash equilibrium of division managers given by (A1) in the proof of Lemma 3. Consider first

$$\begin{aligned}\frac{d\hat{\pi}}{d\beta_d} &= -\hat{q}^{HQ}_d\check{a}_d+\phi_d\check{a}_d\frac{\partial\hat{q}^{HQ}_d}{\partial\beta_d}+\phi_{d'}\check{a}_{d'}\frac{\partial\hat{q}^{HQ}_{d'}}{\partial\beta_d}+\phi_d\hat{q}^{HQ}_d\frac{\partial\check{a}_d}{\partial\beta_d}+\phi_{d'}\hat{q}^{HQ}_{d'}\frac{\partial\check{a}_{d'}}{\partial\beta_d}\\ &\quad+\frac{d\hat{u}_d(\check{a}_d,\hat{q}^d(\check{a}_d,w_d))}{d\beta_d}+\frac{d\hat{u}_{d'}(\check{a}_{d'},\hat{q}^{d'}(\check{a}_{d'},w_{d'}))}{d\beta_d}.\end{aligned}$$

Because $\hat{q}^{HQ}$ solves (40), from the envelope theorem $\phi_d\check{a}_d\frac{\partial\hat{q}^{HQ}_d}{\partial\beta_d}+\phi_{d'}\check{a}_{d'}\frac{\partial\hat{q}^{HQ}_{d'}}{\partial\beta_d}=0$, which, together with (A2) and (A3) from the proof of Theorem 3, gives

$$\frac{d\hat{\pi}}{d\beta_d}=-\check{a}_d\left(\hat{q}^{HQ}_d-\hat{q}^d_d\right)+\phi_d\hat{q}^{HQ}_d\frac{3a_d}{8\beta_d}+\phi_{d'}\hat{q}^{HQ}_{d'}\frac{a_{d'}}{8\beta_d}+\gamma_d\hat{q}^d_{d'}\frac{\check{a}_{d'}}{8\beta_d}+\gamma_{d'}\hat{q}^{d'}_d\frac{3\check{a}_d}{8\beta_d}. \quad \text{(A18)}$$

Consider now γ_d. Applying again the envelope theorem on $\hat{\pi}\left(\hat{q}^{HQ}\right)$, we obtain

$$\begin{aligned}\frac{d\hat{\pi}}{d\gamma_d} &= -\hat{q}^{HQ}_{d'}\check{a}_{d'}+\phi_d\hat{q}^{HQ}_d\frac{\partial\check{a}_d}{\partial\gamma_d}+\phi_{d'}\hat{q}^{HQ}_{d'}\frac{\partial\check{a}_{d'}}{\partial\gamma_d}\\ &\quad+\frac{d\hat{u}_d(\check{a}_d,\hat{q}^d(\check{a}_d,w_d))}{d\gamma_d}+\frac{d\hat{u}_{d'}(\check{a}_{d'},\hat{q}^{d'}(\check{a}_{d'},w_{d'}))}{d\gamma_d}.\end{aligned}$$

Substituting (A5) and (A6) from the proof of Theorem 3 gives

$$\frac{d\hat{\pi}}{d\gamma_d}=-\check{a}_{d'}\left(\hat{q}^{HQ}_{d'}-\hat{q}^d_{d'}\right)+\phi_d\hat{q}^{HQ}_d\frac{3\check{a}_d}{8\gamma_d}+\phi_{d'}\hat{q}^{HQ}_{d'}\frac{\check{a}_{d'}}{8\gamma_d}+\gamma_d\hat{q}^d_{d'}\frac{\check{a}_{d'}}{8\gamma_d}+\gamma_{d'}\hat{q}^{d'}_d\frac{3\check{a}_d}{8\gamma_d}. \quad \text{(A19)}$$

Thus, from (A18) and (A19) we obtain the first-order conditions

$$\frac{d\hat{\pi}}{d\beta_d}=-\check{a}_d\left(\hat{q}^{HQ}_d-\hat{q}^d_d\right)+\frac{\Delta_d}{\beta_d}=0,\quad \frac{d\hat{\pi}}{d\gamma_d}=-\check{a}_{d'}\left(\hat{q}^{HQ}_{d'}-\hat{q}^d_{d'}\right)+\frac{\Delta_d}{\gamma_d}=0,$$

where $\Delta_d\equiv\phi_d\hat{q}^{HQ}_d\frac{3\check{a}_d}{8}+\phi_{d'}\hat{q}^{HQ}_{d'}\frac{\check{a}_{d'}}{8}+\gamma_d\hat{q}^d_{d'}\frac{\check{a}_{d'}}{8}+\gamma_{d'}\hat{q}^{d'}_d\frac{3\check{a}_d}{8}$, giving

$$\beta_d\check{a}_d\left(\hat{q}^{HQ}_d-\hat{q}^d_d\right)=\gamma_{d'}\check{a}_{d'}\left(\hat{q}^{HQ}_{d'}-\hat{q}^d_{d'}\right). \quad \text{(A20)}$$

Because, from Lemma 2, $\beta_d \check{a}_d \hat{q}_d^d = \gamma_d \check{a}_{d'} \hat{q}_{d'}^d$, we have that (A20) implies $\beta_d \check{a}_d \hat{q}_d^{HQ} = \gamma_d \check{a}_{d'} \hat{q}_{d'}^{HQ}$. Because $H_d^{HQ} \in (e^{-\kappa_{HQ}}, e^{\kappa_{HQ}})$, from Lemma 4, $\phi_d a_d \hat{q}_d^{HQ} = \phi_{d'} a_{d'} \hat{q}_{d'}^{HQ}$. Thus, $\frac{a_{d'} \hat{q}_{d'}^{HQ}}{a_d \hat{q}_d^{HQ}} = \frac{\beta_d}{\gamma_d} = \frac{\phi_d}{\phi_{d'}}$. Define m_d such that $\beta_d = m_d \phi_d$, so $\gamma_d = m_d \phi_{d'}$, which implies $\phi_d = 1 - \beta_d - \gamma_{d'} = \frac{1}{1+m_d+m_{d'}}$, and thus $\beta_d = \gamma_d = \frac{m_d}{1+m_d+m_{d'}}$. Substituting in $\gamma_d = \beta_d$ into $\check{a}$ from Lemma 3, we have $\check{a}_d = (Z_d^3 Z_{d'})^{\frac{1}{4}} e^{-\frac{\kappa}{2}} (\beta_d^3 \beta_{d'})^{\frac{1}{4}} (q_d q_{d'})^{\frac{1}{2}}$. Substituting into HQ objective, we obtain

$$\hat{\pi} = (Z_A Z_B)^{\frac{1}{2}} q_A q_B (\beta_A \beta_B)^{\frac{1}{2}} \left[2e^{-\frac{\kappa_{HQ}}{2}} e^{-\frac{\kappa}{2}} (1 - \beta_A - \beta_B) + \frac{3}{2} e^{-\kappa} (\beta_A + \beta_B) \right].$$

Differentiating, we obtain the first-order condition

$$\frac{d\hat{\pi}}{d\beta_d} = (Z_A Z_B)^{\frac{1}{2}} q_d q_{d'} (\beta_d^{-1} \beta_{d'})^{\frac{1}{2}} \left[e^{-\frac{\kappa_{HQ}}{2}} (1 - 3\beta_d - \beta_{d'}) e^{-\frac{\kappa}{2}} + \frac{3}{4} e^{-\kappa} (3\beta_d + \beta_{d'}) \right] = 0,$$

giving

$$e^{\frac{1}{2}(\kappa - \kappa_{HQ})} + 3 \left(\frac{3}{4} - e^{\frac{1}{2}(\kappa - \kappa_{HQ})} \right) \beta_d + \left(\frac{3}{4} - e^{\frac{1}{2}(\kappa - \kappa_{HQ})} \right) \beta_{d'} = 0.$$

Because this holds for both divisions, after solving we obtain

$$\beta_A = \beta_B = \frac{1}{4 - 3e^{\frac{1}{2}(\kappa_{HQ} - \kappa)}} = \frac{1}{1 + 3(1 - \hat{q}_d^d / \hat{q}_d^{HQ})} = \gamma_d,$$

giving (45). Note $\beta < \frac{1}{2}$ because $\kappa_{HQ} < \kappa - 2 \ln \frac{3}{2}$ and $H_d^{HQ} = \eta_d \in (e^{-\kappa_{HQ}}, e^{\kappa_{HQ}})$. This implies that $\check{a}_d = \frac{(Z_d^3 Z_{d'})^{\frac{1}{4}} e^{-\frac{\kappa}{2}} (q_d q_{d'})^{\frac{1}{2}}}{4 - 3e^{\frac{1}{2}(\kappa_{HQ} - \kappa)}}$, and thus that $\hat{q}_d^d = e^{-\frac{\kappa}{2}} q_d \eta_d^{\frac{1}{2}}$ and $\hat{q}_d^{HQ} = e^{-\frac{\kappa_{HQ}}{2}} q_d \eta_d^{\frac{1}{2}}$. Similarly, (43) and (44) follow by direct substitution. ∎

Proof of Theorems 6-8. Because the participation constraint (33) binds, we can express HQ's payoff as

$$\hat{\pi} = \phi_A a_A \hat{q}_A^{HQ} + \phi_B a_B \hat{q}_B^{HQ} + \left(\hat{u}_A(a_A, \hat{q}^A(a_A, w_A)) - s_A \right) + \left(\hat{u}_B(a_B, \hat{q}^B(a_B, w_B)) - s_B \right),$$

where $\phi_d = 1 - \beta_d - \gamma_{d'}$ and $\hat{u}_d(a_d, \hat{q}^d(a_d, w_d)) = \min_{\hat{q}^d \in C^d} \hat{u}_d$, with

$$\hat{u}_d(a_d, \hat{q}^d(a_d, w_d)) = s_d + \beta_d a_d \hat{q}_d^d + \gamma_d a_{d'} \hat{q}_{d'}^d - \frac{r\sigma^2}{2} \left(\beta_d^2 + 2\rho \beta_d \gamma_d + \gamma_d^2 \right) - \frac{a_d^2}{2Z_d} = 0,$$

where $\hat{q}^d$ is from Lemma 2, a_d is from Lemma 3, and $\hat{q}^{HQ}$ is from Lemma 4. Different from Theorem 5, and similar to Theorem 4, because of division manager risk aversion, HQ objective function π admits again multiple strict local maxima. The proof proceeds again in two steps. First, we consider candidate optimal contracts that induce division managers to hold one of four possible configurations of beliefs (implied by Lemma 2) in the same four cases examined in the proof of Theorem 4, Cases (A) to (D). Second, we compare payoffs to HQ from optimal contracts in these regions and we determine the globally optimal contract. Note that optimal contracts falling in Case (A) and Case (B) correspond to Theorem 6, Case (C) corresponds to Theorem 7. Finally, the comparison of payoffs from Case (B) and Case (C) gives Theorem 8.

Case (A): If $|H_d| < e^{-\kappa_d}$, have $\hat{q}_d^d = e^{-\kappa_d} q_d$ and $\hat{q}_{d'}^d = q_{d'}$, which do not depend on γ_d. Similarly, by Lemma 3, $a_d = \beta_d Z_d e^{-\kappa_d} q_d$, which implies that both a_d and $a_{d'}$ do not depend on γ_d. Therefore,

$$\begin{aligned}\frac{d\hat{\pi}}{d\gamma_d} &= -\hat{q}_{d'}^{HQ} a_{d'} + \phi_d a_d \frac{\partial \hat{q}_d^{HQ}}{\partial \gamma_d} + \phi_{d'} a_{d'} \frac{\partial \hat{q}_{d'}^{HQ}}{\partial \gamma_d} + \phi_d \hat{q}_d^{HQ} \frac{\partial a_d}{\partial \gamma_d} + \phi_{d'} \hat{q}_{d'}^{HQ} \frac{\partial a_{d'}}{\partial \gamma_d} \\ &\quad + \frac{d\hat{u}_d(a_d, \hat{q}^d(a_d, w_d))}{d\gamma_d} + \frac{d\hat{u}_{d'}(a_{d'}, \hat{q}^{d'}(a_{d'}, w_{d'}))}{d\gamma_d},\end{aligned}$$

where, by the envelope theorem on $\hat{\pi}$, we have $\phi_d a_d \frac{\partial \hat{q}_d^{HQ}}{\partial \gamma_d} + \phi_{d'} a_{d'} \frac{\partial \hat{q}_{d'}^{HQ}}{\partial \gamma_d} = 0$. In addition, on this region, $\frac{\partial a_d}{\partial \gamma_d} = \frac{\partial a_{d'}}{\partial \gamma_d} = 0$, which implies that $\frac{d\hat{u}_d(a_d, \hat{q}^d(a_d, w_d))}{d\gamma_d} = \frac{\partial \hat{u}}{\partial \gamma_d} = a_{d'} \hat{q}_{d'}^d - r\sigma^2 (\rho\beta_d + \gamma_d)$ and $\frac{d\hat{u}_{d'}(a_{d'}, \hat{q}^{d'}(a_{d'}, w_{d'}))}{d\gamma_d} = \frac{\partial \hat{u}_{d'}}{\partial \gamma_d} = 0$. Thus,

$$\frac{\partial \hat{\pi}}{\partial \gamma_d} = a_{d'} \left(q_{d'} - \hat{q}_{d'}^{HQ} \right) - r\sigma^2 (\rho\beta_d + \gamma_d).$$

Because HQ has long exposure to the symmetric divisions, $\hat{q}_d^{HQ} = \hat{q}_{d'}^{HQ} = e^{-\frac{\kappa_{HQ}}{2}} q$. Thus, $\frac{\partial \hat{\pi}}{\partial \gamma_d} = 0$ if and only if $\gamma = -M\beta$, where $M \equiv \rho - \bar{\rho}$ and $\bar{\rho} \equiv \frac{Z\hat{q}_d^d}{r\sigma^2} \left(q_{d'} - \hat{q}_{d'}^{HQ} \right) = \frac{e^{-\kappa} Z q^2}{r\sigma^2} \left(1 - e^{-\frac{\kappa_{HQ}}{2}} \right)$. Following a similar approach, we obtain

$$\frac{d\hat{\pi}}{d\beta_d} = \hat{q}_d^d \hat{q}_d^{HQ} Z (1 - 2\beta_d) - M\beta_d \left(q_{d'} - \hat{q}_{d'}^{HQ} \right) \hat{q}_d^d Z + \beta_d Z \left(\hat{q}_d^d \right)^2 - r\sigma^2 \beta_d (1 - \rho M).$$

Note $1 - \rho M = 1 - \rho^2 + \rho\bar{\rho}$ and $1 - 2\rho M + M^2 = 1 - \rho^2 + \bar{\rho}^2$, so $1 - \rho M = 1 - 2\rho M + M^2 + \bar{\rho}(\rho - \bar{\rho})$. Also, $r\sigma^2 \beta_d \bar{\rho} (\rho - \bar{\rho}) = Z \left(q_{d'} - \hat{q}_{d'}^{HQ} \right) \hat{q}_d^d (\rho - \bar{\rho})$. Thus, we obtain the first-order condition

$$\begin{aligned}\frac{d\hat{\pi}}{d\beta_d} &= \hat{q}_d^d \hat{q}_d^{HQ} Z (1 - 2\beta_d) + \beta_d Z \left(\hat{q}_d^d \right)^2 \\ &\quad - 2M\beta_d \left(q_{d'} - \hat{q}_{d'}^{HQ} \right) \hat{q}_d^d Z - r\sigma^2 \beta_d \left(1 - 2\rho M + M^2 \right) = 0,\end{aligned}$$

which implies

$$\beta_d^4 \equiv \frac{1}{1+2(\rho-\bar{\rho})\left(\frac{\hat{q}_{d'}^d}{\hat{q}_{d'}^{HQ}}-1\right)+\left(1-\frac{\hat{q}_d^d}{\hat{q}_d^{HQ}}\right)+\frac{r\sigma^2(1-\rho^2+\bar{\rho}^2)}{Z\hat{q}_d^{HQ}\hat{q}_d^d}},$$

giving (46). After substitution, this gives HQ payoff

$$\hat{\pi}^4 \equiv \frac{e^{-\left(\kappa_{HQ}+2\kappa\right)}Z^2q^4}{\left(2M+2\left(1-M\right)e^{-\frac{\kappa_{HQ}}{2}}-e^{-\kappa}\right)e^{-\kappa}Zq^2+r\sigma^2\left(1-2\rho M+M^2\right)}.$$

Case (B): If $H_d \in (e^{-\kappa}, e^{\kappa})$, as in the proof of Theorem 5, applying the envelope theorem on $\hat{\pi}\left(\hat{q}^{HQ}\right)$, we have

$$\begin{aligned}\frac{d\hat{\pi}}{d\beta_d} &= -\hat{q}_d^{HQ}\check{a}_d+\left(1-\beta_d-\gamma_{d'}\right)\hat{q}_d^{HQ}\frac{\partial\check{a}_d}{\partial\beta_d}+\left(1-\beta_{d'}-\gamma_d\right)\hat{q}_{d'}^{HQ}\frac{\partial\check{a}_{d'}}{\partial\beta_d}\\ &\quad+\frac{d\hat{u}_d(\check{a}_d,\hat{q}^d(\check{a}_d,w_d))}{d\beta_d}+\frac{d\hat{u}_{d'}(\check{a}_{d'},\hat{q}^{d'}(\check{a}_{d'},w_{d'}))}{d\beta_d}.\end{aligned}$$

Because in this region $\frac{\partial\check{a}_d}{\partial\beta_d}=\frac{3\check{a}_d}{8\beta_d}$ and $\frac{\partial\check{a}_{d'}}{\partial\beta_d}=\frac{\check{a}_{d'}}{8\beta_d}$, from (A9) and (A3), we have

$$\begin{aligned}\frac{d\hat{\pi}}{d\beta_d} &= -a_d\left(\hat{q}_d^{HQ}-\hat{q}_d^d\right)+\left(1-\beta_d-\gamma_{d'}\right)\hat{q}_d^{HQ}\frac{3\check{a}_d}{8\beta_d}+\left(1-\beta_{d'}-\gamma_d\right)\hat{q}_{d'}^{HQ}\frac{\check{a}_{d'}}{8\beta_d}\\ &\quad-r\sigma^2\left(\beta_d+\rho\gamma_d\right)+\gamma_d\hat{q}_{d'}^d\frac{\check{a}_{d'}}{8\beta_d}+\gamma_{d'}\hat{q}_d^{d'}\frac{3\check{a}_d}{8\beta_d}.\end{aligned} \tag{A21}$$

Consider now γ_d. Applying again the envelope theorem on $\hat{\pi}\left(\hat{q}^{HQ}\right)$, we have

$$\begin{aligned}\frac{d\hat{\pi}}{d\gamma_d} &= -q_{d'}\check{a}_{d'}+\left(1-\beta_d-\gamma_{d'}\right)\hat{q}_d^{HQ}\frac{\partial\check{a}_d}{\partial\gamma_d}+\left(1-\beta_{d'}-\gamma_d\right)\hat{q}_{d'}^{HQ}\frac{\partial\check{a}_{d'}}{\partial\gamma_d}\\ &\quad+\frac{d\hat{u}_d(\check{a}_d,\hat{q}^d(\check{a}_d,w_d))}{d\gamma_d}+\frac{d\hat{u}_{d'}(\check{a}_{d'},\hat{q}^{d'}(\check{a}_{d'},w_{d'}))}{d\gamma_d}.\end{aligned}$$

Because in this region $\frac{\partial\check{a}_d}{\partial\gamma_d}=\frac{3\check{a}_d}{8\gamma_d}$ and $\frac{\partial\check{a}_{d'}}{\partial\gamma_d}=\frac{\check{a}_{d'}}{8\gamma_d}$, from (A11) and (A6), we have that

$$\begin{aligned}\frac{d\hat{\pi}}{d\gamma_d} &= -\check{a}_{d'}\left(q_{d'}-\hat{q}_{d'}^d\right)+\left(1-\beta_d-\gamma_{d'}\right)\hat{q}_d^{HQ}\frac{3\check{a}_d}{8\gamma_d}+\left(1-\beta_{d'}-\gamma_d\right)\hat{q}_{d'}^{HQ}\frac{\check{a}_{d'}}{8\gamma_d}\\ &\quad-r\sigma^2\left(\gamma_d+\rho\beta_d\right)+\gamma_d\hat{q}_{d'}^d\frac{\check{a}_{d'}}{8\gamma_d}+\gamma_{d'}\hat{q}_d^{d'}\frac{3\check{a}_d}{8\gamma_d}.\end{aligned} \tag{A22}$$

Thus, from (A21) and (A22) we obtain the first-order conditions

$$\begin{aligned}\frac{d\hat{\pi}}{d\beta_d} &= -a_d\left(\hat{q}_d^{HQ}-\hat{q}_d^d\right)-r\sigma^2\left(\beta_d+\rho\gamma_d\right)+\frac{\Delta_d}{\beta_d}=0\\ \frac{d\hat{\pi}}{d\gamma_d} &= -a_{d'}\left(\hat{q}_{d'}^{HQ}-\hat{q}_{d'}^d\right)-r\sigma^2\left(\rho\beta_d+\gamma_d\right)+\frac{\Delta_d}{\gamma_d}=0,\end{aligned}$$

where $\Delta_d=\phi_d\hat{q}_d^{HQ}\frac{3a_d}{8}+\phi_{d'}\hat{q}_{d'}^{HQ}\frac{a_{d'}}{8}+\gamma_d\hat{q}_{d'}^d\frac{a_{d'}}{8}+\gamma_{d'}\hat{q}_d^{d'}\frac{3a_d}{8}$, giving

$$\beta_d a_d\left(\hat{q}_d^{HQ}-\hat{q}_d^d\right)+r\sigma^2\left(\beta_d^2+\rho\gamma_d\beta_d\right)=\gamma_d a_{d'}\left(\hat{q}_{d'}^{HQ}-\hat{q}_{d'}^d\right)+r\sigma^2\left(\rho\gamma_d\beta_d+\gamma_d^2\right)$$

From Lemma 2, we have $\beta_d a_d\hat{q}_d^d=\gamma_d a_{d'}\hat{q}_{d'}^d$. Also, because $\phi_d>0$ and HQ has beliefs as in case (ii) of Lemma 4, with $\phi_d a_d\hat{q}_d^{HQ}=\phi_{d'}a_{d'}\hat{q}_{d'}^{HQ}$, we have

$$\beta_d a_d\hat{q}_d^{HQ}+r\sigma^2\beta_d^2=\gamma_d\frac{\phi_d}{\phi_{d'}}a_d\hat{q}_d^{HQ}+r\sigma^2\gamma_d^2. \tag{A23}$$

We now show that $\phi_A=\phi_B$. Suppose to the contrary that $\phi_A>\phi_B$. Because (A23) holds for both divisions, $\beta_A>\gamma_A$ but $\beta_B<\gamma_B$. This would imply, however, that $\phi_A=1-\beta_A-\gamma_B<1-\beta_B-\gamma_A=\phi_B$, which is a contradiction. Similarly, $\phi_A<\phi_B$ would also imply a contradiction. Thus, $\phi_A=\phi_B$. Further, this implies

$$\left(a_d\hat{q}_d^{HQ}+r\sigma^2\left(\beta_d+\gamma_d\right)\right)\left(\beta_d-\gamma_d\right)=0. \tag{A24}$$

Since the first term is strictly positive, $\beta_d=\gamma_d$. Further, because the divisions are symmetric, the first-order conditions are symmetric, which implies the existence of a symmetric solution, $\beta_A=\beta_B$. Because the problem is strictly concave on this region, this must be the unique solution. Thus, $a_A=a_B=e^{-\frac{\kappa}{2}}Z\beta q$. Also, $\hat{q}_d^{HQ}=\hat{q}_{d'}^{HQ}=e^{-\frac{\kappa_{HQ}}{2}}q$ and $\hat{q}_d^d=\hat{q}_{d'}^d=e^{-\frac{\kappa}{2}}q$, so $\Delta_d=(1-2\beta)\,e^{-\frac{\kappa_{HQ}}{2}}q\frac{e^{-\frac{\kappa}{2}}Z\beta q}{2}+\beta e^{-\frac{\kappa}{2}}q\frac{e^{-\frac{\kappa}{2}}Z\beta q}{2}$, which gives the first-order condition

$$\frac{d\hat{\pi}}{d\beta_d}=\frac{1}{2}Z\hat{q}_d^d\hat{q}_d^{HQ}-2\beta Z\hat{q}_d^d\hat{q}_d^{HQ}+\frac{3}{2}Z\beta\left(\hat{q}_d^d\right)^2-r\sigma^2\beta\left(1+\rho\right)=0.$$

and thus

$$\beta_d^5\equiv\frac{1}{1+3\left(1-\hat{q}_d^d/\hat{q}_d^{HQ}\right)+\frac{2r\sigma^2(1+\rho)}{Z\hat{q}_d^{HQ}\hat{q}_d^d}}=\hat{\beta},$$

giving (47). After substitution, this gives HQ payoff

$$\hat{\pi}^5\equiv\frac{Z^2q^4e^{-\left(\kappa_{HQ}+\kappa\right)}}{Zq^2\left(4e^{-\frac{\left(\kappa_{HQ}+\kappa\right)}{2}}-3e^{-\kappa}\right)+2r\sigma^2\left(1+\rho\right)}.$$

Theorem 5 showed that $\gamma_d > 0$ is optimal when $r = 0$. Similarly, $\gamma_d > 0$ when $\rho = 0$. Further, for $\rho < 0$, granting $\gamma_d < 0$ results in a larger risk premium, $\frac{r\sigma^2}{2}\left(\beta_d^2 + 2\rho\beta_d + \gamma_d^2\right)$, than setting $\gamma_d > 0$. Thus, Case (B) dominates Case (C) for all $\rho \leq 0$. To conclude the proof of Theorem 6, note that $\hat{\pi}^5 \geq \hat{\pi}^4$ if and only if $g_L \geq 0$, where

$$\begin{aligned} g_L \equiv & \left(2M + 2\left(1-M\right)e^{-\frac{\kappa_{HQ}}{2}} + 2e^{-\kappa} - 4e^{-\frac{(\kappa_{HQ}+\kappa)}{2}}\right)e^{-\kappa}Zq^2 \\ & + r\sigma^2\left(1 - 2\rho M + M^2 - 2e^{-\kappa}\left(1+\rho\right)\right). \end{aligned}$$

and note that $g_L|_{\kappa=\kappa_{HQ}=0} = -r\sigma^2\left(1+\rho\right)^2 < 0$, which implies that $\hat{\pi}^4 > \hat{\pi}^5$ for $\kappa = \kappa_{HQ} = 0$. Note also that $\frac{\partial g_L}{\partial M} = 2\left(1 - e^{-\frac{\kappa_{HQ}}{2}}\right)e^{-\kappa}Zq^2 + 2r\sigma^2\left(M-\rho\right) = 0$, because $M \equiv \rho - \bar{\rho}$ and $\bar{\rho} \equiv \frac{e^{-\kappa}Zq^2}{r\sigma^2}\left(1 - e^{-\frac{\kappa_{HQ}}{2}}\right)$, and thus that $\frac{\partial g_L}{\partial \kappa} = -g_L + 2\left(e^{-\frac{(\kappa_{HQ}+\kappa)}{2}} - e^{-\kappa}\right)e^{-\kappa}Zq^2 + r\sigma^2\left(1 - 2\rho M + M^2\right) > 0$ for all $g_L < 0$. This implies that, for a given κ_{HQ}, there is a unique $\hat{\kappa}$ so that $g_L\left(\hat{\kappa}, \kappa_{HQ}\right) = 0$, and for all $\kappa > \hat{\kappa}$, it is $g_L > 0$ and thus $\hat{\pi}^5 > \hat{\pi}^4$.

Consider now κ_{HQ}. Note first that $\frac{\partial g_L}{\partial \kappa_{HQ}} = \left(2e^{-\frac{\kappa}{2}} - \left(1-M\right)\right)e^{-\frac{\kappa_{HQ}}{2}}e^{-\kappa}Zq^2 > 0$ for $\kappa < \kappa' \equiv -2\ln\frac{1}{2}\left(1-M\right)$. Substituting κ' in g_L, we obtain

$$g_L|_{\kappa=\kappa'} \equiv \frac{\left(1+M\right)^2\left(1-M\right)^2}{8}Zq^2 + r\sigma^2\left(1 - 2\rho M + M^2 - \frac{\left(1-M\right)^2}{4}\left(1+\rho\right)\right) > 0,$$

where the inequality is obtained by noting that $h\left(\rho\right) \equiv 1 - 2\rho M + M^2 - \frac{(1-M)^2}{4}\left(1+\rho\right)$ is linear in ρ for any given M, thus achieving its minimum at an endpoint. Because $h\left(1\right) = \frac{1}{2}\left(1-M\right)^2 > 0$ and $h\left(-1\right) = \left(1+M\right)^2 > 0$, we have that $h\left(\rho\right) > 0$ for all $\rho \in [-1, 1]$, and thus that $g_L|_{\kappa=\kappa'} > 0$. This implies that in the neighborhood of $g_L = 0$, $\kappa < \kappa'$, and thus that $\frac{\partial g_L}{\partial \kappa_{HQ}} > 0$. Thus, there is a unique $\hat{\kappa}_{HQ}$ (allowing for the possibility that $\hat{\kappa}_{HQ} = 0$) such that $\hat{\pi}^2 > \hat{\pi}^1$ for $\kappa > \hat{\kappa}_{HQ}$, proving Theorem 6.

Case (C): Consider $H_d \in \left(-e^{\kappa}, -e^{-\kappa}\right)$ with $\beta_d > 0 > \gamma_d$. This case gives Theorem 7. As in Case (B),

$$\begin{aligned} \frac{d\hat{\pi}}{d\beta_d} = & -\hat{q}_d^{HQ}\check{a}_d + \left(1 - \beta_d - \gamma_{d'}\right)\hat{q}_d^{HQ}\frac{\partial \check{a}_d}{\partial \beta_d} + \left(1 - \beta_{d'} - \gamma_d\right)\hat{q}_{d'}^{HQ}\frac{\partial \check{a}_{d'}}{\partial \beta_d} \\ & + \frac{d\hat{u}_d(\check{a}_d, \hat{q}^d(\check{a}_d, w_d))}{d\beta_d} + \frac{d\hat{u}_{d'}(\check{a}_{d'}, \hat{q}^{d'}(\check{a}_{d'}, w_{d'}))}{d\beta_d}. \end{aligned}$$

Because $\frac{\partial \check{a}_d}{\partial \beta_d} = \frac{3\check{a}_d}{8\beta_d}$ and $\frac{\partial \check{a}_{d'}}{\partial \beta_d} = \frac{\check{a}_{d'}}{8\beta_d}$, from (A9) and (A3) we have that

$$\begin{aligned}\frac{d\hat{\pi}}{d\beta_d} &= -a_d\left(\hat{q}_d^{HQ} - \hat{q}_d^d\right) + (1-\beta_d-\gamma_{d'})\,\hat{q}_d^{HQ}\frac{3\check{a}_d}{8\beta_d} + (1-\beta_{d'}-\gamma_d)\,\hat{q}_{d'}^{HQ}\frac{\check{a}_{d'}}{8\beta_d} \\ &\quad -r\sigma^2(\beta_d+\rho\gamma_d) + \gamma_d\hat{q}_{d'}^d\frac{\check{a}_{d'}}{8\beta_d} + \gamma_{d'}\hat{q}_d^{d'}\frac{3\check{a}_d}{8\beta_d}. \qquad \text{(A25)}\end{aligned}$$

Consider now γ_d. We have that

$$\begin{aligned}\frac{d\hat{\pi}}{d\gamma_d} &= -\hat{q}_{d'}^{HQ}\check{a}_{d'} + (1-\beta_d-\gamma_{d'})\,\hat{q}_d^{HQ}\frac{\partial\check{a}_d}{\partial\gamma_d} + (1-\beta_{d'}-\gamma_d)\,\hat{q}_{d'}^{HQ}\frac{\partial\check{a}_{d'}}{\partial\gamma_d} \\ &\quad + \frac{d\hat{u}_d(\check{a}_d,\hat{q}^d(\check{a}_d,w_d))}{d\gamma_d} + \frac{d\hat{u}_{d'}(\check{a}_{d'},\hat{q}^{d'}(\check{a}_{d'},w_{d'}))}{d\gamma_d}.\end{aligned}$$

Because $\frac{\partial \check{a}_d}{\partial \gamma_d} = \frac{3\check{a}_d}{8\gamma_d}$, $\frac{\partial \check{a}_{d'}}{\partial \gamma_d} = \frac{\check{a}_{d'}}{8\gamma_d}$, from (A14) and (A6) we obtain

$$\begin{aligned}\frac{d\hat{\pi}}{d\gamma_d} &= -\check{a}_{d'}\left(\hat{q}_{d'}^{HQ} - \hat{q}_{d'}^d\right) + (1-\beta_d-\gamma_{d'})\,\hat{q}_d^{HQ}\frac{3\check{a}_d}{8\gamma_d} \qquad \text{(A26)} \\ &\quad + (1-\beta_{d'}-\gamma_d)\,\hat{q}_{d'}^{HQ}\frac{\check{a}_{d'}}{8\gamma_d} - r\sigma^2(\gamma_d+\rho\beta_d) + \hat{q}_{d'}^d\frac{\check{a}_{d'}}{8} + \gamma_{d'}\hat{q}_d^{d'}\frac{3\check{a}_d}{8\gamma_d}.\end{aligned}$$

From (A25) and (A26) we obtain the first-order conditions

$$\begin{aligned}\frac{d\hat{\pi}}{d\beta_d} &= -\check{a}_d\left(\hat{q}_d^{HQ} - \hat{q}_d^d\right) - r\sigma^2(\beta_d+\rho\gamma_d) + \frac{\Delta_d}{\beta_d} = 0, \\ \frac{d\hat{\pi}}{d\gamma_d} &= -\check{a}_{d'}\left(\hat{q}_{d'}^{HQ} - \hat{q}_{d'}^d\right) - r\sigma^2(\gamma_d+\rho\beta_d) + \frac{\Delta_d}{\gamma_d} = 0,\end{aligned}$$

where $\Delta_d \equiv (1-\beta_d-\gamma_{d'})\,\hat{q}_d^{HQ}\frac{3\check{a}_d}{8} + (1-\beta_{d'}-\gamma_d)\,\hat{q}_{d'}^{HQ}\frac{\check{a}_{d'}}{8} + \gamma_d\hat{q}_{d'}^d\frac{\check{a}_{d'}}{8} + \gamma_{d'}\hat{q}_d^{d'}\frac{3\check{a}_d}{8}$, giving

$$\beta_d\check{a}_d\left(\hat{q}_d^{HQ} - \hat{q}_d^d\right) + r\sigma^2\left(\beta_d^2 + \rho\gamma_d\beta_d\right) = \gamma_d\check{a}_{d'}\left(\hat{q}_{d'}^{HQ} - \hat{q}_{d'}^d\right) + r\sigma^2\left(\gamma_d^2 + \rho\beta_d\gamma_d\right).$$

Because the first-order conditions are symmetric, there exists a symmetric solution: $\beta_A = \beta_B = \beta$ and $\gamma_A = \gamma_B = \gamma$. Thus, $a_d = a = e^{-\frac{\kappa}{2}}Z\beta^{\frac{1}{2}}|\gamma|^{\frac{1}{2}}q$. This also implies that $\phi_A = \phi_B$, so $\hat{q}_d^{HQ} = e^{-\frac{\kappa_{HQ}}{2}}q$. Also, $H_d = \frac{\gamma}{\beta}$, so $\hat{q}_d^d = e^{-\frac{\kappa}{2}}\frac{|\gamma|^{\frac{1}{2}}}{\beta^{\frac{1}{2}}}q$ and $\hat{q}_{d'}^d = (2 - e^{-\frac{\kappa}{2}}\frac{\beta^{\frac{1}{2}}}{|\gamma|^{\frac{1}{2}}})q$. Thus, $\beta a\hat{q}_d^d = e^{-\frac{\kappa}{2}}\beta^{\frac{1}{2}}|\gamma|^{\frac{1}{2}}aq$ and

$$\gamma a\left(\hat{q}_{d'}^{HQ} - \hat{q}_{d'}^d\right) = \gamma a e^{-\frac{\kappa_{HQ}}{2}}q - 2\gamma aq - e^{-\frac{\kappa}{2}}\beta^{\frac{1}{2}}|\gamma|^{\frac{1}{2}}aq,$$

which implies that

$$\beta a e^{-\frac{\kappa_{HQ}}{2}}q + r\sigma^2\beta^2 = |\gamma|\left(2 - e^{-\frac{\kappa_{HQ}}{2}}\right)aq + r\sigma^2\gamma^2 \qquad \text{(A27)}$$

Because $\frac{\gamma}{\beta} \in (-e^{\kappa}, -e^{-\kappa})$, there exists $\hat{\xi} \in (e^{-\kappa}, e^{\kappa})$ such that $\gamma = -\hat{\xi}\beta$. Substituting in $a = e^{-\frac{\kappa}{2}} Z\beta\hat{\xi}^{\frac{1}{2}} q$, (A27) is equivalent to $f\left(\hat{\xi}\right) = 0$, where

$$f\left(\hat{\xi}\right) \equiv \left[\left(2e^{\frac{\kappa_{HQ}}{2}} - 1\right)\hat{\xi} - 1\right] e^{-\frac{\kappa_{HQ}}{2}} e^{-\frac{\kappa}{2}} \hat{\xi}^{\frac{1}{2}} Zq^2 + r\sigma^2\left(\hat{\xi}^2 - 1\right) = 0. \qquad \text{(A28)}$$

Note $f(e^{-\kappa}) < 0 < f(1) = 2\left[e^{\frac{\kappa_{HQ}}{2}} - 1\right] e^{-\frac{\kappa_{HQ}+\kappa}{2}} Zq^2$ and $f' > 0$, so $\hat{\xi} \in (e^{-\kappa}, 1)$ for $\kappa_{HQ} > 0$, but $\hat{\xi} = 1$ if $\kappa_{HQ} = 0$. Comparative statics on $\hat{\xi}$ follow because $max\left\{\frac{\partial f}{\partial r}, \frac{\partial f}{\partial \sigma^2}, \frac{\partial f}{\partial \kappa}\right\} < 0 < \min\left\{\frac{\partial f}{\partial Z}, \frac{\partial f}{\partial q}, \frac{\partial f}{\partial \kappa_{HQ}}\right\}$. Further, $\frac{\partial \hat{\pi}}{\partial \beta} = 0$ if and only if

$$\beta = \frac{1}{1 + \left(\frac{\hat{q}^d_{d'}}{\hat{q}^{HQ}_{d'}} - 1\right)\hat{\xi} + 2\left(1 - \frac{\hat{q}^d_d}{\hat{q}^{HQ}_d}\right) + \frac{2r\sigma^2\left(1-\rho\hat{\xi}\right)}{Z\hat{q}^{HQ}_d \hat{q}^d_d}}; \quad \gamma = -\hat{\xi}\beta < 0,$$

giving (48). After substitution in $\hat{\pi}$, we have

$$\hat{\pi}^6 \equiv \frac{e^{-(\kappa_{HQ}+\kappa)}\hat{\xi} Z^2 q^4}{2e^{-\frac{\kappa_{HQ}}{2}}\left[1 + \left(2e^{\frac{\kappa_{HQ}}{2}} - 1\right)\hat{\xi}\right] e^{-\frac{\kappa}{2}} \hat{\xi}^{\frac{1}{2}} Zq^2 - 3e^{-\kappa}\hat{\xi} q^2 Z + r\sigma^2\left(1 - 2\rho\hat{\xi} + \hat{\xi}^2\right)}.$$

Note $\hat{\pi}^6 \geq \hat{\pi}^4$ if and only if $g_S \geq 0$, where

$$\begin{aligned} g_S \equiv\ & \left(2M + 2(1-M)e^{-\frac{\kappa_{HQ}}{2}} + 2e^{-\kappa}\right) Zq^2 + e^{\kappa} r\sigma^2\left(1 - 2\rho M + M^2\right) \\ & -2e^{-\frac{\kappa_{HQ}}{2}}\left[1 + \left(2e^{\frac{\kappa_{HQ}}{2}} - 1\right)\hat{\xi}\right] e^{-\frac{\kappa}{2}} \hat{\xi}^{-\frac{1}{2}} Zq^2 - r\sigma^2 \frac{\left(1 - 2\rho\hat{\xi} + \hat{\xi}^2\right)}{\hat{\xi}}, \end{aligned}$$

with

$$\frac{\partial g_S}{\partial \kappa} = e^{\kappa} r\sigma^2\left(1 - \rho^2 + (\rho - M)^2\right) + \left\{e^{-\frac{\kappa_{HQ}}{2}} e^{-\frac{\kappa}{2}}\left[1 + \left(2e^{\frac{\kappa_{HQ}}{2}} - 1\right)\hat{\xi}\right]\hat{\xi}^{-\frac{1}{2}} - 2e^{-\kappa}\right\} Zq^2.$$

Note that $\left[1 + \left(2e^{\frac{\kappa_{HQ}}{2}} - 1\right)\hat{\xi}\right]\hat{\xi}^{-\frac{1}{2}}$ is increasing and larger than 2 for $\hat{\xi} \in (e^{-\kappa}, 1)$, so $\frac{\partial g_S}{\partial \kappa} > 0$. Also, $\frac{\partial g}{\partial \kappa_{HQ}} = -(1-M)e^{-\frac{\kappa_{HQ}}{2}} Zq^2 + \left(1 - \hat{\xi}\right) e^{-\frac{\kappa_{HQ}}{2}} e^{-\frac{\kappa}{2}} \hat{\xi}^{-\frac{1}{2}} Zq^2$. Because $M < e^{-\kappa} < \hat{\xi}$, we have that $\frac{\partial g}{\partial \kappa_{HQ}} < 0$. Defining $\hat{\kappa}$, $\hat{\kappa}^{HQ}_1$ so that $g_S\left(\hat{\kappa}, \hat{\kappa}^{HQ}_1\right) = 0$, Theorem 7 is proven.

Case (D): If $\gamma_d > e^{\kappa_d}\beta_d$, $\frac{\partial a_d}{\partial \gamma_d} = 0$, so $\frac{\partial a_{d'}}{\partial \gamma_d} = 0$, and thus $\frac{\partial \hat{\pi}}{\partial \gamma_d} = -a_{d'}\left(\hat{q}^{HQ}_{d'} - \hat{q}^d_{d'}\right) - r\sigma^2(\rho\beta + \gamma) < 0$, so $\gamma \leq e^{\kappa}\beta$. Similarly, if $\gamma_d < -e^{\kappa_d}\beta_d$, $\frac{\partial a_d}{\partial \gamma_d} = \frac{\partial a_{d'}}{\partial \gamma_d} = 0$, so $\frac{d\hat{\pi}}{d\gamma_d} = -a_{d'}\left(\hat{q}^{HQ}_{d'} - \hat{q}^d_{d'}\right) - r\sigma^2\left(\rho\beta_d + \gamma_d\right)$. Because $\phi_{d'} > 0 > \gamma_d$, $\hat{q}^{HQ}_{d'} < q_{d'} < \hat{q}^d_{d'}$. Also, $\rho \in (-1, 1)$. Thus, $\frac{d\hat{\pi}}{d\gamma_d} > 0$ for $\gamma_d < -e^{\kappa_d}\beta_d$, so it must be that $\gamma_d \geq -e^{\kappa_d}\beta_d$.

Therefore, Case (D) is suboptimal.

All that remains to be shown is Theorem 8, by showing that $\hat{\pi}^5 \geq \hat{\pi}^6$ when κ_{HQ} is large enough. Note $\hat{\pi}^5 \geq \hat{\pi}^6$ if and only if $g_E \geq 0$, where

$$\begin{aligned} g_E \equiv \; & 2e^{-\frac{\kappa_{HQ}}{2}}\left[1+\left(2e^{\frac{\kappa_{HQ}}{2}}-1\right)\hat{\xi}\right]e^{-\frac{\kappa}{2}}\hat{\xi}^{-\frac{1}{2}}Zq^2+r\sigma^2\left(1-2\rho\hat{\xi}+\hat{\xi}^2\right)/\hat{\xi} \\ & -4e^{-\frac{(\kappa_{HQ}+\kappa)}{2}}Zq^2-2r\sigma^2\left(1+\rho\right). \end{aligned} \tag{A29}$$

Note $\frac{\partial g_E}{\partial \hat{\xi}} = \frac{f(\hat{\xi})}{\hat{\xi}^2} = 0$. Note that

$$\frac{\partial g_E}{\partial \kappa_{HQ}} = \left[-\left(1-\hat{\xi}\right)\hat{\xi}^{-\frac{1}{2}}+2\right]e^{-\frac{(\kappa_{HQ}+\kappa)}{2}}Zq^2 \geq 0$$

if and only if $\hat{\xi} \geq 3-2\sqrt{2}$. Recall $\hat{\xi}$ is strictly decreasing in κ_{HQ}. This implies that g_E an inverse U-shaped function of κ_{HQ} and that there is a unique κ'_{HQ}, defined by $\hat{\xi}(\kappa'_{HQ}) = 3-2\sqrt{2}$, such that $\frac{\partial g_E}{\partial \kappa_{HQ}} > 0$ for $\kappa_{HQ} < \kappa'_{HQ}$ and $\frac{\partial g_E}{\partial \kappa_{HQ}} < 0$ for $\kappa_{HQ} > \kappa'_{HQ}$. Next, we will show that $g_E > 0$ for all $\kappa_{HQ} \geq \kappa'_{HQ}$ and, thus, for all $\hat{\xi} \leq 3-2\sqrt{2}$. Note that, from (A28), we can express (A29) as

$$g_E = 4e^{-\frac{\kappa_{HQ}}{2}}e^{-\frac{\kappa}{2}}\left(\hat{\xi}^{-\frac{1}{2}}-1\right)Zq^2+\frac{f\left(\hat{\xi}\right)}{\hat{\xi}}+2r\sigma^2\left[\frac{1}{\hat{\xi}}-2\rho-1\right].$$

The first term is positive because $\hat{\xi} < 1$, the second term is zero, and the third term is positive for all $\frac{1}{\hat{\xi}} > 3$, which is satisfied for $\hat{\xi} \leq 3-2\sqrt{2} < \frac{1}{3}$. This implies that $g_E(\kappa_{HQ}) > 0$ for all $\kappa_{HQ} \geq \kappa'_{HQ}$. Thus, if $g_E(0) \geq 0$, $g_E > 0$ for all $\kappa_{HQ} > 0$, and thus define $\hat{\kappa}_2^{HQ} \equiv 0$; otherwise, if $g_E(0) < 0$, there is a unique $\hat{\kappa}_2^{HQ}$ such that $g_E(\hat{\kappa}_2^{HQ}) = 0$, with $\hat{\kappa}_2^{HQ} < \kappa'_{HQ}$, completing the proof of Theorem 8. ■

Proof of Corollary 2. Follows directly from equation (A24). ■